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FITTED NUMERICAL SCHEME FOR SINGULARLY PERTURBED CONVECTION-DIFFUSION REACTION PROBLEMS INVOLVING DELAYS

Mesfin Woldaregay, Worku Aniley, and Gemechis Duressa

Abstract. This paper deals with solution methods for singularly perturbed delay differential equations having delay on the convection and reaction terms. The considered problem exhibits an exponential boundary layer on the left or right side of the domain. The terms with the delay are treated using Taylor's series approximation and the resulting singularly perturbed boundary value problem is solved using a specially designed exponentially finite difference method. The stability of the scheme is analysed and investigated using a comparison principle and solution bound. The formulated scheme converges uniformly with linear order of convergence. The theoretical findings are validated using three numerical test examples.

1. Introduction

Differential equations relate an unknown function to its derivatives evaluated at the same instance, but delay differential equations (DDEs) model physical problems for which the evaluation does not only depend on the present state of the system but also on the past history. In real time, the future behaviour of many phenomena is assumed to be described by the solution to a differential equation. Implicitly this assumption means that the future behaviour is uniquely determined by the present and independent of the past. However, in delay differential equations, the past exerts its influence in a significant manner upon the future. Many models are better represented by delay differential equations than differential equations. DDEs can be classified as a retarded or neutral type. We call DDEs retarded type if the delay argument does not occur in the highest order derivative term, otherwise they are known as neutral DDEs. DDEs arise in the mathematical modelling of various physical phenomena, for example in micro scale heat transfer [20], fluid dynamics [6], diffusion in polymers [10], reaction-diffusion equations [3], a lot of models in diseases or physiological processes [11, 21] etc.

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Singularly perturbed delay differential equations are differential equations in which their highest order derivative term is multiplied by a small perturbation parameter ε and involving at least one term with delay. In general, when the perturbation parameter tends to zero, the smoothness of the solution to the singularly perturbed delay differential equations (SPDDEs) deteriorates and it forms boundary layers [16]. The presence of the singular perturbation parameter ε leads to oscillation in the computed solution while using standard numerical methods in the FDM, FEM and Spline method [4]. To avoid this oscillation, a large number of mesh points are required when ε is very small. This is not practical and leads to a round-off error. So, to overcome the drawbacks associated with standard numerical methods, we developed a scheme using an exponentially fitted operator finite difference method which treats the problem without creating oscillation.

Solution methods for singularly perturbed delay differential equations have received great attention in the recent past because of their applicability in different research areas. Here, we give some recent findings on numerical treatment of the considered problem. In [7] Kadelbajoo and Ramesh used Taylor's series approximation for the delay terms and converted the problem into equivalent BVPs. The authors used upwind, midpoint upwind and hybrid of midpoint upwind on the regular region and central difference on the boundary layer region using piecewise uniform Shishkin mesh. In [9] Kumar and Kadalbajoo used Taylor's series approximation for the delay terms and converted the problem into equivalent BVPs. The authors computed the numerical solution using a B-spline collocation method on Shishkin mesh. In [2] Bahgat and Hafiz used Taylor's series approximation for the delay terms and applied fifth and sixth order finite difference approximation for the derivative terms and developed a finite difference scheme. In [15] Phaneendra and Lulu used a Gaussian quadrature integration method with an exponential fitting parameter. In [17] Ranjan and Prasad developed a FDM using an exponentially fitted finite difference method. In [1] Adilaxmi et al. used an initial value technique using an exponentially fitted non-standard FDM. In [4] Duressa and Reddy used a domain decomposition method for solving the problem. The contribution of this paper is developing a uniformly convergent numerical scheme using an exponentially fitted finite difference method and formulating the uniform convergence analysis of the scheme.

Notations: The symbol C (in some cases indexed) denotes a positive constant independent of ε and the number of mesh intervals N. The norm $\|.\|$ denotes the maximum norm.

2. Continuous problem

Consider a class of singularly perturbed delay differential equations having delay on the convection and reaction terms of the form:

$$\begin{cases} -\varepsilon u''(x) + a(x)u'(x-\delta) + \beta(x)u(x) + \omega(x)u(x-\delta) = f(x), \quad x \in \Omega = (0,1), \\ u(x) = \phi(x), \quad x \in \Omega_L = [-\delta, 0],\ u(1) = \gamma, \end{cases} \tag{2.1}$$

where $\varepsilon, 0 < \varepsilon \ll 1$ is a singular perturbation parameter and δ is a delay parameter satisfying $\delta < \varepsilon$. The functions $a(x), \beta(x), \omega(x)$, and $f(x)$ are assumed to be smooth, bounded and not a function of ε. The values of $\phi(x)$ and γ are assumed to be finite constants. We assume also that the coefficients of non-derivative terms $\beta(x)$ and $\omega(x)$ satisfy

$$\beta(x) + \omega(x) \geqslant \theta > 0, \quad \forall x \in \bar{\Omega},$$

for some constant θ. This condition ensures that the solution to (2.1) exhibits a boundary layer in the neighborhood of $x = 0$ or $x = 1$ depending on the sign of the convective term $a(x)$. When $a(x) < 0$, the regular boundary layer appears near $x = 0$ and for $a(x) > 0$ corresponds to the existence of a boundary layer near $x = 1$. If $a(x)$ changes sign, shock layer will appear in the middle of the domain [22]. The layer is maintained for $\delta \neq 0$, but sufficiently small.

2.1. Estimate for terms with the delay. For $\delta < \varepsilon$, using Taylor's series approximation for the terms with the shifts is valid [19]. So, we approximate $u(x - \delta)$ and $u'(x - \delta)$ as

$$\begin{aligned} u'(x-\delta) &\approx u'(x) - \delta u''(x) + O(\delta^2), \\ u(x-\delta) &\approx u(x) - \delta u'(x) + \frac{\delta^2}{2} u''(x) + O(\delta^3). \end{aligned} \tag{2.2}$$

Substituting the approximations in (2.2) into (2.1) results in

$$\begin{cases} -c_\varepsilon(x) u''(x) + p(x) u'(x) + q(x) u(x) = f(x), & x \in \Omega = (0,1), \\ u(0) = \phi(0), \ u(1) = \gamma, \end{cases} \tag{2.3}$$

where $c_\varepsilon(x) = \varepsilon + \delta a(x) - \frac{\delta^2}{2}\omega(x)$, $p(x) = a(x) - \delta\omega(x)$ and $q(x) = \beta(x) + \omega(x)$. For small values of ε, (2.1) is asymptotically equivalent to (2.3). We assume $0 < c_\varepsilon(x) \leqslant \varepsilon - \delta M_1 - \delta^2 M_2 = c_\varepsilon$, where $a(x) \geqslant M_1$ and $\omega(x) \geqslant 2M_2$ for M_1 and M_2 are constants. We consider first the case $p(x) \leqslant p^* < 0$, which implies occurrence of the boundary layer of thickness $O(c_\varepsilon)$ near the left side of the domain, the other case $p(x) \geqslant p^* > 0$ implies the occurrence on the right side of the domain.

Setting $c_\varepsilon = 0$ in (2.3), we obtain a problem which is called reduced problem for $p(x) > 0$ given as

$$\begin{cases} p(x) u_0'(x) + q(x) u_0(x) = f(x), & \forall x \in \Omega, \\ u_0(0) = \phi(0). \end{cases} \tag{2.4}$$

It is first order IVPs. For small values of c_ε the solution to (2.3) is very close to the solution to (2.4).

2.2. Bounds and properties of the continuous solution.

LEMMA 2.1 (The Maximum Principle). *Let z be a sufficiently smooth function defined on Ω which satisfies $z(0) \geqslant 0$ and $z(1) \geqslant 0$. Then $Lz(x) > 0$, $\forall x \in \Omega$ implies that $z(x) \geqslant 0$, $\forall x \in \bar{\Omega}$, where L stands for the differential operator $Lz = -c_\varepsilon z''(x) + p(x) z'(x) + q(x) z(x)$.*

Proof. Let x^* be such that $z(x^*) = \min_{(x)\in\bar{\Omega}} z(x)$ and suppose that $z(x^*) < 0$. It is clear that $x^* \notin \{0, 1\}$. Since $z(x^*) = \min_{(x)\in\bar{\Omega}} z(x)$, which implies $z'(x^*) = 0$ and $z''(x^*) \geqslant 0$ and implies that $Lz(x^*) < 0$, which is a contradiction of the assumption made above that $Lz(x^*) > 0$, $\forall x \in \Omega$. Therefore $z(x) \geqslant 0$, $\forall x \in \Omega$. □

Lemma 2.2 (Stability estimate). *The solution to* (2.3) *satisfies the bound*

$$|u(x)| \leqslant \theta^{-1}\|f\| + \max\{|\phi(0)|, |\gamma|\}.$$

Proof. Let us define barrier functions $\vartheta^{\pm}(x,t) = \theta^{-1}\|f\| + \max\{\phi(0), \gamma\} \pm u(x)$. At the boundary points, we obtain

$$\vartheta^{\pm}(0) = \theta^{-1}\|f\| + \max\{\phi(0), \gamma\} \pm u(0) \geqslant 0,$$
$$\vartheta^{\pm}(1) = \theta^{-1}\|f\| + \max\{\phi(0), \gamma\} \pm u(1) \geqslant 0.$$

On the operator L we have

$$\begin{aligned} L\vartheta^{\pm}(x) &= -c_\varepsilon \vartheta''_{\pm}(x) + p(x)\vartheta'_{\pm}(x) + q(x)\vartheta_{\pm}(x) \\ &= -c_\varepsilon(0 \pm u''(x)) + p(x)\big(0 \pm u'(x)\big) + q(x)\big(\theta^{-1}\|f\| + \max\{\phi(0),\gamma\} \pm u(x)\big) \\ &= q(x)\big(\theta^{-1}\|f\| + \max\{\phi(0),\gamma\}\big) \pm f(x) \geqslant 0, \end{aligned}$$

since $q(x) \geqslant \theta > 0$, which implies $L\vartheta_{\pm}(x) \geqslant 0$. Hence by maximum principle in Lemma 2.1 we obtain $\vartheta_{\pm}(x) \geqslant 0$, $\forall x \in \bar{\Omega}$. □

Lemma 2.3. *The derivatives of the solutions to the problem in* (2.3) *satisfy the bound*

$$|u^{(k)}(x)| \leqslant C\Big(1 + c_\varepsilon^{-k} \exp\Big(-\frac{p^* x}{c_\varepsilon}\Big)\Big), \quad x \in \bar{\Omega},\ 0 \leqslant k \leqslant 4,$$

for the left boundary layer,

$$|u^{(k)}(x)| \leqslant C\Big(1 + c_\varepsilon^{-k} \exp\Big(-\frac{p^*(1-x)}{c_\varepsilon}\Big)\Big), \quad x \in \bar{\Omega},\ 0 \leqslant k \leqslant 4,$$

for the right boundary layer.

Proof. See [8], [12]. □

3. Numerical scheme

There are two major techniques for designing numerical schemes which give small errors in the boundary layer region. The first approach is the class of fitted mesh methods which uses fine mesh in the boundary layer region and coarse mesh in the outer layer region. The second approach is the exponentially fitted operator methods, which use uniform mesh and an exponentially fitting factor is induced on the term containing the singular perturbation parameter for stabilizing the influence of the singular perturbation parameter. In this approach the difference schemes reflect the qualitative behaviour of the solution inside the boundary layer region. In this manuscript, we apply the exponentially fitted operator methods for solving the considered problem.

First discretize the domain $\bar{\Omega} = [0, 1]$ into the equal number of subintervals N with the mesh length h. Let $\bar{\Omega}^N = \{x_i = ih\}_0^N$, where $h = \frac{1}{N}$ is the discretized domain. Let $u(x)$ be a smooth function on the domain $\bar{\Omega} = [0, 1]$. Then, by using Taylor's series we have

$$\begin{aligned} u(x_{i+1}) = u_{i+1} = u_i + hu'_i + \frac{h^2}{2!}u''_i + \frac{h^3}{3!}u^{(3)}_i + \frac{h^4}{4!}u^{(4)}_i + \frac{h^5}{5!}u^{(5)}_i + \frac{h^6}{6!}u^{(6)}_i + O(h^7), \\ u(x_{i-1}) = u_{i-1} = u_i - hu'_i + \frac{h^2}{2!}u''_i - \frac{h^3}{3!}u^{(3)}_i + \frac{h^4}{4!}u^{(4)}_i - \frac{h^5}{5!}u^{(5)}_i + \frac{h^6}{6!}u^{(6)}_i + O(h^7). \end{aligned} \tag{3.1}$$

Using the difference in (3.1) we obtain

$$u_{i-1} - 2u_i + u_{i+1} = \frac{2h^2}{2!}u''_i + \frac{2h^4}{4!}u^{(4)}_i + \frac{2h^6}{6!}u^{(6)}_i + O(h^8). \tag{3.2}$$

Differentiating (3.2) twice gives

$$u''_{i-1} - 2u''_i + u''_{i+1} = \frac{2h^2}{2!}u^{(4)}_i + \frac{2h^4}{4!}u^{(6)}_i + \frac{2h^6}{6!}u^{(8)}_i + O(h^8). \tag{3.3}$$

Multiplying (3.3) by $-\frac{h^2}{12}$ and adding to (3.2) to eliminate the term with $u^{(4)}_i$ gives

$$u_{i-1} - 2u_i + u_{i+1} = \frac{h^2}{12}[u''_{i-1} + 10u''_i + u''_{i+1}] + O(h^6). \tag{3.4}$$

Now solving (2.3) at x_{i-1}, x_i and x_{i+1}, we get

$$\begin{aligned} -c_\varepsilon u''_{i-1} &= -p(x_{i-1})u'_{i-1} - q(x_{i-1})u_{i-1} + f(x_{i-1}), \\ -c_\varepsilon u''_i &= -p(x_i)u'_i - q(x_i)u_i + f(x_i), \text{ and} \\ -c_\varepsilon u''_{i+1} &= -p(x_{i+1})u'_{i+1} - q(x_{i+1})u_{i+1} + f(x_{i+1}). \end{aligned} \tag{3.5}$$

We approximate the first derivative terms u'_{i-1}, u'_i and u'_{i+1} in (3.5) respectively as

$$\begin{aligned} u'_{i-1} &= \frac{u_{i-1} - 4u_i + 3u_{i+1}}{2h} + \frac{u_{i-1} - 2u_i + u_{i+1}}{h} + O(h^2), \\ u'_i &= \frac{u_{i+1} - u_{i-1}}{2h} + O(h^2), \text{ and} \\ u'_{i+1} &= \frac{-3u_{i-1} + 4u_i - u_{i+1}}{2h} - \frac{u_{i-1} - 2u_i + u_{i+1}}{h} + O(h^2). \end{aligned} \tag{3.6}$$

Let us denote by U_i the approximate solution to $u(x_i)$ in the above discretization. Substituting (3.6) into (3.5), then (3.5) into (3.4), we obtain

$$\begin{aligned} L^h U_i \equiv &- \Big(c_\varepsilon - \frac{hp_{i-1}}{12} + \frac{hp_{i+1}}{12}\Big)\Big(\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2}\Big) \\ &+ \frac{p_{i-1}}{24h}(-3U_{i-1} + 4U_i - U_{i+1}) + \frac{10p_i}{24h}(U_{i+1} - U_{i-1}) \\ &+ \frac{p_{i+1}}{24h}(U_{i-1} - 4U_i + 3U_{i+1}) + \frac{q_{i-1}}{12}U_{i-1} + \frac{10q_i}{12}U_i \\ &+ \frac{q_{i+1}}{12}U_{i+1} = \frac{1}{12}[f_{i-1} + 10f_i + f_{i+1}], \ i = 1, 2, \ldots, N-1, \end{aligned} \tag{3.7}$$

where p_{i-1}, p_i and p_{i+1} denote $p(x_{i-1}), p(x_i)$ and $p(x_{i+1})$ respectively, similarly for q and f.

We apply an exponentially fitted operator finite difference method (FOFDM) to find the numerical solution to the problem in (2.3). We use the theory developed in the asymptotic method for solving singularly perturbed BVPs. Let us consider and treat separately the left and the right boundary layer cases.

3.1. Left boundary layer problems. For left boundary layer problems, from the theory of singular perturbation in [14], the zeroth order asymptotic solution to (2.3) is given by

$$u(x) = u_0(x) + \frac{p(0)}{p(x)}(\phi(0) - u_0(0)) \exp\Big(- \int_0^x \Big(\frac{p(x)}{c_\varepsilon} - \frac{q(x)}{p(x)}\Big) dx\Big), \tag{3.8}$$

where u_0 is the solution to the reduced problem. Using Taylor's series approximation for $u_0(x), p(x)$ and $q(x)$ centring at $x_i = ih$ up to first order and considering $c_\varepsilon \to 0$, the discretized form of (3.8) becomes

$$u(ih) = u_0(ih) + (\phi(0) - u_0(0)) \exp(-p(x_i)i\rho),$$

where $\rho = h/c_\varepsilon$, $h = 1/N$. Similarly we write

$$\begin{aligned} u((i+1)h) &= u_0(ih) + (\phi(0) - u_0(0)) \exp(-p(x_i)(i+1)\rho), \\ u((i-1)h) &= u_0(ih) + (\phi(0) - u_0(0)) \exp(-p(x_i)(i-1)\rho). \end{aligned}$$

To handle the effect of the perturbation parameter, the exponentially fitting factor $\sigma_1(\rho)$ is multiplied on the term containing the perturbation parameter as

$$\begin{aligned} L_1^h U_i \equiv &- \Big[\sigma_1(\rho)c_\varepsilon - \frac{hp_{i-1}}{12} + \frac{hp_{i+1}}{12}\Big]\Big(\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2}\Big) \\ &+ \frac{p_{i-1}}{24h}(-3U_{i-1} + 4U_i - U_{i+1}) + \frac{10p_i}{24h}(U_{i+1} - U_{i-1}) \\ &+ \frac{p_{i+1}}{24h}(U_{i-1} - 4U_i + 3U_{i+1}) + \frac{q_{i-1}}{12}U_{i-1} + \frac{10q_i}{12}U_i \\ &+ \frac{q_{i+1}}{12}U_{i+1} = \frac{1}{12}[f_{i-1} + 10f_i + f_{i+1}], \quad i = 1, 2, \ldots, N-1. \end{aligned} \tag{3.9}$$

Multiplying both sides of (3.9) by h, substituting $h/c_\varepsilon =: \rho$ and taking the limit as $h \to 0$ gives

$$\begin{aligned} - \lim_{h\to 0} \frac{\sigma_1(\rho)}{\rho}(U_{i-1} - 2U_i + U_{i+1}) + \frac{p_0}{24} \lim_{h\to 0}(-3U_{i-1} + 4U_i - U_{i+1}) \\ + \frac{10p_0}{24} \lim_{h\to 0}(U_{i+1} - U_{i-1}) + \frac{p_0}{24} \lim_{h\to 0}(U_{i-1} - 4U_i + 3U_{i+1}) = 0. \end{aligned}$$

Simplifying, the exponential fitting factor is obtained as

$$\sigma_1(\rho) = \frac{\rho p(0)}{2} \coth(\frac{\rho p(0)}{2}).$$

Hence, the finite difference scheme becomes

$$\begin{aligned} L_1^h U_i \equiv &- \Big[\sigma_1(\rho)c_\varepsilon - \frac{hp_{i-1}}{12} + \frac{hp_{i+1}}{12}\Big]\Big(\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2}\Big) \\ &+ \frac{p_{i-1}}{24h}(-3U_{i-1} + 4U_i - U_{i+1}) + \frac{10p_i}{24h}(U_{i+1} - U_{i-1}) \\ &+ \frac{p_{i+1}}{24h}(U_{i-1} - 4U_i + 3U_{i+1}) + \frac{q_{i-1}}{12}U_{i-1} + \frac{10q_i}{12}U_i \end{aligned} \tag{3.10}$$

$$+\frac{q_{i+1}}{12}U_{i+1}=\frac{1}{12}[f_{i-1}+10f_i+f_{i+1}],\quad i=1,2,\dots,N-1.$$

3.2. Right boundary layer problems. From [14] for right boundary layer problems, the zeroth order asymptotic solution to (2.3) is given as

$$u(x)=u_0(x)+\frac{p(1)}{p(x)}(\gamma-u_0(1))\exp\Big(-\int_x^1\Big(\frac{p(x)}{c_\varepsilon}-\frac{q(x)}{p(x)}\Big)dx\Big).$$

Using Taylor's series approximation at $x=1$ for $p(x)$ and $q(x)$ and by considering $c_\varepsilon\to 0$, we obtain

$$u(x)=u_0(x)+(\gamma-u_0(1))\exp\Big(-\frac{p(1)(1-x)}{c_\varepsilon}\Big),$$

where u_0 is the solution to the reduced problem. The exponential fitting factor is obtained as

$$\sigma_2(\rho)=\frac{\rho p(1)}{2}\coth\Big(\frac{\rho p(1)}{2}\Big).$$

Hence, the required finite difference scheme becomes

$$\begin{aligned}(3.11)\qquad L_2^hU_i\equiv&-\Big[\sigma_2(\rho)c_\varepsilon-\frac{hp_{i-1}}{12}+\frac{hp_{i+1}}{12}\Big]\Big(\frac{U_{i-1}-2U_i+U_{i+1}}{h^2}\Big)\\&+\frac{p_{i-1}}{24h}(-3U_{i-1}+4U_i-U_{i+1})+\frac{10p_i}{24h}(U_{i+1}-U_{i-1})\\&+\frac{p_{i+1}}{24h}(U_{i-1}-4U_i+3U_{i+1})+\frac{q_{i-1}}{12}U_{i-1}+\frac{10q_i}{12}U_i+\frac{q_{i+1}}{12}U_{i+1}\\=&\frac{1}{12}[f_{i-1}+10f_i+f_{i+1}],\quad i=1,2,\dots,N-1.\end{aligned}$$

3.3. Stability and uniform convergence analysis. In this section, we show the convergence analysis for the left boundary layer problem and in a similar manner it is shown for the right boundary layer problems. First, we need to prove the discrete maximum principle for the scheme in (3.10) for guaranteeing the existence of a unique discrete solution.

LEMMA 3.1 (Discrete comparison principle). *Assume that for the mesh function U_i there exists a comparison function V_i such that $L_1^hU_i\leqslant L_1^hV_i$, $i=1,2,\dots,N-1$ and if $U_0\leqslant V_0$ and $U_N\leqslant V_N$ then $U_i\leqslant V_i$, $\forall i, i=0,1,2,\dots,N$.*

PROOF. The matrix associated with the operator L_1^h is of size $(N+1)\times(N+1)$ and satisfies the property of the M-matrix. See the detailed proof in [8]. □

LEMMA 3.2 (Discrete stability estimate). *The solution U_i to the discrete scheme in (3.10) satisfies the following bound*

$$|U_i|\leqslant\theta^{-1}\|L_1^hU_i\|+\max\{|U_0|,|U_N|\}.$$

PROOF. Let $s=\theta^{-1}\|L_1^hU_i\|+\max\{U_0,U_N\}$ and define the barrier function $\vartheta_i^\pm$ by $\vartheta_i^\pm=s\pm U_i$. On the boundary points, we obtain

$$\begin{aligned}\vartheta_0^\pm&=s\pm U_0=\theta^{-1}\|L_1^hU_i\|+\max\{U_0,U_N\}\pm\phi(0)\geqslant 0,\\\vartheta_N^\pm&=s\pm U_N=\theta^{-1}\|L_1^hU_i\|+\max\{U_0,U_N\}\pm\gamma\geqslant 0.\end{aligned}$$

On the discretized spatial domain x_i, $1 < i < N-1$, we obtain

$$\begin{aligned}
L_1^h \vartheta_i^{\pm} =& -\Big(c_\varepsilon \sigma(\rho) - \frac{hp_{i-1}}{12} + \frac{hp_{i+1}}{12}\Big)\Big(\frac{s \pm U_{i+1} - 2(s \pm U_i) + s \pm U_{i-1}}{h^2}\Big) \\
&+ \frac{p_{i-1}}{24h}(-3(s+U_{i-1}) + 4(s+U_i) - (s+U_{i+1})) + \frac{10p_i}{24h}((s+U_{i+1}) \\
&- (s+U_{i-1})) + \frac{p_{i+1}}{24h}((s+U_{i-1}) - 4(s+U_i) + 3(s+U_{i+1})) \\
&+ \frac{q_{i-1}}{12}(s+U_{i-1}) + \frac{10q_i}{12}(s+U_i) + \frac{q_{i+1}}{12}(s+U_{i+1}) \\
=& \mp \Big(c_\varepsilon \sigma(\rho) - \frac{hp_{i-1}}{12} + \frac{hp_{i+1}}{12}\Big)\Big(\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}\Big) \\
&\pm \frac{p_{i-1}}{24h}(-3U_{i-1} + 4U_i - U_{i+1}) \pm \frac{10p_i}{24h}(U_{i+1} - U_{i-1}) \\
&\pm \frac{p_{i+1}}{24h}(U_{i-1} - 4U_i + 3U_{i+1}) \pm \frac{q_{i-1}}{12}U_{i-1} \\
&\pm \frac{10q_i}{12}U_i \pm \frac{q_{i+1}}{12}U_{i+1} + \Big(\frac{q_{i-1}}{12} + \frac{10q_i}{12} + \frac{q_{i+1}}{12}\Big)s \\
=& \Big(\frac{q_{i-1}}{12} + \frac{10q_i}{12} + \frac{q_{i+1}}{12}\Big)\Big(\theta^{-1}\|L_L^h U_i\| + \max\{U_0, U_N\}\Big) \\
&\pm \frac{1}{12}[f_{i-1} + 10f_i + f_{i+1}] \geqslant 0, \quad \text{since} \quad q_i \geqslant \theta.
\end{aligned}$$

Using the discrete comparison principle in Lemma 3.1, we obtain $\vartheta_i^{\pm} \geqslant 0, \forall x_i \in \bar{\Omega}^N$. Hence, the required bound is obtained. □

First, let us denote the left shifted, central and right shifted finite differences respectively as

$$\begin{aligned}
\Delta_L u(x_i) &= \frac{u_{i-1} - 4u_i + 3u_{i+1}}{2h} + \frac{u_{i-1} - 2u_i + u_{i+1}}{h}, \\
\Delta_C u(x_i) &= \frac{u_{i+1} - u_{i-1}}{2h}, \text{ and} \\
\Delta_R u(x_i) &= \frac{-3u_{i-1} + 4u_i - u_{i+1}}{2h} - \frac{u_{i-1} - 2u_i + u_{i+1}}{h}.
\end{aligned}$$

Using Taylor's series expansion, the approximation for the differences of the derivatives is bounded as

$$\begin{aligned}
|u''(x_i) - D^+D^-u(x_i)| &\leqslant Ch^2\|u^{(4)}(x_i)\|, \\
|D^+D^-u(x_i)| &\leqslant C\ u''(x_i)\|,
\end{aligned} \tag{3.12}$$

and for the first derivative approximations

$$\begin{aligned}
|u'_{i-1} - \Delta_L u(x_i)| &\leqslant Ch^2\|u^{(4)}(x_i)\|, \\
|u'_i - \Delta_C u(x_i)| &\leqslant Ch^2\|u''(x_i)\|, \\
|u'_{i+1} - \Delta_R u(x_i)| &\leqslant Ch^2\|u^{(4)}(x_i)\|,
\end{aligned} \tag{3.13}$$

where $\|u^{(k)}(x_i)\| = \max_{0 \leqslant i \leqslant N} |u^{(k)}(x_i)|, k = 2, 4.$

Now for $z > 0$, C_1 and C_2 are constants we have,

$$C_1\frac{z^2}{z+1} \leqslant z\coth(z) - 1 \leqslant C_2\frac{z^2}{z+1} \quad \text{and} \quad c_\varepsilon\frac{(h/c_\varepsilon)^2}{h/c_\varepsilon+1} = \frac{h^2}{h+c_\varepsilon} \tag{3.14}$$

giving

$$\Big| - c_\varepsilon\Big[p(1)\frac{\rho}{2}\coth\Big(p(1)\frac{\rho}{2}\Big) - 1\Big]D^+D^-u(x_i)\Big| \leqslant \frac{Ch^2}{h+c_\varepsilon}\|u''(x_i)\|.$$

In this paper, the aim is to show the scheme convergence independent of the perturbation parameter c_ε. The following theorem gives a truncation error bound of the proposed scheme.

THEOREM 3.1. *Let $u(x_i)$ and U_i be the exact and numerical solutions to* (2.4) *and* (3.10) *respectively. Then the following error estimate holds:*

$$|L_1^h(u(x_i) - U_i)| \leqslant \frac{Ch^2}{h+c_\varepsilon}\Big(1 + c_\varepsilon^{-4}\max_{x_1\leqslant x_i\leqslant x_{N-1}}\exp\Big(-\frac{p^*x_i}{c_\varepsilon}\Big)\Big).$$

PROOF. Consider the truncation error

$$\begin{aligned}
|L^hu(x_i) - L_1^hU_i| &\leqslant | - c_\varepsilon\sigma(\rho)(u''(x_i) - D^+D^-u(x_i))| + \Big|\frac{p_{i-1}}{12}(u'_{i-1} - \Delta_Lu(x_i))\Big| \\
&\quad + \Big|\frac{10p_i}{12}(u'_i - \Delta_Cu(x_i))\Big| + \Big|\frac{p_{i+1}}{12}(u'_{i+1} - \Delta_Ru(x_i))\Big| \\
&\leqslant \Big| - c_\varepsilon\Big[p(1)\frac{\rho}{2}\coth\Big(p(1)\frac{\rho}{2}\Big) - 1\Big]D^+D^-u(x_i)\Big| \\
&\quad + |c_\varepsilon(u''(x_i) - D^+D^-u(x_i))| + \Big|\frac{p_{i-1}}{12}(u'_{i-1} - \Delta_Lu(x_i))\Big| \\
&\quad + \Big|\frac{10p_i}{12}(u'_i - \Delta_Cu(x_i))\Big| + \Big|\frac{p_{i+1}}{12}(u'_{i+1} - \Delta_Ru(x_i))\Big|.
\end{aligned}$$

Using the bounds in (3.14), (3.13) and (3.12), we obtain

$$\begin{aligned}
|L^h(u(x_i) - U_i)| &\leqslant \frac{Ch^2}{h+c_\varepsilon}\|u''(x_i)\| + c_\varepsilon Ch^2\|u^{(4)}(x_i)\| + Ch^2\|u^{(4)}(x_i)\| \\
&\leqslant \frac{Ch^2}{h+c_\varepsilon}\|u''(x_i)\| + Ch^2\|u^{(4)}(x_i)\|.
\end{aligned}$$

Using the bounds for the derivatives of the solution in Lemma 2.3 gives

$$\begin{aligned}
|L^h(u(x_i) - U_i)| &\leqslant \frac{Ch^2}{h+c_\varepsilon}\Big(1 + c_\varepsilon^{-2}\exp\Big(\frac{-p^*x_i}{c_\varepsilon}\Big)\Big) + Ch^2\Big(1 + c_\varepsilon^{-4}\exp\Big(\frac{-p^*x_i}{c_\varepsilon}\Big)\Big) \\
&\leqslant \frac{Ch^2}{h+c_\varepsilon}\Big(1 + c_\varepsilon^{-4}\max_{x_1\leqslant x_i\leqslant x_{N-1}}\exp\Big(\frac{-p^*x_i}{c_\varepsilon}\Big)\Big), \text{ since } c_\varepsilon^{-4} \geqslant c_\varepsilon^{-2}. \ \square
\end{aligned}$$

LEMMA 3.3. *For $c_\varepsilon \to 0$ and for a given fixed N, we obtain*

$$\lim_{c_\varepsilon\to 0}\max_j\frac{\exp\big(\frac{-p^*x_j}{c_\varepsilon}\big)}{c_\varepsilon^m} = 0, \quad \lim_{c_\varepsilon\to 0}\max_j\frac{\exp\big(\frac{-p^*(1-x_j)}{c_\varepsilon}\big)}{c_\varepsilon^m} = 0, \qquad m = 1,2,3,\ldots$$

where $x_j = jh$, $h = 1/N$, $\forall j = 1,2,\ldots,N-1$.

PROOF. See [22]. $\square$

THEOREM 3.2. *Under the hypothesis of boundedness of discrete solution, the solution to the discrete schemes in* (3.10) *satisfies the following uniform error bound*

$$\sup_{0<c_\varepsilon \ll 1} \|u(x_i) - U_i\| \leqslant Ch.$$

PROOF. Substituting the results in Lemma 3.3 into Theorem 3.1 and then applying the discrete comparison principle gives the required bound. □

4. Numerical results and discussion

To demonstrate the efficiency of the proposed scheme, we solved model examples having boundary layer behaviour.

EXAMPLE 4.1. We consider the problem

$$-\varepsilon u''(x) + (1+x)u'(x-\delta) - e^{-2x}u(x-\delta) + e^{-x}u(x) = 0$$

with interval-boundary conditions $u(x) = 1,\ -\delta \leqslant x < 0$ and $u(1) = -1$.

EXAMPLE 4.2. We consider the problem

$$-\varepsilon u''(x) + (1+x)u'(x-\delta) - e^{-2x}u(x-\delta) + e^{-x}u(x) = e^{x-1}$$

with interval-boundary conditions $u(x) = 1,\ -\delta \leqslant x < 0$ and $u(1) = -1$.

EXAMPLE 4.3. We consider the problem

$$\varepsilon u''(x) + (1+x)u'(x-\delta) + \sin(2x)u(x-\delta) - e^{-x}u(x) = \sin(2x) + 3e^{-x}$$

with interval-boundary conditions $u(x) = -1,\ -\delta \leqslant x < 0$ and $u(1) = 1$.

Since the exact solutions for the considered problems are not known, the maximum absolute errors are estimated using the double mesh principle and they are defined as

$$E_\varepsilon^N = \max_{0\leqslant i\leqslant N} |U_i^N - U_i^{2N}|,$$

where U_i^N stands for the solution to the problem on the number of mesh points N and U_i^{2N} stands for the numerical solution to the problem on the number of mesh points $2N$ by including the mid-points $x_{i+1/2}$ into the mesh numbers. The uniform error estimate is defined as

$$E^N = \max_{\varepsilon} |E_\varepsilon^N|.$$

The rate of convergence of the scheme is given by

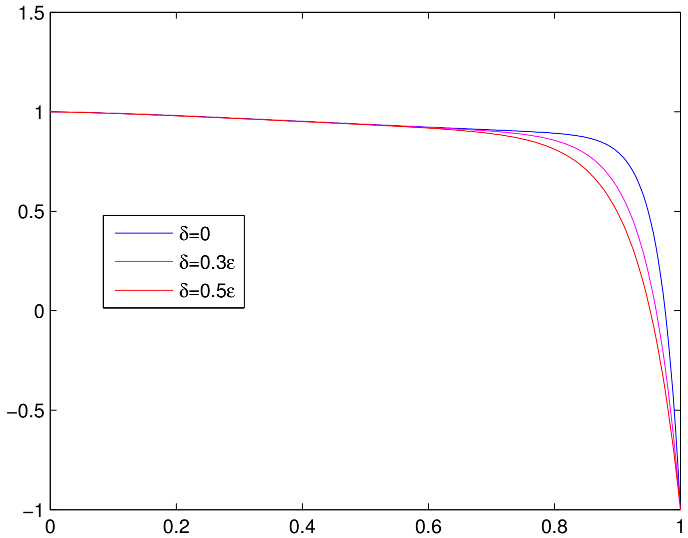
$$R_\varepsilon^N = \log_2(E_\varepsilon^N / E_\varepsilon^{2N}),$$

and the uniform rate of convergence is given by

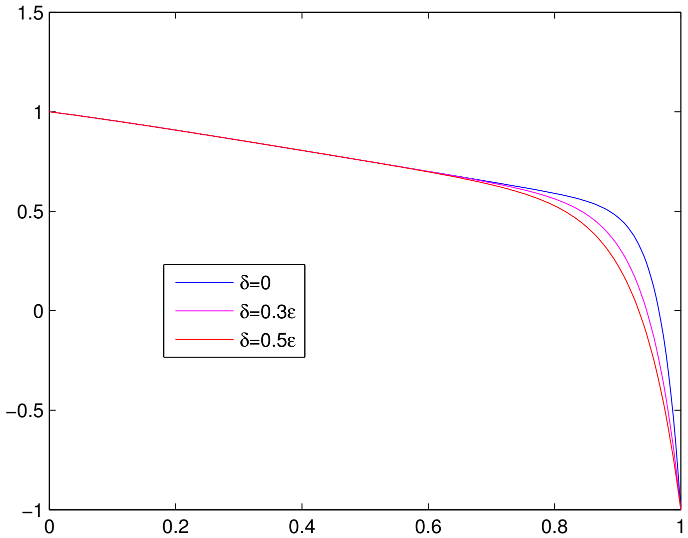
$$R^N = \log_2(E^N / E^{2N}).$$

In Figure 1, the influence of the delay parameter on the behaviour of the solution for Examples 4.1 and 4.2 is shown for $\varepsilon = 2^{-4}$ and $\delta = 0, 0.3\varepsilon$ and 0.5ε.

From these figures we observe that for the right boundary layer problem as the values of the delay parameter increase, the thickness of the boundary layer increases and vice versa for the left layer problem. From Figures 2 and 3, we observe the numerical solution for Examples 4.2 and 4.3 in (a) using the scheme without the exponential fitting factor which gives bad approximation and in (b) using the proposed exponentially fitted scheme which gives good approximation.



(a)



(b)

FIGURE 1. Effect of the delay on the solution (a) Example 4.1, (b) Example 4.2 for $\varepsilon = 2^{-4}$.

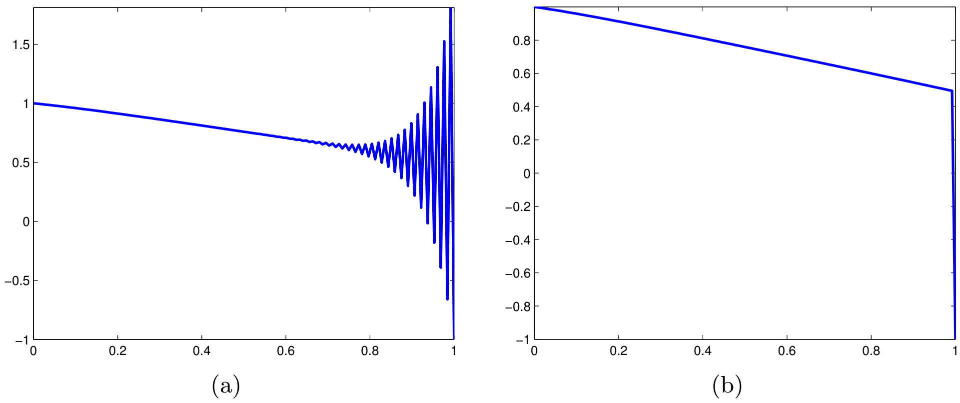


FIGURE 2. Computed solution for Example 4.2 in (a) using scheme (3.7) without the exponential fitting in (b) fitted scheme in (3.11) for $\varepsilon = 2^{-12}$.

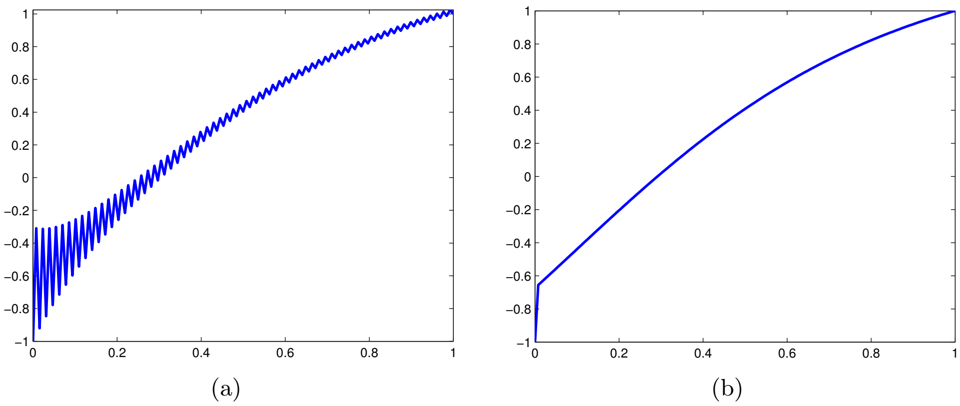


FIGURE 3. Computed solution for Example 4.3 in (a) using scheme (3.7) without the exponential fitting in (b) fitted scheme in (3.11) for $\varepsilon = 2^{-12}$.

In Table 1, the maximum absolute error of the proposed scheme and schemes in [7] and [9] of Example 4.1 is given. In Table 2, the maximum absolute error of the proposed scheme and schemes in [7] of Example 4.2 is given. As can be seen in these tables, for each N as $\varepsilon \to 0$, the maximum absolute error of the proposed scheme is stable and uniform. In the second section of Tables 1 and 2, the maximum absolute error of the upwind FDM scheme on Shishkin mesh from [7] is given. As we can see from the results in this table, the proposed scheme gives more accurate results than the one in [7] and [9]. In Table 3, the maximum absolute error of the proposed scheme of Example 4.3 is given. In Table 4, the ε-uniform error and the ε-uniform rate of convergence of the scheme are given. As one can observe, the proposed scheme gives linear order of convergence.

TABLE 1. Maximum absolute error of Example 4.1 using the proposed scheme and result in [7] and [9].

$\varepsilon \downarrow$	$N \to 32$	64	128	256	512	1024
	Proposed	Scheme				
2^{-12}	1.6980e-03	8.3814e-04	4.1610e-04	2.0761e-04	1.0149e-04	4.0067e-05
2^{-16}	1.6981e-03	8.3819e-04	4.1612e-04	2.0729e-04	1.0345e-04	5.1673e-05
2^{-20}	1.6981e-03	8.3820e-04	4.1613e-04	2.0729e-04	1.0345e-04	5.1673e-05
2^{-24}	1.6981e-03	8.3820e-04	4.1613e-04	2.0729e-04	1.0345e-04	5.1673e-05
2^{-28}	1.6981e-03	8.3820e-04	4.1613e-04	2.0729e-04	1.0345e-04	5.1673e-05
2^{-32}	1.6981e-03	8.3820e-04	4.1613e-04	2.0729e-04	1.0345e-04	5.1673e-05
	Result	in [9]				
2^{-12}	3.6669e-02	1.2541e-02	4.2449e-03	1.4298e-03	4.7508e-04	1.5924e-04
2^{-16}	3.6701e-02	1.2463e-02	4.1666e-03	1.4086e-03	4.7513e-04	1.5946e-04
2^{-20}	3.6709e-02	1.2478e-02	4.1763e-03	1.3898e-03	4.5369e-04	1.5379e-04
2^{-24}	3.6710e-02	1.2479e-02	4.1785e-03	1.3936e-03	4.5637e-04	1.4895e-04
2^{-28}	3.6710e-02	1.2479e-02	4.1786e-03	1.3939e-03	4.5697e-04	1.4992e-04
2^{-32}	3.6710e-02	1.2479e-02	4.1786e-03	1.3939e-03	4.5701e-04	1.5000e-04
	Result	in [7]				
2^{-12}	3.93e-2	2.36e-2	1.38e-2	7.89e-3	4.49e-3	4.49e-3
2^{-16}	3.90e-2	2.33e-2	1.35e-2	7.66e-3	4.30e-3	2.39e-3
2^{-20}	3.90e-2	2.33e-2	1.35e-2	7.65e-3	4.28e-3	2.37e-3
2^{-24}	3.90e-2	2.33e-2	1.35e-2	7.65e-3	4.28e-3	2.37e-3

TABLE 2. Maximum absolute error of Example 4.2 using the proposed scheme and upwind FDM on Shishkin mesh in [7].

$\varepsilon \downarrow$	$N \to 32$	64	128	256	512	1024
	Proposed	Scheme				
2^{-12}	1.2262e-03	6.0983e-04	3.0392e-04	1.5105e-04	7.0666e-05	2.9267e-05
2^{-16}	1.2264e-03	6.0994e-04	3.0398e-04	1.5171e-04	7.5784e-05	3.7874e-05
2^{-20}	1.2264e-03	6.0995e-04	3.0398e-04	1.5172e-04	7.5785e-05	3.7874e-05
2^{-24}	1.2264e-03	6.0995e-04	3.0398e-04	1.5172e-04	7.5785e-05	3.7874e-05
2^{-28}	1.2264e-03	6.0995e-04	3.0398e-04	1.5172e-04	7.5785e-05	3.7874e-05
2^{-32}	1.2264e-03	6.0995e-04	3.0398e-04	1.5172e-04	7.5785e-05	3.7874e-05
	Result	in [7]				
2^{-12}	3.22e-2	1.92e-2	1.12e-2	6.39e-3	3.63e-3	2.04e-3
2^{-16}	3.19e-2	1.90e-2	1.10e-2	6.21e-3	3.47e-3	1.93e-3
2^{-20}	3.19e-2	1.90e-2	1.09e-2	6.19e-3	3.46e-3	1.91e-3
2^{-24}	3.19e-2	1.90e-2	1.09e-2	6.19e-3	3.46e-3	1.91e-3

TABLE 3. Maximum absolute error of Example 4.3 using the proposed scheme.

$\varepsilon \downarrow$	$N \rightarrow 32$	64	128	256	512	1024
2^{-12}	1.4173e-02	7.1502e-03	3.5899e-03	1.7984e-03	8.9997e-04	4.4677e-04
2^{-16}	1.4172e-02	7.1502e-03	3.5899e-03	1.7984e-03	9.0005e-04	4.5024e-04
2^{-20}	1.4172e-02	7.1502e-03	3.5899e-03	1.7984e-03	9.0005e-04	4.5024e-04
2^{-24}	1.4172e-02	7.1502e-03	3.5899e-03	1.7984e-03	9.0005e-04	4.5024e-04
2^{-28}	1.4172e-02	7.1502e-03	3.5899e-03	1.7984e-03	9.0005e-04	4.5024e-04
2^{-32}	1.4172e-02	7.1502e-03	3.5899e-03	1.7984e-03	9.0005e-04	4.5024e-04

TABLE 4. ε-uniform error and ε-uniform rate of convergence of the proposed scheme.

$\varepsilon \downarrow$	$N \rightarrow 32$	64	128	256	512	1024
	Example 4.1					
E^N	1.6981e-03	8.3820e-04	4.1613e-04	2.0729e-04	1.0345e-04	5.1673e-05
R^N	1.0186	1.0103	1.0054	1.0027	1.0015	-
	Example 4.2					
E^N	1.2264e-03	6.0995e-04	3.0398e-04	1.5172e-04	7.5785e-05	3.7874e-05
R^N	1.0077	1.0047	1.0026	1.0014	1.0007	-
	Example 4.3					
E^N	1.4172e-02	7.1502e-03	3.5899e-03	1.7984e-03	9.0005e-04	4.5024e-04
R^N	0.9870	0.9940	0.9972	0.9986	0.9993	-

5. Conclusion

In this paper, singularly perturbed delay differential equations having small delay on the convection and reaction terms of the equation are considered. A numerical scheme is developed using an exponentially fitted finite difference method. Stability and uniform convergence of the scheme are investigated theoretically. Three test examples having boundary layers are considered to validate the theoretical findings. The findings in the tables indicate the uniform convergence of the scheme with linear rate of convergence.

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НУМЕРИЧКА СХЕМА ЗА СИНГУЛАРНО ПЕРТУРБОВАНЕ ПРОБЛЕМЕ РЕАКЦИЈА КОНВЕКЦИЈЕ-ДИФУЗИЈЕ КОЈЕ УКЉУЧУЈУ КАШЊЕЊА

Резиме. Овај рад се бави методама решвања за сингуларно поремећене диференцијалне једначине кашњења које имају кашњење у члановима конвекције и реакције. Разматрани проблем показује експоненцијални гранични слој на левој или десној страни домена. Чланови са кашњењем су разматрани у апроксимацији помоћу Таилоровог реда, а резултујући сингуларно поремећени гранични проблем је решаван помоћу посебно дизајниране методе експоненцијалних коначних разлика. Стабилност шеме је анализиран и истраживан применом принципа поређења и везаног решења. Формулисана шема конвергира равномерно са линеарним редом конвергенције. Теоријски налази су потврђени помоћу три нумеричка тест примера.

Department of Applied Mathematics
Adama Science and Technology University
Adama
Ethiopia
msfnmkr02@gmail.com

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Department of Mathematics
Jimma University
Jimma
Ethiopia
workutil12@gmail.com

Department of Mathematics
Jimma University
Jimma
Ethiopia
gammeef@gmail.com