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NOETHER'S THEOREM FOR HERGLOTZ TYPE VARIATIONAL PROBLEMS UTILIZING COMPLEX FRACTIONAL DERIVATIVES

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Abstract. This is a review article which elaborates the results presented in [1], where the variational principle of Herglotz type with a Lagrangian that depends on fractional derivatives of both real and complex orders is formulated and the invariance of this principle under the action of a local group of symmetries is determined. The conservation law for the corresponding fractional Euler Lagrange equation is obtained and a sequence of approximations of a fractional Euler–Lagrange equation by systems of integer order equations established and analyzed.

1. Introduction

In this review article, the results obtained in [1] are elaborated. Namely, variational principles of Hamilton type are suited for conservative systems, while for systems with non-conservative forces, variational principles of Hamilton type are rare (see [2, 3]). An important advantage of the Herglotz variational principle (HVP) [4] is that it could be formulated for a non-conservative dynamical system. It leads to Nöther's type theorems and first integrals of the non-conservative dynamical systems [5].

In [1] symmetry properties of the HVP with real and complex order fractional derivatives are derived. Concerning the variational principle of Hamilton type with real fractional derivatives, important contributions are by Riewe [6, 7] and recent [8–16]. For the Hamilton variational principle with complex order derivatives, important contributions are in [11]. Mathematical models of mechanical systems involving complex order derivatives are connected with viscoelasticty. A generalized wave equation related to a viscoelastic body with the complex order fractional derivatives in a constitutive equation is considered in [17]. Also, Noether's theory

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for the fractional variational principle of Hamilton type was presented in many publications (c.f. [18] with the references therein).

The HVP and corresponding Noether's theorems for systems involving integer order derivatives were studied in a series of papers by Georgieva and Guenther, [19–22]. The symmetry properties which correspond to a Lagrangian with integer and real fractional order derivatives were recently presented in [5, 19, 20, 23, 24]. The HVP for Birkhoffian systems with combined fractional derivatives, introduced in [25], was treated in [26].

The real and complex order fractional derivatives in the HVP are introduced in [12], where generalized Euler–Lagrange equations are introduced, with different assumptions imposed on the Lagrangian (see also [11]).

In [1], a HVP in the case when its Lagrangian involves real and complex order fractional derivatives is introduced, where the results are continuation of the work presented in [12]. The actions of a local one-parameter group of transformations related to the corresponding Euler–Lagrange equation are formulated and the infinitesimal criteria obtained, so that the results presented in [13] and [18] are generalized. Moreover, using the expansion of fractional derivative of a function into series, the approximations of the already established Euler–Lagrange equation are invoked as well as the infinitesimal criteria and Noether's type theorem. The convergence in a weak sense within the dual pairing of corresponding topological spaces is proved. Finally, two examples are given, generalizing known results.

1.1. Notation. The left and right Riemann–Liouville fractional derivatives of order $\alpha \in (0,1)$ are defined by

$$ {}_aD_t^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t \frac{u(\tau)}{(t-\tau)^{\alpha}}\,d\tau, $$

$$ {}_tD_T^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\Big(-\frac{d}{dt}\Big)\int_t^T \frac{u(\tau)}{(\tau-t)^{\alpha}}\,d\tau, t\in[a,T]. $$

The same definition can be extended for complex α ($\alpha \in \mathbb{C}$) with $\operatorname{Re}\alpha \in [0,1)$, cf. [27–31]. However, in this case, the result of the Riemann–Liouville fractional operator of complex order applied to a real valued function is a complex valued function. Therefore, in order to obtain as a result a real valued function, we use the fractional derivatives of complex order defined as

$$ {}_a\mathcal{D}_t^{\alpha} = \frac{1}{2}({}_aD_t^{\alpha} + {}_0D_t^{\bar{\alpha}}), \quad \text{and} \quad {}_t\mathcal{D}_T^{\alpha} = \frac{1}{2}({}_tD_T^{\alpha} + {}_tD_T^{\bar{\alpha}}), $$

where $\bar{\alpha}$ is the complex conjugate of α. In this way ${}_a\mathcal{D}_t^{\alpha}u$ and ${}_t\mathcal{D}_T^{\alpha}u$, $u \in \mathcal{U}$, have real values.

Let $f, g \in AC([a,b])$, $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha \in (0,1), \operatorname{Im}\alpha \geqslant 0$. Then, the fractional integration by parts formula [11, 12] reads

$$ \int_a^b f(t)\,{}_a\mathcal{D}_t^{\alpha}g(t)dt = \int_a^b g(t)\,{}_t\mathcal{D}_b^{\alpha}f(t)dt. \tag{1.1} $$

1.2. HVP. Let $u_0, a, T \in \mathbb{R}$, $T > a$, and let

$$\mathcal{U} := \{u \in C^1([a,T]) \mid u \text{ is real}, \ u(a) = u_0\} \tag{1.2}$$

be the set of admissible functions; $C^1 = C^1([a,T])$ denotes the space of continuous functions with continuous first derivatives on $[a,T]$. We consider the following first order equation for a real function z (with $z(a) = z_0$),

$$\dot{z}(t) = \frac{dz}{dt} = L(t, u, {}_aD_t^{\alpha}u, {}_a\mathcal{D}_t^{\gamma}u, z), \quad t \in [a,T], \ \ u \in \mathcal{U}, \tag{1.3}$$

$\alpha \in (0,1)$, $\gamma \in C \setminus \mathbb{R}$ with $\operatorname{Re}\gamma \in [0,1)$.

The Lagrangian L is an absolutely continuous function over $[a,T] \times \mathbb{R}^4$ ($L \in AC([a,T] \times \mathbb{R}^4)$) as well as

$$t \mapsto \partial_5 L\big(t, u, {}_0D_t^{\alpha}u, {}_0\mathcal{D}_t^{\gamma}u, z\big) \in L^1([a,T]); \tag{1.4}$$

$$t \mapsto {}_tD_T^{\alpha}(\lambda(t)\partial_3 L(t, u, {}_0D_t^{\alpha}u, {}_0\mathcal{D}_t^{\gamma}u, z)) \in L^1([a,T]); \tag{1.5}$$

$$t \mapsto {}_t\mathcal{D}_T^{\gamma}(\lambda(t)\partial_4 L(t, u, {}_0D_t^{\alpha}u, {}_0\mathcal{D}_t^{\gamma}u, z)) \in L^1([a,T]), \tag{1.6}$$

where

$$\lambda(t) = \exp\left(- \int_0^t \partial_5 L(\tau, u, {}_0D_\tau^{\alpha}u, {}_0\mathcal{D}_\tau^{\gamma}u, z)d\tau \right), \tag{1.7}$$

and ∂_i denotes differentiation with respect to the i-th variable.

Denote by S the set of solutions to equation (1.3). The HVP is related to a mapping $Z\colon \mathcal{U} \to S$, $u \mapsto Z(u) = z$. The HVP is stated as:

Find $u \in \mathcal{U}$ such that the solution to (1.3) takes the extremal value (*extr*) at the point T, that is,

$$z(T) \to \text{extr}, \ \text{so that (1.3) holds.} \tag{1.8}$$

Remark 1.1. Concerning the admissible set $\mathcal{U}$, one can make additional assumptions. Actually, this was done in the section 5. Also, additional assumptions are included in the section 4.

We recall our result from [12], related to functions defined on $[0,T]$.

Theorem 1.1. [12] *Let $u^* \in \mathcal{U}$ such that $z(t), t \in [0,T]$, takes its extremal value $z(T)$ at T, and z satisfies (1.3). Then u^* satisfies the following generalized Euler–Lagrange equation:*

$$\lambda(t)\frac{\partial L}{\partial u} + {}_tD_T^{\alpha}(\lambda(t)\partial_3 L) + {}_t\mathcal{D}_T^{\gamma}(\lambda(t)\partial_4 L) = 0, \qquad \text{on } \ [0,T], \tag{1.9}$$

where

$$\lambda(t) = \exp\left(- \int_0^t \partial_z L(\tau)d\tau \right), \quad t \in [0,T]. \tag{1.10}$$

2. Local group of symmetries for HVP

In [1], the following results are formulated and proved, so we elaborate on those. We list those together with proofs in order to illustrate the concept used. Consider a local one-parameter group of transformations acting on real functions u and t:

$$\bar{t} = \phi(t, u, \eta), \quad \bar{u}(\bar{t}) = \psi(t, u, \eta),$$

where $\eta \in (-\varepsilon, \varepsilon)$ is a parameter of the group, and $\phi(t, u, 0) = t, \psi(t, u, 0) = u$. The infinitesimal generators are $\tau(t, u) = \frac{d}{d\eta}\phi(t, u, \tau)|_{\eta=0}, \xi(t, u) = \frac{d}{d\eta}\psi(t, u, \tau)|_{\eta=0}$ so that

$$\bar{t} = t + \eta\tau(t, u) + o(\eta), \quad \bar{u}(\bar{t}) = u(t) + \eta\xi(t, u) + o(\eta). \tag{2.1}$$

We assume that $\tau, \xi \in C^1([a, t] \times \mathbb{R})$. In Section 4 we will assume that these functions are analytic ones. Let

$$\Delta t = \lim_{\eta \to 0} \frac{\bar{t}(\eta) - t}{\eta}, \quad \Delta u = \lim_{\eta \to 0} \frac{\bar{u}(\bar{t}, \eta) - u(t)}{\eta}, \quad \delta u = \lim_{\eta \to 0} \frac{\bar{u}(t) - u(t)}{\eta}.$$

Then, it could be shown that the following proposition holds, see [13, 21].

PROPOSITION 2.1.

$$\begin{aligned} &\Delta t = \tau, && \Delta u = \xi, && \Delta u = \delta u + \dot{u}\tau, \\ &\delta u = \xi - \tau\dot{u}, && \Delta\dot{u} = (\Delta u)^{\dot{}} - \dot{u}\dot{\tau}, && (\delta u)^{\dot{}} = \delta\dot{u}. \end{aligned} \tag{2.2}$$

In particular, $d\bar{t}/dt_{|\eta=0} = 1$, $(d\bar{t}/d\eta)(d\bar{t}/dt)_{|\eta=0} = \dot{\tau}$.

The action of the local symmetry on Riemann–Liouville fractional derivatives was presented in our paper [13] and, in a different way, in [18]. Recall

$$\Delta_a D_t^\alpha u = {}_aD_t^\alpha \delta u + ({}_aD_t^\alpha u)^{\dot{}}\tau, \quad \Delta_a \mathcal{D}_t^\alpha u = {}_a\mathcal{D}_t^\alpha \delta u + ({}_a\mathcal{D}_t^\alpha u)^{\dot{}}\ \tau. \tag{2.3}$$

Note that in (2.3) the terms $({}_aD_t^\alpha u)^{\dot{}}$ and $({}_a\mathcal{D}_t^\alpha u)^{\dot{}}$ must be calculated taking into account that the lower bound in the fractional derivatives $t = a$ is also a subject which should be transformed so that in (2.3) we have

$$\begin{aligned} ({}_aD_t^\alpha u)^{\dot{}}\tau &= \frac{\partial}{\partial t}[{}_aD_t^\alpha u(t)]\tau(t, u(t)) + \frac{\partial}{\partial a}[{}_aD_t^\alpha u(t)]\tau(a, u(t)), \\ ({}_a\mathcal{D}_t^\alpha u)^{\dot{}}\tau &= \frac{\partial}{\partial t}[{}_a\mathcal{D}_t^\alpha u(t)]\tau(t, u(t)) + \frac{\partial}{\partial a}[{}_a\mathcal{D}_t^\alpha u(t)]\tau(a, u(t)). \end{aligned}$$

Thus, we have

$$\begin{aligned} 00\Delta_a D_t^\alpha u &= {}_aD_t^\alpha \delta u + \frac{d}{dt}({}_aD_t^\alpha u(t))\tau(t, u(t)) + R(a, t), \\ \Delta_a \mathcal{D}_t^\alpha u &= {}_a\mathcal{D}_t^\alpha \delta u + \frac{d}{dt}({}_a\mathcal{D}_t^\alpha u(t))\tau(t, u(t)) + R(a, t), \end{aligned} \tag{2.4}$$

where

$$R(a, t) = \frac{\alpha}{\Gamma(1 - \alpha)} \frac{u(a)}{(t - a)^{\alpha+1}} \tau(a, u(a)). \tag{2.5}$$

Let us turn to the invariance of the solution to the variational problem 1.3 under the action of the local group. Every element of the group transforms $\mathcal{U}$ bijectively to $\bar{\mathcal{U}}$ consisting of C^1-real functions defined on $[\bar{a}, \bar{T}]$ with the same initial value u_0. The function $\bar{t} \mapsto \bar{u}(\bar{t}) \in \bar{\mathcal{U}}$, satisfying

$$\frac{d\bar{z}}{d\bar{t}} = L(\bar{t}, \bar{u}, {}_{\bar{a}}D^{\alpha}_{\bar{t}}\bar{u}, {}_{\bar{a}}\mathcal{D}^{\gamma}_{\bar{t}}\bar{u}, \bar{z}), \tag{2.6}$$

(in the transformed domain) is the solution to the variational problem, i.e. $\bar{z}(\bar{T}) = z(T)$, where z solves (1.8).

Remark 2.1. The definition of symmetries implies that for any sub-interval $[A, B] \subset [a, T]$ we have

$$\Delta z(t) = \lim_{\eta \to 0} \frac{\bar{z}(\bar{t}, \eta) - z(t)}{\eta} = 0, \quad t \in [A, B], \tag{2.7}$$

that is, $\bar{z}(\bar{t}) = z(t)$, $t \in [a, T]$.

Now we give the infinitesimal criteria:

Theorem 2.1. *Infinitesimals τ, ξ define a local symmetry group of* (1.3) *if and only if*

$$\begin{aligned}\tau\frac{\partial L}{\partial t} + \xi\frac{\partial L}{\partial u} &+ ({}_aD^{\alpha}_t(\xi - \dot{u}\tau) + \dot{({}_aD^{\alpha}_t u)}\tau + R(a,t))\frac{\partial L}{\partial_a D^{\alpha}_t u} \\ &+ ({}_a\mathcal{D}^{\gamma}_t(\xi - \dot{u}\tau) + \dot{({}_a\mathcal{D}^{\gamma}_t u)}\tau + R(a,t))\frac{\partial L}{\partial_a \mathcal{D}^{\gamma}_t u} + L\dot{\tau} = 0,\end{aligned} \tag{2.8}$$

for all $u \in \mathcal{U}$.

Proof. Suppose that τ, ξ define a symmetry group. From $\dot{z} = L$ we have $\Delta\dot{z}(t) = \Delta L$. So, $(2.2)_4$ implies $\Delta\dot{z} = \dot{(\Delta z)} - \dot{z}\dot{\tau}$. Therefore,

$$\begin{aligned}\dot{(\Delta z)} &= \Delta L + L\dot{\tau} \\ &= \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial u}\xi + \frac{\partial L}{\partial_a D^{\alpha}_t u}\Delta_a D^{\alpha}_t u + \frac{\partial L}{\partial_a \mathcal{D}^{\gamma}_t u}\Delta_a \mathcal{D}^{\gamma}_t u + \frac{\partial L}{\partial z}\Delta z + L\dot{\tau}.\end{aligned}$$

Since $\Delta_a D^{\alpha}_t u = \delta_a D^{\alpha}_t u + \dot{({}_aD^{\alpha}_t u)}\tau ={}_a D^{\alpha}_t \delta u + \dot{({}_aD^{\alpha}_t u)}\tau$ and a similar expression holds for $\Delta_a \mathcal{D}^{\gamma}_t u$, we obtain

$$\begin{aligned}\dot{(\Delta z)} - \frac{\partial L}{\partial z}\Delta z = \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial u}\xi &+ \frac{\partial L}{\partial_a D^{\alpha}_t u}[{}_aD^{\alpha}_t(\xi - \dot{u}\tau) + \dot{({}_aD^{\alpha}_t u)}\tau + R(a,t)] \\ &+ \frac{\partial L}{\partial_a \mathcal{D}^{\gamma}_t u}[{}_a\mathcal{D}^{\gamma}_t u(\xi - \dot{u}\tau) + \dot{({}_a\mathcal{D}^{\gamma}_t u)}\tau + R(a,t)] + L\dot{\tau}.\end{aligned}$$

Multiplying by $\lambda(t)$ and integrating, we obtain

$$\begin{aligned}&[\lambda(t)\Delta z(t) - \Delta z(a)\lambda(a)] \\ &\quad = \int_a^t \lambda(s)\Big[\frac{\partial L}{\partial s}\tau + \frac{\partial L}{\partial u}\xi + \frac{\partial L}{\partial_a D^{\alpha}_s u}[{}_aD^{\alpha}_s(\xi - \dot{u}\tau) + \dot{({}_aD^{\alpha}_s u)}\,\tau + R(a,s)] \\ &\qquad + \frac{\partial L}{\partial_a \mathcal{D}^{\gamma}_s u}[{}_a\mathcal{D}^{\gamma}_s u(\xi - \dot{u}\tau) + \dot{({}_a\mathcal{D}^{\gamma}_s u)}\tau + R(a,s)] + L\dot{\tau}\Big]ds, \quad t \in [a, T].\end{aligned}$$

An initial value $z(a) = u_0$ implies $\Delta z(a) = 0$, so that $\Delta z(t) = 0$ leads to

$$\begin{aligned}(2.9)\qquad \int_a^t \lambda(s)\Big[\frac{\partial L}{\partial s}\tau + \frac{\partial L}{\partial u}\xi + \frac{\partial L}{\partial {}_aD_t^\alpha u}[{}_aD_t^\alpha(\xi-\dot u\tau) + ({}_aD_t^\alpha u)\dot{}\,\tau + R(a,t)] \\ + \frac{\partial L}{\partial {}_a\mathcal{D}_t^\gamma u}[{}_a\mathcal{D}_t^\gamma u(\xi-\dot u\tau) + ({}_a\mathcal{D}_t^\gamma u)\dot{}\,\tau + R(a,t)] + L\dot\tau\Big]ds = 0, \quad t\in[a,T].\end{aligned}$$

Relation (2.8) follows from (2.9) since

$$\begin{aligned}\int_a^t \lambda(s)\Big[\frac{\partial L}{\partial s}\tau + \frac{\partial L}{\partial u}\xi + \frac{\partial L}{\partial {}_aD_s^\alpha u}[{}_aD_s^\alpha(\xi-\dot u\tau) + ({}_aD_s^\alpha u)\dot{}\,\tau + R(a,s)] \\ + \frac{\partial L}{\partial {}_a\mathcal{D}_s^\gamma u}[{}_a\mathcal{D}_s^\gamma u(\xi-\dot u\tau) + ({}_a\mathcal{D}_s^\gamma u)\dot{}\,\tau + R(a,s)] + L\dot\tau\Big]ds = 0, \quad t\in[a,T].\end{aligned}$$

Next, we show that (2.8) is also a sufficient condition for τ and ξ to be the infinitesimal generators for the local group of symmetries for which $\bar z(\bar t) = z(t), t\in [a,T]$. Starting with $\bar t = t+\eta\tau$, and $\bar u(\bar t) = u(t)+\eta\xi$, (2.7), we obtain

$$(2.10)\qquad \bar z(\bar t,\eta) = z(t) + \Delta z(t)\eta, \quad t\in[a,T].$$

Multiplying (2.8) by λ, integrating respect to t from $t=a$ and using $\Delta z(a)=0$, it follows that $\Delta z(t) = 0$. Substituting this result in (2.10), we obtain $\bar z(\bar t) = z(t)$, $t\in[a,T]$. □

REMARK 2.2. If L does not contain fractional derivatives but contains the first derivative of u, that is, $\dot z(t) = \frac{dz}{dt} = L(t,u,\dot u,z)$, then (2.8) becomes

$$(2.11)\qquad \tau\frac{\partial L}{\partial t} + \xi\frac{\partial L}{\partial u} + ((\xi-\dot u\tau)\dot{} + \ddot u\tau)\frac{\partial L}{\partial \dot u} + L\dot\tau = 0.$$

This result is in agreement with [21].

3. Nöther's theorem

We determine now the conservation law (CL) for the Euler–Lagrange equation (1.9).

THEOREM 3.1. *Suppose that τ,ξ define a symmetry group for (1.3). Then, the Euler–Lagrange equation (1.9) has the first integral:*

$$\begin{aligned}(3.1)\qquad \lambda L\tau + \int_a^t \Big[-(\xi-\dot u\tau)\Big\{{}_sD_T^\alpha\Big(\lambda\frac{\partial L}{\partial {}_aD_s^\alpha u}\Big) + {}_s\mathcal{D}_T^\gamma\Big(\lambda\frac{\partial L}{\partial {}_a\mathcal{D}_s^\gamma u}\Big)\Big\} \\ + \lambda\Big\{\frac{\partial L}{\partial {}_aD_s^\alpha u}{}_aD_s^\alpha(\xi-\dot u\tau) + \frac{\partial L}{\partial {}_a\mathcal{D}_s^\gamma u}{}_a\mathcal{D}_t^\gamma u(\xi-\dot u\tau)\Big\} \\ + R(a,s)\Big(\frac{\partial L}{\partial {}_aD_s^\alpha u} + \frac{\partial L}{\partial {}_a\mathcal{D}_s^\gamma u}\Big) + L\dot\tau\Big]ds = const.\end{aligned}$$

PROOF. Condition $\Delta z(t) = 0$, $t\in[a,T]$, given by (2.9), may be written as

$$\int_a^t \lambda(s)\Big[\frac{\partial L}{\partial s}\tau + \frac{\partial L}{\partial u}\xi + \frac{\partial L}{\partial {}_aD_s^\alpha u}[{}_aD_s^\alpha(\xi-\dot u\tau) + ({}_aD_s^\alpha u)\dot{}\,\tau + R(a,s)] + \dot u\tau\frac{\partial L}{\partial u}$$

$$- \dot{u}\tau \frac{\partial L}{\partial u} + \frac{\partial L}{\partial_a \mathcal{D}_s^\gamma u}[{}_a\mathcal{D}_s^\gamma u(\xi - \dot{u}\tau) + \dot{({}_a\mathcal{D}_s^\gamma u)}\tau + R(a,s)] + L\dot{\tau}\Big] ds = 0.$$

Using (2.4), we obtain

$$\int_a^t \lambda(s)\Big[\tau\Big(\frac{\partial L}{\partial s} + \dot{u}\frac{\partial L}{\partial u} + \frac{\partial L}{\partial_a D_s^\alpha u}\dot{({}_aD_s^\alpha u)} + \frac{\partial L}{\partial_a \mathcal{D}_s^\gamma u}\dot{({}_a\mathcal{D}_s^\gamma u)} + \frac{\partial L}{\partial z}\dot{z}\tau - \frac{\partial L}{\partial z}\dot{z}\tau\Big)$$
$$+ \frac{\partial L}{\partial u}(\xi - \dot{u}\tau) + \frac{\partial L}{\partial_a D_s^\alpha u}[{}_aD_s^\alpha(\xi - \dot{u}\tau)] + \frac{\partial L}{\partial_a \mathcal{D}_s^\gamma u}{}_a\mathcal{D}_s^\gamma u(\xi - \dot{u}\tau)$$
$$+ R\Big(\frac{\partial L}{\partial_a D_s^\alpha u} + \frac{\partial L}{\partial_a \mathcal{D}_s^\gamma u}\Big) + L\dot{\tau}\Big] ds = 0,$$

or

$$\int_a^t \lambda(s)\Big[\frac{d}{ds}(L\tau) + \frac{\partial L}{\partial u}(\xi - \dot{u}\tau) + \frac{\partial L}{\partial_a D_s^\alpha u}{}_aD_s^\alpha(\xi - \dot{u}\tau) + \frac{\partial L}{\partial_a \mathcal{D}_s^\gamma u}{}_a\mathcal{D}_s^\gamma u(\xi - \dot{u}\tau)$$
$$- \frac{\partial L}{\partial z}\dot{z}\tau + R(a,s)\Big(\frac{\partial L}{\partial_a D_s^\alpha u} + \frac{\partial L}{\partial_a \mathcal{D}_s^\gamma u}\Big)\Big] = 0.$$

Since $\frac{d}{dt}\lambda(t) = \lambda(t)\frac{\partial L}{\partial z}\dot{z}$ (see (1.10)), we obtain $\lambda(t)\frac{d}{dt}(L\tau) - \lambda(t)\frac{\partial L}{\partial z}\dot{z}\tau = \frac{d}{dt}(\lambda\tau L)$ so that the last equation becomes

$$\int_a^t \frac{d}{ds}(\lambda L\tau) + \lambda(s)\Big[\frac{\partial L}{\partial u}(\xi - \dot{u}\tau) + \frac{\partial L}{\partial_a D_s^\alpha u}{}_aD_s^\alpha(\xi - \dot{u}\tau)$$
$$+ \frac{\partial L}{\partial_a \mathcal{D}_s^\gamma u}{}_a\mathcal{D}_s^\gamma u(\xi - \dot{u}\tau) + R(a,s)\Big(\frac{\partial L}{\partial_a D_s^\alpha u} + \frac{\partial L}{\partial_a \mathcal{D}_s^\gamma u}\Big)\Big] = 0,$$

or

$$\lambda(t)L(t)\tau(t) + \int_a^t \lambda(s)\Big[\frac{\partial L}{\partial u}(\xi - \dot{u}\tau) + \frac{\partial L}{\partial_a D_s^\alpha u}{}_aD_s^\alpha(\xi - \dot{u}\tau) + \frac{\partial L}{\partial_a \mathcal{D}_s^\gamma u}{}_a\mathcal{D}_s^\gamma u(\xi - \dot{u}\tau)$$
$$+ R(a,s)\Big(\frac{\partial L}{\partial_a D_s^\alpha u} + \frac{\partial L}{\partial_a \mathcal{D}_s^\gamma u}\Big)\Big] ds = const.$$

By the Euler–Lagrange equation (1.9), we can replace the term $\lambda\frac{\partial L}{\partial u}$ with the expression

$$-[{}_sD_T^\alpha(\lambda(t)\partial_3 L) + {}_s\mathcal{D}_T^\gamma(\lambda(t)\partial_4 L)]$$

so that (3.1) follows. □

Remark 3.1. If the Lagrangian depends on the integer order derivatives only (as in Remark 2.2) and $\frac{\partial L}{\partial_a \mathcal{D}_t^\gamma u} = 0$, then the CL (3.1) becomes

$$\lambda(t)L(t)\tau(t) + \int_a^t \Big[(\xi - \dot{u}\tau)\dot{\Big(\lambda\frac{\partial L}{\partial \dot{u}}\Big)} + \lambda\frac{\partial L}{\partial \dot{u}}\dot{(\xi - \dot{u}\tau)}\Big] ds \tag{3.2}$$
$$= \lambda(t)L(t)\tau(t) + \lambda\frac{\partial L}{\partial \dot{u}}(\xi - \dot{u}\tau)$$
$$= \lambda\Big[\Big(L - \frac{\partial L}{\partial \dot{u}}\dot{u}\Big)\tau + \frac{\partial L}{\partial \dot{u}}\xi\Big] = const.$$

Again, it is in agreement with [21] and [22].

4. Approximations

In this section we elaborate on a procedure that generates an approximation of the HVP, fully presented in [1]. The procedure utilizes classical (integer order) derivatives of the order $N \in \mathbb{N}$. The infinitesimal criteria as well as the conservation law CL_N for the mentioned approximate problem invoked in [1] are presented. The convergence of CL_N to CL, $N \to \infty$, in the weak sense, proved in [1], is displayed.

For the sake of simplicity, we assume that $\tau(a, u(a)) = 0$, so that the term $R(a,t)$ (given by (2.5)) vanishes. Let (c,d) be an open interval in $\mathbb{R}$ containing $[a,T]$, such that the close ball $B(t, T-a)$ lies in (c,d). Let f be an analytic function in (c,d) and $\alpha \in (0,1)$. Then, (see [31, Lemma 15.3])

$$ {}_aD_t^\alpha f(t) = \sum_{i=0}^{\infty} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} f^{(i)}(t), \quad t \in B(t, T-a) \subset (c,d), \tag{4.1} $$

with $\binom{\alpha}{i} = \frac{(-1)^{i-1}\alpha\Gamma(i-\alpha)}{\Gamma(1-\alpha)\Gamma(i+1)}$. In this section we further assume that:

a) $L \in C^\infty([a,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$;
b) Solution u to the problem (1.8) is analytic in (c,d);
c) $\partial_3^{(i)} L(T, b_0, p, z_0) = 0$, $p \in \mathbb{R}$, $i \in \mathbb{N}$.

Consider

$$ \dot{z}(t) = L_N(t, u(t), u'(t), \dots, u^{(N)}(t), z(t)), \quad t \in [a,T],\ z(a) = z_0, \tag{4.2} $$

where

$$ L_N(t, u(t), u'(t), \dots, u^{(N)}(t), z(t)) = L\left(t, u(t), \sum_{i=0}^{N} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} u^{(i)}(t), z(t)\right) $$

i.e. L_N is obtained from L in (1.8) replacing ${}_aD_t^\alpha u$ by its approximation (4.1). Then,

$$ \begin{aligned} \frac{\partial L_N}{\partial t} &= \partial_1 L_N, \\ \frac{\partial L_N}{\partial u} &= \partial_2 L_N + \binom{\alpha}{0} \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} \partial_3 L_N, \\ \frac{\partial L_N}{\partial u^{(i)}} &= \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} \partial_3 L_N, \end{aligned} \tag{4.3} $$

where $\partial_i L_N$ is obtained from $\partial_i L$, $i = 1,2,3,4$, replacing ${}_aD_t^\alpha u$ by its approximation (4.1). Denote by HVP_N the corresponding problem for (4.2): Find $u \in \mathcal{U}$ such that the solution to (4.2) takes its extreme value at T .

We recall some basic facts from the abstract functional analysis. Let $\mathcal{A}((c,d))$ be the space of real analytic functions with the family of semi-norms

$$ p_{[m,n]}(\varphi) = \sup_{t \in [m,n]} |\varphi(t)|, \quad \varphi \in \mathcal{A}((c,d)). \tag{4.4} $$

where $[m,n]$ run through all closed sub-intervals of (c,d). (For the sake of completeness, we note that it is a Fréchet space.) Every function $f \in C([a,T])$ extended by zero in $(c,d) \smallsetminus [a,T]$ defines an element of the topological dual $\mathcal{A}'((c,d))$ by

$$\varphi \mapsto \langle f, \varphi \rangle = \int_a^T f(t)\varphi(t)dt, \quad \varphi \in \mathcal{A}((c,d)). \tag{4.5}$$

We need the following simple result (cf. [13,15]).

Proposition 4.1. *Let $F \in C^\infty([a,T])$, such that $F^{(i)}(T) = 0$, for $i \in \mathbb{N}_0$, and $F \equiv 0$, on $(c,d) \smallsetminus [a,T]$. Let ${}_tD_T^\alpha F$ be extended by zero in $(c,d) \smallsetminus [a,T]$. Then:*

i) *For every $i \in \mathbb{N}$, the $i-1$-th derivative of $t \mapsto (t-a)^{i-\alpha}F(t)$ is continuous at $t = a$ and $t = T$. Also it is integrable in (c,d) and supported by $[a,T]$.*

ii) *Function*

$$t \mapsto S_N(t) = \begin{cases} \sum_{i=0}^N \left(-\frac{d}{dt}\right)^i \left(F \cdot \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\right), & t \in [a,T] \\ 0, & t \in (c,d) \smallsetminus [a,T] \end{cases} \tag{4.6}$$

is integrable in (c,d) and supported by $[a,T]$.

iii) *$S_N \in \mathcal{A}'((c,d))$, $N \in \mathbf{N}$ and, in the sense of weak topology,*

$${}_tD_T^\alpha F = \sum_{i=0}^\infty \left(-\frac{d}{dt}\right)^i \left(F \cdot \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\right). \tag{4.7}$$

We will use now

$$\lambda_N(t) = \exp\left(-\int_0^t \partial_z L_N(\tau)d\tau\right), \quad t \in [0,T], \tag{4.8}$$

Clearly, it converges uniformly to λ over $[a,T]$.

We fix $N \in \mathbb{N}$. Then, $\partial_3 L_N, \dots, \partial_{N+2} L_N \in C^\infty([a,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$. The Euler–Lagrange equation for HVP_N (4.2) reads (see [32, Theorem 4]):

$$\lambda_N(t)\frac{\partial L_N}{\partial u} + \sum_{i=0}^\infty \left(-\frac{d}{dt}\right)^i \left(\lambda_N(t)\frac{\partial L_N}{\partial u^{(i)}}\right) = 0, \tag{4.9}$$

where λ_N is given by (4.8). Due to (4.3), this is equivalent to

$$\lambda_N(t)\frac{\partial L_N}{\partial u} + \sum_{i=0}^\infty \left(-\frac{d}{dt}\right)^i \left(\lambda_N(t)\partial_3 L_N \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\right) = 0. \tag{4.10}$$

Denote by τ_N and ξ_N infinitesimal generators of the local symmetry group for HVP_N. We will assume in this section:

$$\tau, \xi, \tau_N, \xi_N \in \mathcal{A}((c,d)), \quad N \in \mathbb{N}; \tag{4.11}$$

$$\tau_N \to \tau,\ \xi_N \to \xi_N \text{ uniformly on every compact set of } (c,d). \tag{4.12}$$

These assumptions imply that one can use the series expansions (4.1) for the quoted infinitesimal generators. Now, we will recall some assertions from [13]. Moreover, in the Appendix of [1], it is proved that $v_N = \tau_N \frac{\partial}{\partial t} + \xi_N \frac{\partial}{\partial u}$ generates a local one-parameter symmetry group of HVP_N (4.2) if and only if:

$$\frac{\partial L_N}{\partial t}\tau_N + \frac{\partial L_N}{\partial u}\xi_N + \sum_{i=1}^{N}\frac{\partial L_N}{\partial u^{(i)}}((\xi_N - \dot{u}\tau_N)^{(i)} + u^{(i+1)}\tau_N) + L_N\dot{\tau}_N = 0, \quad t\in[a,T], \tag{4.13}$$

which is due to (4.3) equivalent to the following infinitesimal criteria IC_N (see also [13, Theorem 2.4]):

$$\frac{\partial L_N}{\partial t}\tau_N + \frac{\partial L_N}{\partial u}\xi_N + \left[\sum_{i=1}^{N}\binom{\alpha}{i}\frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}(\xi_N - \dot{u}\tau_N)^{(i)} + \frac{d}{dt}\left(\sum_{i=1}^{N}\binom{\alpha}{i}\frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)u^{(i)}}\right)\tau_N\right]\partial_3 L_N + L_N\dot{\tau}_N = 0.$$

Note that the infinitesimal criterion IC given by (2.8) is the same as the one given in [13, Theorem 5].

It is shown in [13, Theorem 19], that assumptions i), ii) and iii) for L as well as assumptions (4.11) and (4.12) for τ_N and ξ_N imply

$$IC_N \to IC, \text{ uniformly on } [a,T], \quad N\to\infty.$$

In [1], the following is proved:

PROPOSITION 4.2. *Let P_N denote the Euler–Lagrange equation for HVP_N (4.2), given by (4.9), and P denote the Euler–Lagrange equation of HVP (1.8), given by (1.9). Then, $P_N \to P$, as $N\to\infty$, in the weak sense.*

Also, in [1], the convergence of the conservation law CL_N for the HVP_N given by (4.2), ($N\in\mathbb{N}$) to CL of the original fractional problem in the weak sense, as $N\to\infty$, is proved. We elaborate on the mentioned statement as well as the proof.

PROPOSITION 4.3. *Assumptions on infinitesimal generators (4.11) and (4.12) imply the weak convergence*

$$CL_N \to CL, \quad N\to\infty.$$

PROOF. First, we derive the conservation law for CL_N, $N\in\mathbb{N}$ for the approximate problem (4.2). We obtain from (4.13):

$$\begin{aligned}
&\frac{\partial L_N}{\partial t}\tau_N \pm \partial_2 L_N\dot{u}\tau_N + \partial_3 L_N\frac{d}{dt}\left(\sum_{i=0}^{N}\binom{\alpha}{i}\frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}u^{(i)}\right)\tau_N \pm \partial_4 L_N\dot{z}\tau_N\\
&+\partial_2 L_N\xi_N + \partial_3 L_N\sum_{i=0}^{N}\binom{\alpha}{i}\frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}(\xi_N - \dot{u}\tau_N)^{(i)} + L_N\dot{\tau}_N = 0, \quad t\in[a,T]\\
&\Leftrightarrow \dot{L}_N\tau_N + L_N\dot{\tau}_N - \partial_2 L\dot{u}\tau_N - \partial_4 L_N\dot{z}\tau_N + \partial_2 L_N\xi_N\\
&\quad + \partial_3 L_N\sum_{i=0}^{N}\binom{\alpha}{i}\frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}(\xi_N - \dot{u}\tau_N)^{(i)} = 0, \quad t\in[a,T],
\end{aligned}\tag{4.14}$$

where we use $\dot{L}_N = \partial_1 L_N + \partial_2 L_N \dot{u} + \partial_3 L_N \frac{d}{dt}\big(\sum_{i=0}^{N} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} u^{(i)}\big) + \partial_4 L_N \dot{z}$. Multiplying (4.14) by λ_N (see (4.8)) and using $\dot{z} = L_N$, we obtain

$$\begin{aligned}
&\lambda_N \dot{L}_N \tau_N + \lambda_N L_N \dot{\tau}_N - L_N \partial_4 L_N \lambda \tau_N + \lambda_N (\xi_N - \dot{u}\tau_N)\partial_2 L_N \\
&\qquad + \lambda_N \partial_3 L_N \sum_{i=0}^{N} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} (\xi_N - \dot{u}\tau_N)^{(i)} \\
&\qquad \pm (\xi_N - \dot{u}\tau_N) \sum_{i=0}^{N} \Big(-\frac{d}{dt}\Big)^i \Big(\lambda_N \partial_3 L_N \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\Big) = 0.
\end{aligned}$$

Using $\frac{d}{dt}(\lambda_N L_N \tau_N) = \dot{\lambda}_N L_N \tau_N + \lambda_N \dot{L}_N \tau_N + \lambda_N L_N \dot{\tau}_N$ and $\dot{\lambda}_N = -\partial_4 L_N \lambda_N$, we continue to obtain

$$\begin{aligned}
(4.15)\quad &\frac{d}{dt}(\lambda_N L_N \tau_N) \\
&+ (\xi_N - \dot{u}\tau_N)\Big[\lambda_N \partial_2 L_N + \sum_{i=0}^{N} \Big(-\frac{d}{dt}\Big)^i \Big(\lambda_N \partial_3 L_N \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\Big)\Big] \\
&+ \lambda_N \partial_3 L_N \sum_{i=0}^{N} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} (\xi_N - \dot{u}\tau_N)^{(i)} \\
&- (\xi_N - \dot{u}\tau_N) \sum_{i=0}^{N} \Big(-\frac{d}{dt}\Big)^i \Big(\lambda_N \partial_3 L_N \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\Big) = 0 \\
\Leftrightarrow\ &\frac{d}{dt}(\lambda_N L_N \tau_N) + \lambda_N \partial_3 L_N \sum_{i=0}^{N} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} (\xi_N - \dot{u}\tau_N)^{(i)} \\
&- (\xi_N - \dot{u}\tau_N) \sum_{i=0}^{N} \Big(-\frac{d}{dt}\Big)^i \Big(\lambda_N \partial_3 L_N \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\Big) = 0 \\
\Leftrightarrow\ &\frac{d}{dt}\Big[\lambda_N L_N \tau_N + \int_a^t \Big\{\lambda_N \partial_3 L_N \sum_{i=0}^{N} \binom{\alpha}{i} \frac{(s-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} (\xi_N - \dot{u}\tau_N)^{(i)} \\
&- (\xi_N - \dot{u}\tau_N) \sum_{i=0}^{N} \Big(-\frac{d}{ds}\Big)^i \Big(\lambda_N \partial_3 L_N \binom{\alpha}{i} \frac{(s-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\Big)\Big\} ds\Big] = 0.
\end{aligned}$$

Above we use the fact that the Euler–Lagrange equation (4.10) is satisfied so that the expression in brackets in the first line of (4.15) vanishes. Thus, we obtain the conservation law CL_N for the HVP$_N$ (4.2), for fixed $N \in \mathbb{N}$ formulated as:

$$\begin{aligned}
&\lambda_N L_N \tau_N + \int_a^t \Big\{\lambda_N \partial_3 L_N \sum_{i=0}^{N} \binom{\alpha}{i} \frac{(s-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} (\xi_N - \dot{u}\tau_N)^{(i)} \\
&\qquad - (\xi_N - \dot{u}\tau_N) \sum_{i=0}^{N} \Big(-\frac{d}{ds}\Big)^i \Big(\lambda_N \partial_3 L_N \binom{\alpha}{i} \frac{(s-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\Big)\Big\} ds = const
\end{aligned}$$

Next, we prove the weak convergence $CL_N \to CL$ (where we considered partial derivatives of L extended to zero outside $[a,T]$). Let $\varphi \in \mathcal{A}((c,d))$. Then,

$$\Big\langle \int_a^t (\xi_N - \dot u \tau_N) \sum_{i=0}^{N} \Big(-\frac{d}{ds}\Big)^i \Big(\lambda_N \partial_3 L_N \binom{\alpha}{i} \frac{(s-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\Big) ds, \varphi \Big\rangle \tag{4.16}$$

$$= \int_a^T dt \Big(\int_a^t ds(\xi_N(s) - \dot u(s)\tau_N(s)) \sum_{i=0}^{N} \Big(-\frac{d}{ds}\Big)^i \Big(\lambda_N \partial_3 L_N \binom{\alpha}{i} \frac{(s-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\Big)\Big)\varphi(t)$$

$$= \int_a^T ds \int_s^T dt \Big((\xi_N(s) - \dot u(s)\tau_N(s)) \sum_{i=0}^{N} \Big(-\frac{d}{ds}\Big)^i \Big(\lambda_N \partial_3 L_N \binom{\alpha}{i} \frac{(s-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\Big)\varphi(t)\Big)$$

$$= \int_a^T ds(\xi_N(s) - \dot u(s)\tau_N(s)) \sum_{i=0}^{N} \Big(-\frac{d}{ds}\Big)^i \Big(\lambda_N \partial_3 L_N \binom{\alpha}{i} \frac{(s-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\Big)\psi(s) = J_N,$$

with $\psi(s) = \int_s^T \varphi(t)dt$, where $\psi \in \mathcal{A}((c,d))$ and we used Fubini's theorem. Next we put $\psi_N = (\xi_N - \dot u \tau_N)\psi$ and obtain $\psi_N \in \mathcal{A}((c,d))$ since, by assumptions, $u, \tau_N, \xi_N \in \mathcal{A}((c,d))$. Thus

$$J_N = \Big\langle \sum_{i=0}^{N} \Big(-\frac{d}{ds}\Big)^i \Big(\lambda_N \partial_3 L_N \binom{\alpha}{i} \frac{(s-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\Big), \psi_N \Big\rangle.$$

Since $\psi_N \to (\xi - \dot u\tau)\psi$ as $N \to \infty$, uniformly on compact sets of (c,d), respectively on $[a,T]$, we obtain, by Montel's theorem, that $(\xi - \dot u \tau)\psi \in \mathcal{A}((c,d))$. Also using the uniform convergence of $\lambda_N \to \lambda, N \to \infty$, we have

$$\lim_N J_N = \langle {}_tD_T^\alpha(\lambda_N \partial_3 L), (\xi - \dot u\tau)\psi\rangle = \Big\langle {}_tD_T^\alpha(\lambda \partial_3 L), (\xi - \dot u\tau) \int_s^T \varphi(t)dt \Big\rangle$$

$$= \int_a^T ds {}_tD_T^\alpha(\lambda\partial_3 L)(\xi - \dot u\tau) \int_s^T \varphi(t)dt = \int_a^T dt \int_a^t (\xi - \dot u\tau){}_tD_T^\alpha(\lambda\partial_3 L)\varphi(t)$$

$$= \Big\langle \int_a^t (\xi - \dot u\tau){}_tD_T^\alpha(\lambda\partial_3 L), \varphi \Big\rangle,$$

where we apply Fubini's Theorem. This completes the proof. □

5. Examples

In this section we list the examples given in [1] which support the previously invoked theory.

1. We consider the case treated in [22],

$$\begin{aligned} &\dot z = \tfrac{1}{2}(\dot u^2 - \omega^2 u^2) - kz, && k = const \geqslant 0, && \omega = const. \\ &z(0) = 0, && t \in [0,T], && T > 0, \end{aligned} \tag{5.1}$$

and $u(0) = 0, \dot u(T) = 0$. Here (1.10) leads to $\lambda = \exp(kt)$ and the Euler–Lagrange equation becomes

$$\exp(kt)u\omega^2 + [\exp(kt)\dot u]^{\cdot} = 0 \ \Leftrightarrow\ \ddot u + k\dot u + \omega^2 u = 0. \tag{5.2}$$

Let the transformation generators be given as $\tau = A = const.$, $\xi = 0$. Then the invariance condition (2.11) is satisfied. The conservation law (3.2) is

$$\exp(kt)[\tfrac{1}{2}(\dot{u}(t)^2 + \omega^2 u(t)) + kz(t)] = const. \tag{5.3}$$

Expression (5.3) is a conserved quantity for the system (5.1) and (5.2), since it has both dependent variables u and z. If we use (5.1) to determine z as

$$z(t) = \frac{1}{2}\int_0^t \exp(k\tau)(\dot{u}(\tau)^2 - \omega^2 u(\tau)^2)d\tau,$$

then (5.3) becomes

$$\frac{1}{2}\exp(kt)\Big[(\dot{u}(t)^2 + \omega^2 u(t)) + k\int_0^t \exp(k\tau)(\dot{u}(\tau)^2 - \omega^2 u(\tau)^2)d\tau\Big] = const.$$

Finally, note that

$$\dot{u}(t)u(t) = \int_0^t \exp(k\tau)(\dot{u}(\tau)^2 - \omega^2 u(\tau)^2)d\tau, \quad t \in [0, T],$$

so that

$$\tfrac{1}{2}\exp(kt)[(\dot{u}(t)^2 + \omega^2 u(t)) + k\dot{u}u] = const. \tag{5.4}$$

is in agreement with [2, p. 64], where conservation law (5.4) was obtained by a different method.

2. Consider a problem of finding a function u over $[0, 1]$, $u(0) = 0, \dot{u}(0) = 1$ such that $z(1)$ is an extreme (see [12]) for the case when

$$\dot{z}(t) = \frac{{}_0D_t^{\alpha}u(t)^2}{2} - \omega^2\frac{u(t)^2}{2} - k{}_0D_t^{\beta}z(t), \ \ \alpha, \beta \in (0, 1], \ \omega \in \mathbb{R}, \ k > 0, \ t \in [0, 1]. \tag{5.5}$$

The Euler–Lagrange equation for (5.5) is

$$\begin{aligned} E_{1-\beta}(-k(1-t)^{1-\beta})\omega^2 u(t) - {}_tD_T^{\alpha}(E_{1-\beta}(-k(1-t)^{1-\beta}){}_0D_t^{\alpha}u(t)) = 0,& \\ \alpha, \beta \in (0, 1], \ \omega \in \mathbb{R}, \ k > 0, \ t \in [0, 1],& \end{aligned} \tag{5.6}$$

where E_{γ} is the Mittag-Leffler function [33]. Let $\beta = 0$ in (5.5). Then,

$$\dot{z}(t) = \frac{{}_0D_t^{\alpha}u(t)^2}{2} - \omega^2\frac{u(t)^2}{2} - kz(t), \quad \alpha, \beta \in (0, 1], \ \omega \in \mathbb{R}, \ k > 0, \ t \in [0, 1].$$

and the Euler–Lagrange equation becomes

$$\begin{aligned} \exp(-k(1-t))\omega^2 u(t) - {}_tD_1^{\alpha}(\exp(-k(1-t))\dot{u}(t)) = 0,& \\ \beta \in (0, 1], \ \omega \in \mathbb{R}, \ k > 0, \ t \in [0, 1].& \end{aligned} \tag{5.7}$$

Assume that the infinitesimal generators are $\tau = A$, $\xi = 0$. Then, condition (2.8) is satisfied since $u(0) = 0$ implies $\frac{d}{dt}{}_0D_t^{\alpha}u(t) = {}_0D_t^{\alpha}\dot{u}(t)$. Conservation law (3.1) becomes

$$\begin{aligned} \exp kt\Big[\frac{{}_0D_t^{\alpha}u(t)^2}{2} - \omega^2\frac{u(t)^2}{2} - kz(t)\Big] + \int_0^t \dot{u}(t)\,{}_tD_1^{\alpha}[\exp(kt){}_0D_t^{\alpha}u(t)]& \\ - \exp(kt){}_0D_t^{\alpha}u(t)[{}_0D_t^{\alpha}\dot{u}(t)] = const., \quad t \in [0, 1].& \end{aligned}$$

6. Conclusions

In this review article, the results obtained and presented in [1] are elaborated. Noether's theorem for the Herglotz variational principle when the Lagrangian contains fractional derivatives of real and complex orders is presented. Conditions for the invariance of the action integral invoked in [1] are also elaborated as well as the conservation law generated by invariance conditions. Also, approximations of the EL equation and the conservation law invoked in [1] are reviewed. In the approximation scheme, based on the approximation of the fractional derivative, the systems of differential equations containing integer order derivatives for the corresponding EL equation, as well as the conservation law are obtained. It is shown that those converge in the weak sense to the corresponding original fractional EL equation, as well as the conservation law, respectively. Two examples supporting the theory presented are elaborated.

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НЕТЕРИНА ТЕОРЕМА ЗА ВАРИЈАЦИОНЕ ПРОБЛЕМЕ ХЕРГЛОЦОВОГ ТИПА СА ФРАКЦИОНИМ ИЗВОДИМА КОМПЛЕКСНОГ РЕДА

Резиме. Ово је прегледни рад који елаборира резултате добијене у [1], где је формулисан варијациони принцип Херглоцовог типа са Лагранжианом који зависи од фракционих извода реалног и комплексног реда. Ту су такође одређени и услови инвариајнтности поменутог принципа под акцијом локалне групе симетрије. Добијен је и закон конзервације који одговара Ојлер-Лагранжовој једначини фракционог реда као и низ апроксимативних система једначина који одговарају фракционој Ојлер-Лагранжовој једначини, где је анализирана и конвергенција.

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