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ANALYTICAL SOLUTIONS FOR THE EFFECTIVE VISCOELASTIC PROPERTIES OF COMPOSITE MATERIALS WITH DIFFERENT SHAPES OF INCLUSIONS

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Abstract. This paper aims to model the effect of different shapes of inclusions on the homogenized viscoelastic properties of composite materials made of a viscoelastic matrix and inclusion particles. The viscoelastic behavior of the matrix phase is modeled by the Generalized Maxwell rheology. The effective properties are firstly derived by combining the homogenization theory of elasticity and the correspondence principle. Then, the effective rheological properties in time space are explicitly derived without using the complex inverse Laplace–Carson transformation (LC). Closed-form solutions for the effective bulk and shear rheological viscoelastic properties, the relaxation and creep moduli as well as the Poisson ratio are obtained for the isotropic case with random orientation distribution and different shapes of inclusions: spherical, oblate and elongate inclusions. The developed approach is validated against the exact solutions obtained by the classical inverse LC method. It is observed that the homogenized viscoelastic moduli are highly sensitive to different shapes of inclusions.

1. Introduction

A large number of models are based on the homogenization theory of continuum elasticity and the correspondence principle in order for the effective viscoelastic properties of composite materials to be estimated [6, 8, 15–18]. The solutions in time space (or real space) are usually determined by the inverse Laplace-Carson (LC) transform. However, for most of the cases, the inverse LC requires numerical calculations that are normally complex [10, 28, 34]. Besides, such a technique provides just the effective relaxation or creep moduli, not the effective rheological properties. Fortunately, it is demonstrated by short and long term analyses that the effective viscoelastic behavior of a composite made of a rheological viscoelastic matrix and elastic inclusion can be approximated by an effective rheology, with a very high accuracy, without using the inverse LC transform [5, 22]. However,

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these studies are limited to simple inclusion shapes such as spherical, cuboidal or penny-shaped inclusions. They can be extended to a complex inclusion shape. The case of a 2D periodic medium containing rectangular or elliptical inclusions is considered [32]. It is well-known that the shape of inclusions can strongly affect the effective properties of elastic materials [4, 13, 14, 25, 26, 29, 30, 33]. These characteristics are also observed for viscoelastic materials [1, 9, 35]. The inclusions' shape effect is also analyzed in several numerical studies [2, 19]. This work provides closed-form solutions for effective viscoelastic properties of composite materials containing different shapes of inclusions: spherical, oblate and elongate inclusions. It is organized as follows. First, the theoretical basis of the homogenization technique for viscoelastic composite materials and closed-form solutions for the effective viscoelastic rheological moduli are developed in Section 2. Then, in Section 3, the developed solutions are validated against the "exact" solutions obtained by the classical inverse LC transform. Finally, the effects of the inclusion shape on the relaxation and creep moduli as well as the effective rheological properties are discussed in Section 4. Concluding remarks are given in Section 5. Complex mathematical calculations are given in Appendix A to D.

Nomenclature

$*$	Laplace–Carson transform
0 and ∞	Short term and long term parameters, respectively
e and v	Elastic and viscous properties, respectively
hom	Homogenized (effective) properties
t and p	Time and Laplace–Carson variables, respectively
$\boldsymbol{\sigma}$	Microscopic stress tensor
$\mathbf{\Sigma}$	Macroscopic stress tensor
$\boldsymbol{\varepsilon}$	Microscopic strain tensor
$\mathbf{E}$	Macroscopic strain tensor
$\mathbf{1}$	Second-order identity tensor
ϖ	Volume fraction of inclusions
w	Aspect ratio of inclusions
γ	Ratio between creep and relaxation Poisson's ratios
$\mathbb{C}$	4^{th} order stiffness tensor
$\mathbb{A}$	4^{th} order concentration tensors
$\mathbb{S}$	4^{th} order compliance tensor
$\mathcal{O}(.)$	Negligible term
$\langle . \rangle$	Average overall the REV

2. The homogenization method for viscoelastic composite materials

2.1. Effective properties in LC space.

2.1.1. *Effective stiffness tensors.* We consider a viscoelastic composite material that is a mixture of a generalized Maxwell (GM) viscoelastic matrix (Figure 1) and elastic inclusions or void inclusions that are assumed to have a spheroidal shape. In LC space, the Mori–Tanaka homogenization scheme is appropriate for such a matrix-inclusion system [3, 20]. The homogenized apparent stiffness tensor in LC space is

$$\mathbb{C}^{\mathrm{hom}\,*} = [(1-\varphi)\mathbb{C}^* + \varphi\mathbb{C}_I : \mathbb{A}^*] : [(1-\varphi)\mathbb{I} + \varphi A^*]^{-1} \tag{2.1}$$

where $\mathbb{C}^*$ is the apparent stiffness tensor of the matrix phase, φ and $\mathbb{C}_I$ the volume fraction and the elastic stiffness tensor of the elastic inclusions, $\mathbb{A}^*$ the apparent strain concentration tensor of a single inclusion surrounded by an infinite matrix [12] and $\mathbb{I}$ the fourth order identity tensor.

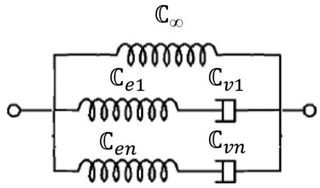

FIGURE 1. The generalized Maxwell rheology with an $n+1$ element.

It is well-known that the homogenized relaxation and creep properties in time domain can be obtained by the classical inverse LC transform method. However, the latter is complex and analytical solutions can be obtained for several limited simple examples where a limited number of Maxwell chains are involved to describe the matrix behavior [31]. To avoid such a problem, we assume that the effective viscoelastic behavior of the particulate medium can also be modeled by the GM rheology (Figure 1). Previous studies have shown that such an assumption is appropriate for a mixture of a viscoelastic matrix and elastic or void inclusions [21–24,32]. By approximating the homogenized effective viscoelastic behavior by an effective GM rheology, it is possible to define the effective rheological properties in detail as in Appendix A. Using the homogenization estimation (2.1), the effective instantaneous and long term stiffness tensors $\mathbb{C}_0^{\mathrm{hom}}$ and $\mathbb{C}_\infty^{\mathrm{hom}}$ can be derived as follows

$$\mathbb{C}_0^{\mathrm{hom}} = [(1-\varphi)\mathbb{C}_0 + \varphi\mathbb{C}_I : \mathbb{A}_0] : \mathbb{D}_0 \tag{2.2}$$

$$\mathbb{C}_\infty^{\mathrm{hom}} = [(1-\varphi)\mathbb{C}_\infty + \varphi\mathbb{C}_I : \mathbb{A}_\infty] : \mathbb{D}_\infty \tag{2.3}$$

with

$$\mathbb{D}_0 = [(1-\varphi)\mathbb{I} + \varphi\mathbb{A}_0]^{-1}; \quad \mathbb{D}_\infty = [(1-\varphi)\mathbb{I} + \varphi\mathbb{A}_\infty]^{-1} \tag{2.4}$$

where the short- and long-term tensors with index 0 and ∞ are defined in Appendix A. Now, to determine the viscous term $\mathbb{C}_v^{\mathrm{hom}}$, let us consider the derivation versus the Laplace variable of $\mathbb{C}^{\mathrm{hom}\,*}$ that was given by equation (2.1)

$$\begin{aligned} \frac{d\mathbb{C}^{\mathrm{hom}\,*}}{dp} = &\left[(1-\varphi)\frac{d\mathbb{C}^*}{dp} + \varphi\mathbb{C}_I : \frac{d\mathbb{A}^*}{dp}\right] : [(1-\varphi)\mathbb{I} + \varphi\mathbb{A}^*]^{-1} \\ &-[(1-\varphi)\mathbb{C}^* + \varphi\mathbb{C}_I : \mathbb{A}^*] : \left\{[(1-\varphi)\mathbb{I} + \varphi\mathbb{A}^*]^{-1} : \frac{\varphi d\mathbb{A}^*}{dp} : [(1-\varphi)\mathbb{I} + \varphi\mathbb{A}^*]^{-1}\right\} \end{aligned}$$

Then setting the limit at $p \to 0$ (using equations (A.2) and (A.4) from Appendix A for the limits of $\mathbb{C}^*$ and $\mathbb{A}^*$), we obtain

$$\mathbb{C}_v^{\mathrm{hom}} = [(1-\varphi)\mathbb{C}_v + \varphi\mathbb{C}_I : \mathbb{A}_v] : \mathbb{D}_\infty - [(1-\varphi)\mathbb{C}_\infty + \varphi\mathbb{C}_I : \mathbb{A}_\infty] : (\mathbb{D}_\infty : \varphi\mathbb{A}_v : \mathbb{D}_\infty)$$

which can be recast in the form

$$\mathbb{C}_v^{\text{hom}} = (1-\varphi)\mathbb{C}_v : \mathbb{D}_v + \varphi\mathbb{C}_I : \mathbb{A}_v : \mathbb{D}_\infty(\mathbb{I} - \varphi\mathbb{A}_\infty : \mathbb{D}_\infty) \tag{2.5}$$

where for such a simplification, we noted

$$\mathbb{D}_v = (\mathbb{I} - \varphi\mathbb{C}_v^{-1} : \mathbb{C}_\infty : \mathbb{D}_\infty : \mathbb{A}_v) : \mathbb{D}_\infty \tag{2.6}$$

By decomposing the first term $\mathbb{C}_v$ of equation (2.5) (see equation (A.3) in Appendix A, we obtain:

$$\begin{aligned} \mathbb{C}_v^{\text{hom}} = \sum_{i=1}^{n} \mathbb{C}_{vi}^{\text{hom}} &= (1-\varphi)\sum_{i=1}^{n}(\mathbb{C}_{vi} : \mathbb{D}_v) \\ &\quad + \varphi\sum_{i=1}^{n}(\mathbb{C}_I : \mathbb{A}_{vi} : \mathbb{D}_\infty(\mathbb{I} - \varphi\mathbb{A}_\infty : \mathbb{D}_\infty)) \end{aligned} \tag{2.7}$$

where we assume that the concentration tensor $\mathbb{A}_v$ can be decomposed as

$$\mathbb{A}_v = \sum_{i=1}^{n} \mathbb{A}_{vi}$$

With respect to equation (2.7), the following estimation for the effective viscous tensor $\mathbb{C}_{vi}^{\text{hom}}$ of each Maxwell series can be obtained

$$\mathbb{C}_{vi}^{\text{hom}} = (1-\varphi)\mathbb{C}_{vi} : \mathbb{D}_v + \varphi\mathbb{C}_I : \mathbb{A}_{vi} : \mathbb{D}_\infty(\mathbb{I} - \varphi\mathbb{A}_\infty : \mathbb{D}_\infty), \quad \forall i = 1 \text{ to } n \tag{2.8}$$

In a similar way, starting from the solutions (2.2) and (2.3), we can estimate the effective elastic stiffness tensor of each Maxwell series as follows

$$\begin{aligned} \mathbb{C}_{ei}^{\text{hom}} &= (1-\varphi)\mathbb{C}_{ei} : (\mathbb{C}_0 - \mathbb{C}_\infty)^{-1} : (\mathbb{C}_0 : \mathbb{D}_0 - \mathbb{C}_\infty : \mathbb{D}_\infty) \\ &\quad + \varphi\mathbb{C}_I : \mathbb{C}_{ei} : (\mathbb{C}_0 - \mathbb{C}_\infty)^{-1} : (\mathbb{A}_0 : \mathbb{D}_0 - \mathbb{A}_\infty : \mathbb{D}_\infty), \quad \forall i = 1 \div n \end{aligned} \tag{2.9}$$

2.1.2. *Viscoelastic concentration tensors.* Using the solutions (2.3), (2.8) and (2.9) to calculate the effective rheological tensor of the homogenized viscoelastic composite materials in time domain requires the knowledge of the strain concentration tensors $\mathbb{A}_0$, $\mathbb{A}_\infty$ and $\mathbb{A}_{vi}$ with $i = 1$ to n. To determine these tensors, let us consider a combination of Eshelby's solution [12] and the correspondence principle that allows writing the apparent concentration tensor, in LC space, of an ellipsoidal elastic inclusion located in an infinite viscoelastic matrix with a prescribed homogeneous macroscopic strain as follows

$$\mathbb{A}^* = [\mathbb{I} - \mathbb{P}^* : (\mathbb{C}^* - \mathbb{C}_I)]^{-1}$$

where $\mathbb{P}^*$ is apparent Hill's tensor in LC space. Asymptotic responses at $p \to \infty$ and $p \to 0$ yield

$$\mathbb{A}_0 = [\mathbb{I} - \mathbb{P}_0 : (\mathbb{C}_0 - \mathbb{C}_I)]^{-1}; \quad \mathbb{A}_\infty = [\mathbb{I} - \mathbb{P}_\infty : (\mathbb{C}_\infty - \mathbb{C}_I)]^{-1} \tag{2.10}$$

$$\mathbb{A}_{vi} = \mathbb{A}_\infty : [\mathbb{P}_{vi} : (\mathbb{C}_\infty - \mathbb{C}_I) + \mathbb{P}_\infty : \mathbb{C}_{vi}] : \mathbb{A}_\infty \tag{2.11}$$

where

$$\mathbb{P}_0 = \lim_{p\to\infty} \mathbb{P}^*; \quad \mathbb{P}_\infty = \lim_{p\to 0} \mathbb{P}^*; \quad \sum_{i=1}^{n} \mathbb{P}_{vi} = \mathbb{P}_v = \lim_{p\to 0} \frac{d\mathbb{P}}{dp}$$

The detailed expressions of $\mathbb{P}_0$, $\mathbb{P}_\infty$ and $\mathbb{P}_{vi}$ are given in Appendix B.

2.2. The isotropic case with random orientation distribution of spheroidal inclusions.

2.2.1. *Effective rheological properties.* Let us consider a case of viscoelastic composite materials containing the inclusions randomly oriented which are spheroidal isotropic elastic particles and an isotropic viscoelastic GM matrix. The average overall directions of the viscoelastic concentration tensors can be decomposed into spherical and deviatoric parts as follows

$$\langle \mathbb{A}_\alpha \rangle = a_J^{(\alpha)} \mathbb{J} + a_K^{(\alpha)} \mathbb{K}$$

where α is 0, ∞ or v; the notation $\langle . \rangle$ stands for an average over the orientation distribution of the particles; $\mathbb{J}$ and $\mathbb{K}$ are the spherical and deviatoric parts of the fourth order identity tensor. The parameters $a_J^{(\alpha)}$ and $a_K^{(\alpha)}$ can be calculated from the components of the concentration tensors $\mathbb{A}_\alpha$ (see Appendix D). Knowing $a_J^{(\alpha)}$ and $a_K^{(\alpha)}$, the correspondent spherical and deviatoric coefficients $d_J^{(\alpha)}$ and $d_K^{(\alpha)}$ of the tensors $\mathbb{D}_0$ and $\mathbb{D}_\infty$ that are defined by equation (2.4) are

$$d_J^{(\alpha)} = \frac{1}{1 - \varphi + \varphi a_J^{(\alpha)}}; \quad d_K^{(\alpha)} = \frac{1}{1 - \varphi + \varphi a_K^{(\alpha)}}$$

with $\alpha = 0$ or ∞. Then the terms $d_J^{(\infty)}$ and $d_K^{(\infty)}$ of the viscous tensor $\mathbb{D}_v$ which is defined by equation (2.6) are

$$d_J^{(v)} = d_J^{(\infty)} \left[1 - \varphi \frac{k_\infty d_J^{(\infty)} a_J^{(v)}}{k_v} \right]; \quad d_K^{(v)} = d_K^{(\infty)} \left[1 - \varphi \frac{\mu_\infty d_K^{(\infty)} a_K^{(v)}}{\mu_v} \right]$$

where k_∞ and μ_∞ are the long-term bulk and shear elastic moduli of the matrix; k_v and μ_v are the bulk and shear viscosity of the tensor $\mathbb{C}_v$ (see also equation (B.3) in Appendix B. Finally, the effective viscoelastic bulk and shear moduli can be estimated using equations (2.3), (2.8) and (2.9) as follows

$$\begin{aligned}
k_\infty^{\text{hom}} &= [(1-\varphi) k_\infty + \varphi k_I a_J^{(\infty)}] d_J^{(\infty)} \\
\mu_\infty^{\text{hom}} &= [(1-\varphi) \mu_\infty + \varphi \mu_I a_K^{(\infty)}] d_K^{(\infty)} \\
k_{vi}^{\text{hom}} &= (1-\varphi) k_{vi} d_J^{(v)} + \varphi k_I a_J^{(vi)} d_J^{(\infty)} (1 - \varphi a_J^{(\infty)} d_J^{(\infty)}) \\
\mu_{vi}^{\text{hom}} &= (1-\varphi) \mu_{vi} d_K^{(v)} + \varphi \mu_I a_K^{(vi)} d_K^{(\infty)} (1 - \varphi a_K^{(\infty)} d_K^{(\infty)}) \\
k_{ei}^{\text{hom}} &= [(1-\varphi) k_{ei} + \varphi k_I a_J^{(ei)}] d_J^{(e)} \\
\mu_{ei}^{\text{hom}} &= [(1-\varphi) \mu_{ei} + \varphi \mu_I a_K^{(ei)}] d_K^{(e)}, \quad \forall i = 1 \text{ to } n
\end{aligned} \tag{2.12}$$

with

$$d_J^{(e)} = \frac{k_0 d_J^{(0)} - k_\infty d_J^{(\infty)}}{k_0 - k_\infty}; \quad d_K^{(e)} = \frac{\mu_0 d_K^{(0)} - \mu_\infty d_K^{(\infty)}}{\mu_0 - \mu_\infty}$$

and

$$a_J^{(ei)} = k_{ei} \frac{a_J^{(0)} d_J^{(0)} - a_J^{(\infty)} d_J^{(\infty)}}{k_0 d_J^{(0)} - k_\infty d_J^{(\infty)}}; \quad a_K^{(ei)} = \mu_{ei} \frac{a_K^{(0)} d_K^{(0)} - a_K^{(\infty)} d_K^{(\infty)}}{\mu_0 d_K^{(0)} - \mu_\infty d_K^{(\infty)}}$$

Note that the macroscopic relaxation and creep moduli can be easily computed from the effective rheological parameters obtained by equation (2.12) by using some well-known expressions given in Appendix C.

3. Validation of the proposed model

3.1. Comparison with the inverse LC method. To obtain the "exact" results of the effects of ellipsoidal shapes of the particulate phases on the macroscopic material response, the analytical inverse LC transform method is used. One key feature of this method is that the analytical expressions of time-dependent effective mechanical properties are derived in time space as a function of the properties of the components, and that these expressions are exact in spherical cases [31]. The exact results can be extended for the microstructure with ellipsoidal inclusions in the cases of oblate and elongate inclusions.

In the considered case of a heterogeneous material in which M ellipsoidal inclusion phases (phase i) are dispersed in the matrix phase (phase m), the effective bulk and shear moduli given in detail by the MT scheme [3] depend on the volume fraction, aspect ratio, bulk and shear moduli of components (see Appendix B). Moreover, taking the same number of Maxwell chains for the two moduli is not necessary, but allows simplifying the developments. The bulk and shear moduli of the matrix phase can then be expressed in time domain by:

$$k^m(t) = k_0^m + \sum_{i=1}^{N} k_i^m e^{-\frac{t}{\tau_i^{m,k}}} \quad \text{and} \quad \mu^m(t) = \mu_0^m + \sum_{i=1}^{N} \mu_i^m e^{-\frac{t}{\tau_i^{m,\mu}}}$$

where k_0^m, k_i^m, μ_0^m and μ_i^m are the elastic moduli of the matrix in the Maxwell chains, $\tau_i^{m,k}$ and $\tau_i^{m,\mu}$ are its relaxation times defined by $\tau_i^{m,k} = \eta_i^{m,k}/k_i^m$ and $\tau_i^{m,\mu} = \eta_i^{m,\mu}/\mu_i^m$, with $\eta_i^{m,k}$ and $\eta_i^{m,\mu}$ as viscosities of the dashpots. We make the assumption that $\tau_i^{m,k} = \tau_i^{m,\mu} = \tau_i^m$, i.e. the relaxation times are identical for $k^m(t)$ and $\mu^m(t)$, and that they are well separated for the consecutive values of i.

The expressions of k^{m*} and μ^{m*} are obtained in LC domain as

$$k^{m*} = k_0^m + \sum_{i=1}^{N} k_i^m \frac{p}{p + 1/\tau_i^m}, \quad \mu^{m*} = \mu_0^m + \sum_{i=1}^{N} \mu_i^m \frac{p}{p + 1/\tau_i^m}$$

where p is a variable in LC domain, t is a variable in time domain.

By using the same decomposition method as [31], the following formulae involving polynomials of the variable p can be derived for the effective moduli of the material:

$$\begin{aligned} k_{MT}^{\text{hom}\,*} &= \frac{\left(\sum_{i=0}^{N} A_i^k p^i\right)\left(\sum_{i=0}^{M^k} B_i^k p^i\right)}{\left(\sum_{i=0}^{N} C_i^k p^i\right)\left(\sum_{i=0}^{M^k} D_i^k p^i\right)}, \\ \mu_{MT}^{\text{hom}\,*} &= \frac{\left(\sum_{i=0}^{N} A_i^\mu p^i\right)\left(\sum_{i=0}^{M^\mu} B_i^\mu p^i\right)}{\left(\sum_{i=0}^{N} C_i^\mu p^i\right)\left(\sum_{i=0}^{M^\mu} D_i^\mu p^i\right)} \end{aligned} \tag{3.1}$$

where A_i^k, A_i^μ, B_i^k, B_i^μ, C_i^k, C_i^μ, D_i^k and D_i^μ are coefficients depending explicitly on the parameters k_0^m, k_i^m, τ_i^m, μ_0^m, μ_i^m and the volume fractions φ, aspect ratio

w; M^k and M^μ are the integer function of the number of Maxwell chains N and the number of particle phases M.

The expressions of the effective viscoelastic properties $k_{MT}^{\text{hom}}(t)$ and $\mu_{MT}^{\text{hom}}(t)$ of composite materials in time domain are easily delivered by inverting Eq. (3.1)

$$k_{MT}^{\text{hom}}(t) = k_0 + \sum_{i=1}^{N} k_i e^{-t/\tau_i^k} \qquad \mu_{MT}^{\text{hom}}(t) = \mu_0 + \sum_{i=1}^{N} \mu_i e^{-t/\tau_i^\mu}$$

where k_i and μ_i are the moduli associated with the relaxation times τ_i^k and τ_i^μ. It should be pointed out that the above exact analytical formulation of the inverse LC problem has been obtained in the case of two-phase matrix/inclusions microstructures and by applying the MT scheme and using a GM with $n = 2$.

Let us consider an example of a GM rheology with $n = 2$ whose viscoelastic properties of the viscoelastic matrix and the elastic inclusions are given in Table 1. Figure 2 shows the effective shear creep modulus as a function of time for the case of oblate inclusions with an aspect ratio of $w = 0.1$.

The relative error between the present approximation and the analytical results is defined by

$$E(t) = \frac{p_{\text{appx}} - p_{\text{ana}}}{p_{\text{ana}}} 100\%$$

where p_{appx} stands for the bulk or shear modulus calculated by the proposed method and p_{ana} is the one calculated by the classical inverse LC method. Figure 3 shows the errors between the two methods for the case of oblate inclusions with an aspect ratio of $w = 0.01$ to ∞. A very good agreement between the proposed analytical solution and the classical inverse Laplace–Carson method is observed. The highest error is observed at a transient time and is less than 4%. Such an error is generally acceptable for engineering application.

TABLE 1. Viscoelastic properties of a composite material with porosity $\varphi = 0.42$ [31] and inclusions.

Elements	∞	1	2	Inclusions
k_i (GPa)	2.55	5.14	1.41	37.8
μ_i (GPa)	2.33	2.34	1.00	44.3
k_{vi} (GPa.days)	5.81	28.72		
μ_{vi} (GPa.days)	2.26	17.37		

3.2. The particular case of void inclusions. The case of void inclusions corresponds to zero bulk and shear elastic moduli $K_I = \mu_I = 0$; the solutions (2.12) are simplified to

$$\begin{aligned} k_\infty^{\text{hom}} &= (1-\varphi)k_\infty d_J^{(\infty)}; & \mu_\infty^{\text{hom}} &= (1-\varphi)\mu_\infty d_K^{(\infty)} \\ k_{vi}^{\text{hom}} &= (1-\varphi)k_{vi} d_J^{(v)}; & \mu_{vi}^{\text{hom}} &= (1-\varphi)\mu_{vi} d_K^{(v)} \\ k_{ei}^{\text{hom}} &= (1-\varphi)k_{ei} d_J^{(e)}; & \mu_{ei}^{\text{hom}} &= (1-\varphi)\mu_{ei} d_K^{(e)} \end{aligned}$$

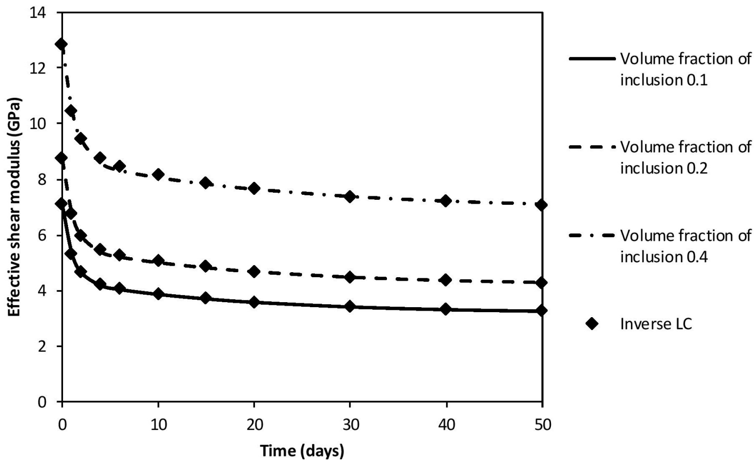

FIGURE 2. Effective relaxation shear modulus of a viscoelastic composite material: a comparison between the proposed method (line curve) and the inverse Laplace–Carson method (symbols) for the case of oblate inclusions with an aspect ratio $w = 0.1$ and volume fraction of inclusion $\varphi = 0.1$ to 0.4.

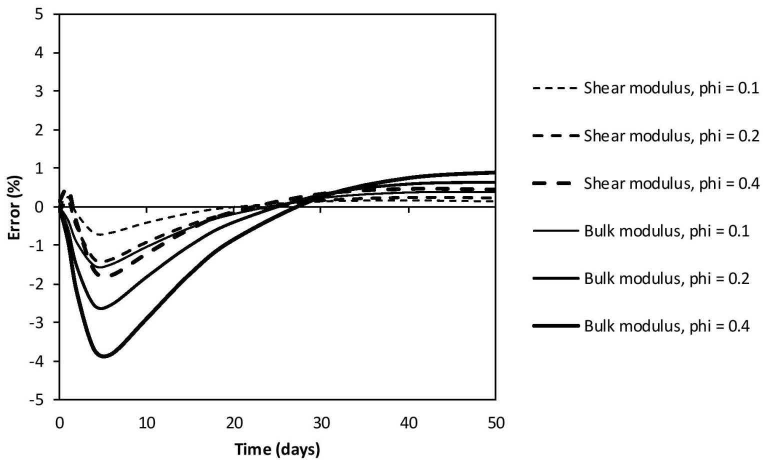

FIGURE 3. Errors between the proposed method and the inverse Laplace–Carson method for the range of volume fraction of inclusions $\varphi = 0.1$ to 0.4 and the aspect ratio of the inclusions $w = 0.01$ to ∞.

4. Results and discussion

Figure 4 shows the evolution of the effective viscoelastic properties (normalized to those of the matrix phase) versus the volume fraction of inclusions with a fixed

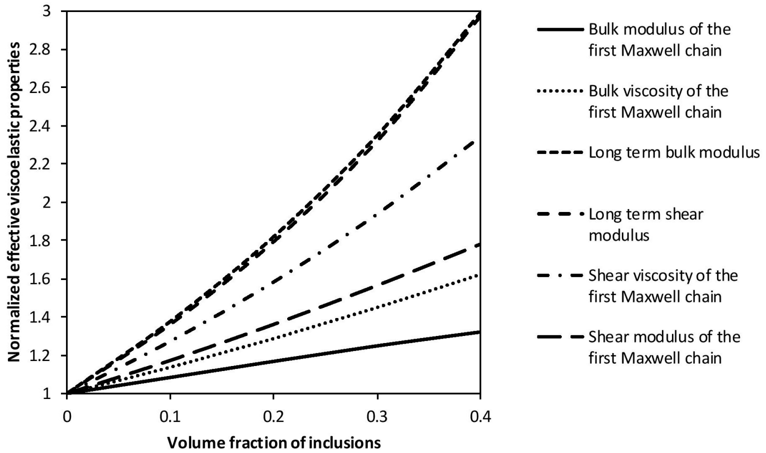

FIGURE 4. Effective viscoelastic properties, normalized to the correspondent properties of the matrix phase, of a particulate medium containing oblate inclusions with the aspect ratio $w = 0.1$.

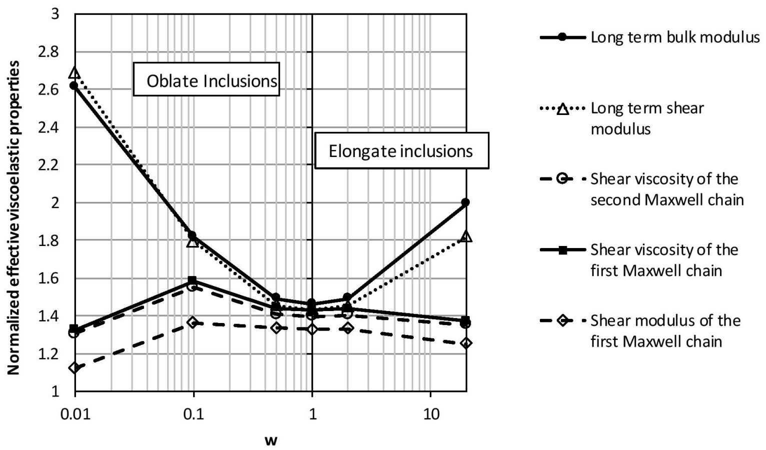

FIGURE 5. Effect of the aspect ratio of the inclusions, $\varphi = 0.2$. The points represent the numerical simulations.

aspect ratio $w = 0.1$ and the parameters given in Table 1. We can observe that all the bulk and shear elastic moduli and viscosities increase with increasing volume fraction of inclusions because the inclusion is stiffer than the matrix phase. Figure 5 shows the effect of the aspect ratio of the inclusions with a fixed volume fraction $\varphi = 0.2$. A strong effect of the inclusion's shape on the overall viscoelastic properties of the mixture can also be observed in Figure 5.

Similarly, Figure 6–8 show a strong effect of inclusions' shape on the effective relaxation bulk and shear moduli and the correspondent Poisson ratio. For example,

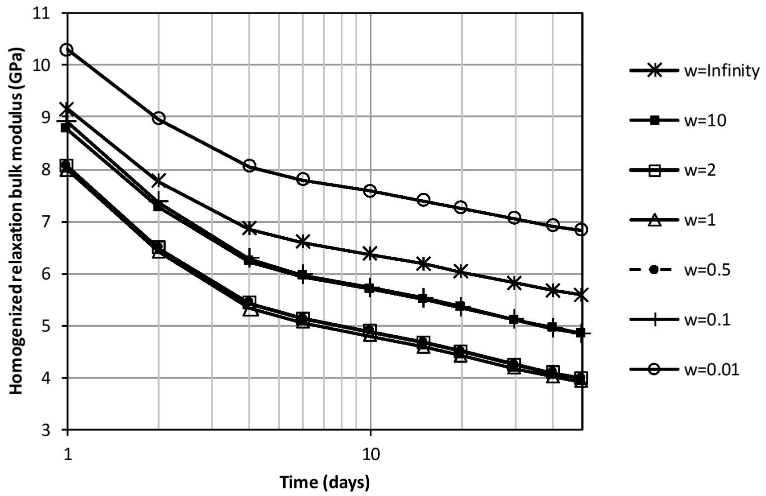

FIGURE 6. Effect of the inclusion aspect ratio w on the relaxation bulk modulus, $\varphi = 0.2$.

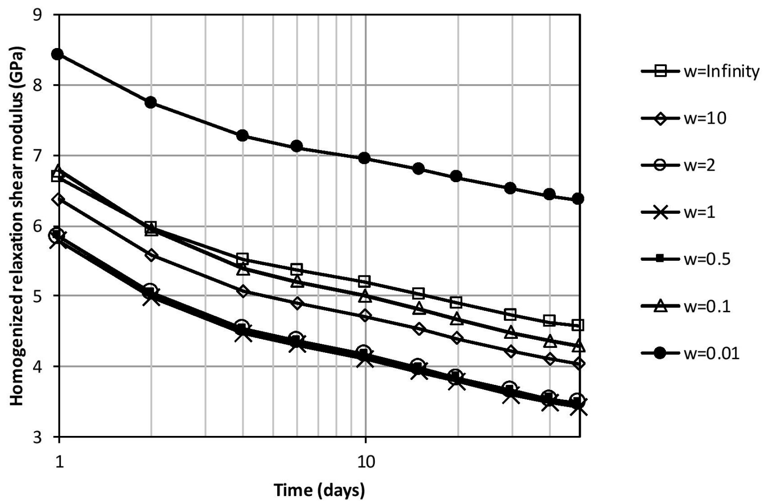

FIGURE 7. Effect of the inclusion aspect ratio w on the relaxation shear modulus, $\varphi = 0.2$.

a composite with inclusions' aspect ratio of $w = 0.01$ has bulk and shear relaxation moduli of about 1.5 to 2 times higher than a composite with spherical inclusions. We can also observe that Poisson's ratio rapidly decreases for the first 4–5 days, then much more at the end of the test. This evolution confirms the need of different time functions for the macroscopic bulk and shear moduli in the model.

Figure 9 shows a comparison between the relaxation and creep Poisson ratios for a particulate medium with an inclusion volume fraction and aspect ratio fixed

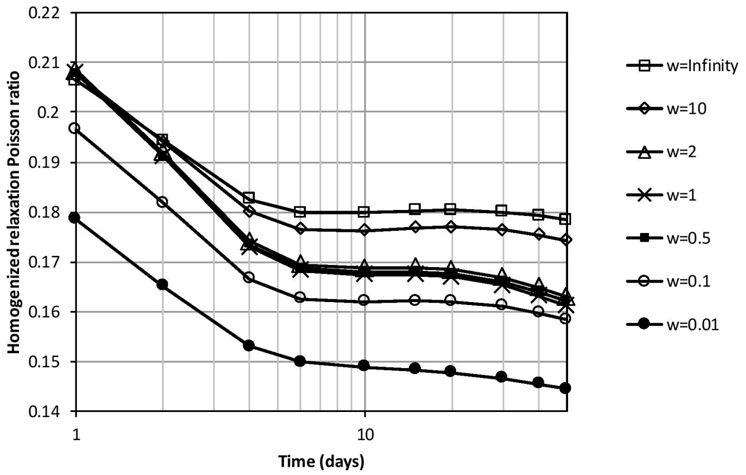

Figure 8. Effect of the inclusion aspect ratio w on the relaxation Poisson ratio, $\varphi = 0.2$.

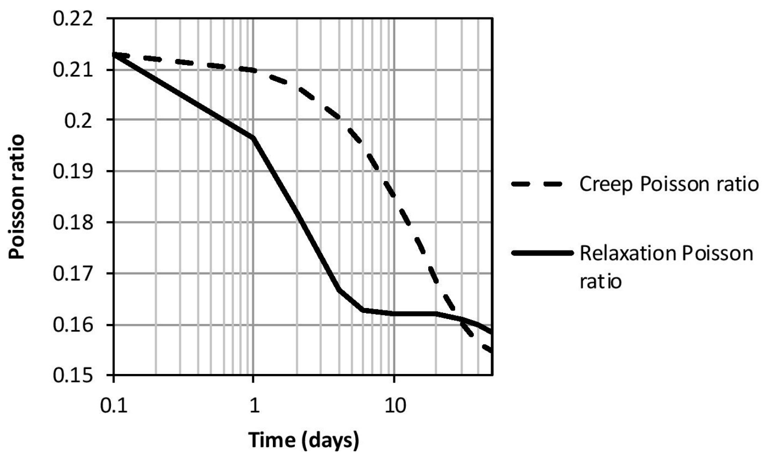

Figure 9. A comparison between the creep and relaxation Poisson ratios calculated by the present method with $\varphi = 0.2$ and $w = 0.1$.

at $\varphi = 0.2$ and $w = 0.1$. The viscoelastic properties of the matrix phase are given in Table 1. The creep and relaxation Poisson ratios are close for short term and long-term behavior. But they are very different at transient time. Figure 10 and Figure 11 show the effect of the volume fraction of inclusions and the aspect ratio of the inclusions on the ratio between creep and relaxation Poisson ratios, noted by $\gamma(t)$. We observe that $\gamma(t)$ is smaller for a higher volume fraction of inclusions. The effect of the aspect ratio of the inclusion shown in Figure 11 is less significant than the effect of the volume fraction of inclusions.

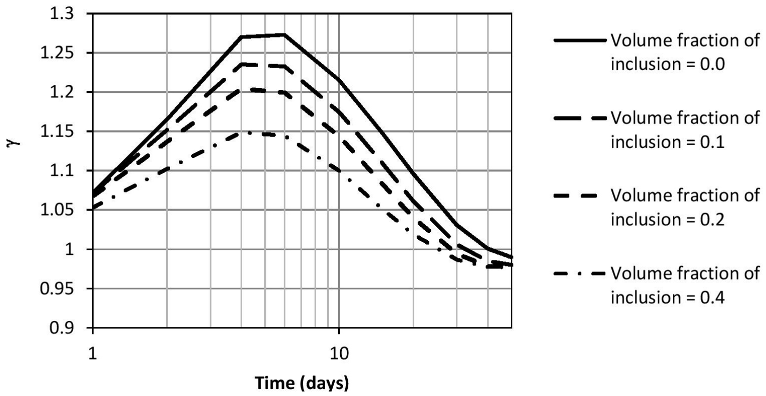

FIGURE 10. Effect of the volume fraction of inclusions on the ratio γ between the creep and relaxation Poisson ratios, $w = 0.1$.

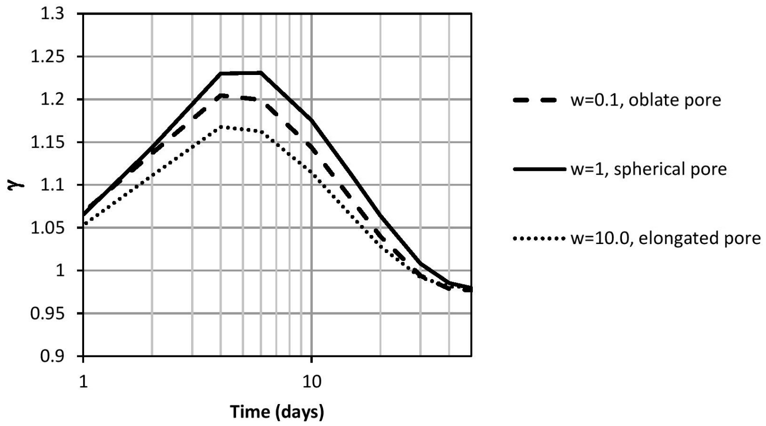

FIGURE 11. Effect of the aspect ratio of the inclusions on the ratio γ between the creep and relaxation Poisson ratios, $\varphi = 0.2$.

5. Conclusions

We have presented in this paper a study on the influence of the aspect ratio of the inclusions: spherical, oblate and elongate inclusions on the effective rheological properties of viscoelastic composite materials made of a generalized Maxwell matrix and inclusions. Explicit analytical solutions are obtained for the effective rheological properties as well as the relaxation and creep bulk and shear moduli and the correspondent Poisson ratio. These results are obtained without considering the complex inverse LC method and they are perfectly validated against the exact solutions obtained by the latter. It is of interest to remark that the inverse LC method provides just the relaxation and creep moduli, while the present method provides also the effective rheological properties that can be flexibly used when modeling structures made of a viscoelastic material under complex loading histories. The results of the simulation indicate that the aspect ratio of the inclusions can have a significant effect on the effective rheological viscoelastic properties, the bulk and shear creep and relaxation moduli and the correspondent Poisson ratio. The ratio between creep and relaxation Poisson ratio is also strongly influenced by both the volume fraction and the aspect ratio of the inclusions.

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Appendix A. Generalized Maxwell model for the matrix phase

The generalized Maxwell (GM) model is schematically expressed by $2n+1$ rheological elements in which the n Maxwell series is parallel with a spring that characterizes the long term elastic behavior by the asymptotic elastic stiffness tensor $\mathbb{C}_\infty$ (Figure 1). The elastic stiffness tensor and the viscosity tensor of a Maxwell series i (with $i = 1 \div n$) are noted by $\mathbb{C}_{ei}$ and $\mathbb{C}_{vi}$, respectively. In Laplace–Carson space, the apparent stiffness tensor of the GM model, noted by $\mathbb{C}^*$ (the superscript $*$ stands for the LC transform), can be expressed by:

$$\mathbb{C}^* = \mathbb{C}_\infty + \sum_{i=1}^{n} \left(\mathbb{S}_{ei} + \frac{1}{p}\mathbb{S}_{vi} \right)^{-1} \tag{A.1}$$

where p is the LC variable; $\mathbb{S}_{ei}$ and $\mathbb{S}_{vi}$ are the elastic and viscous compliance tensors that are the inverses of the stiffness tensor $\mathbb{C}_{ei}$ and the viscosity tensor $\mathbb{C}_{vi}$, respectively.

The long term and short term behaviors that correspond to the limit at $p \to 0$ and $p \to \infty$ ($p \to 0$ corresponds to $t \to \infty$ and vice versa) can be obtained using equation (A.1) as follows:

$$\lim_{p\to\infty} \mathbb{C}^* = \mathbb{C}_0 + \mathcal{O}\left(\frac{1}{p}\right); \quad \lim_{p\to 0} \mathbb{C}^* = \mathbb{C}_\infty + p\mathbb{C}_v + \mathcal{O}(p^2) \tag{A.2}$$

where the terms noted by $\mathcal{O}(.)$ are the negligible terms. We noted also:

$$\mathbb{C}_0 = \mathbb{C}_\infty + \sum_{i=1}^{n} \mathbb{C}_{ei}; \quad \mathbb{C}_v = \sum_{i=1}^{n} \mathbb{C}_{vi} \tag{A.3}$$

The first one, noted by $\mathbb{C}_0$, is the instantaneous elastic stiffness tensor of the matrix phase. The second one has no physical meaning. It is shown that using the approximations presented by equation (A.2) instead of the exact equation (A.1) allows avoiding the complex inverse Laplace–Carson method when passing from LC space to real space [22].

Similar to equation (A.2) which was derived for the matrix phase, short and long terms of the homogeneous apparent stiffness tensor $\mathbb{C}^{\text{hom}\,*}$ can be expressed by:

$$\lim_{p\to\infty} \mathbb{C}^{\text{hom}\,*} = \mathbb{C}_0^{\text{hom}} + \mathcal{O}\left(\frac{1}{p}\right); \qquad \lim_{p\to 0} \mathbb{C}^{\text{hom}\,*} = \mathbb{C}_\infty^{\text{hom}} + p\mathbb{C}_v^{\text{hom}} + \mathcal{O}(p^2)$$

with

$$\mathbb{C}_0^{\text{hom}} = \mathbb{C}_\infty^{\text{hom}} + \sum_{i=1}^{n} \mathbb{C}_{ei}^{\text{hom}}; \qquad \mathbb{C}_v^{\text{hom}} = \sum_{i=1}^{n} \mathbb{C}_{vi}^{\text{hom}}$$

where $\mathbb{C}_0^{\text{hom}}$ is the effective homogeneous instantaneous stiffness tensor (in time space) and $\mathbb{C}_\infty^{\text{hom}}$ the correspondent long term stiffness tensor (that is the effective elastic stiffness tensor of the last spring); $\mathbb{C}_{ei}^{\text{hom}}$ and $\mathbb{C}_{vi}^{\text{hom}}$ the effective elastic stiffness and viscosity tensors of a Maxwell series i (with $i = 1 \div n$).

Alternatively, it will be demonstrated in the next sections that the concentration tensor $\mathbb{A}^*$ can also be decomposed to:

$$\lim_{p\to\infty} \mathbb{A}^* = \mathbb{A}_0 + \mathcal{O}\Big(\frac{1}{p}\Big); \qquad \lim_{p\to 0} \mathbb{A}^* = \mathbb{A}_\infty + p\mathbb{A}_v + \mathcal{O}(p^2) \tag{A.4}$$

where $\mathbb{A}_0$ and $\mathbb{A}_\infty$ are the instantaneous and long-term concentration tensors and $\mathbb{A}_v$ is a certain viscous concentration tensor.

Appendix B. Viscoelastic Hill's tensors for spheroidal inclusions in the GM viscoelastic matrix

For the case of elastic composite materials containing spheroidal inclusions, it is well-known that Hill's tensor is transversely isotropic, i.e. it can be defined by five independent parameters. Equivalently, for the case of viscoelastic composite materials containing spheroidal inclusions, apparent Hill's tensor in LC space can be expressed, in a matrix form, by five independent transformed parameters as follows:

$$\mathbb{P}^* = \begin{bmatrix} P^*_{1111} & P^*_{1122} & P^*_{1133} & 0 & 0 & 0 \\ P^*_{1122} & P^*_{1111} & P^*_{1133} & 0 & 0 & 0 \\ P^*_{1133} & P^*_{1133} & P^*_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2P^*_{3131} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2P^*_{3131} & 0 \\ 0 & 0 & 0 & 0 & 0 & P^*_{1111} - P^*_{1122} \end{bmatrix} \tag{B.1}$$

where the parameters P^*_{ijkl} can be determined using the classical well-known solutions obtained for equivalent linear elastic materials [7]:

$$\begin{aligned} P^*_{1111} &= \frac{3(6w^2 - 5h - 4w^2h)k^* + (6w^2 - 29h + 20w^2h)\mu^*}{16\xi^*} \\ P^*_{1122} &= \frac{3(2w^2 + h - 4w^2h)k^* + (2w^2 + h - 4w^2h)\mu^*}{16\xi^*} \\ P^*_{3333} &= \frac{3(2w^2 - h - 2w^2h)k^* + (-6 + 8w^2 + 5h - 8w^2h)\mu^*}{2\xi^*} \\ P^*_{3131} &= \frac{3(-2 - 2w^2 + 3h + 3w^2h)k^* + 2(-4 + 2w^2 + 3h)\mu^*}{8\xi^*} \end{aligned} \tag{B.2}$$

where k^* and μ^* are the apparent bulk and shear moduli of the solid matrix; w the aspect ratio of the spheroidal inclusion (the ratio between the maximal dimension in the evolution direction of the inclusions and the diameter of its plan), and h a geometrical function that is determined by:

$$h = \begin{cases} \dfrac{w\big(\arccos(w) - w\sqrt{1-w^2}\big)}{(1-w^2)^{3/2}}, & \text{for } w < 1 \\ \dfrac{w\big(w\sqrt{w^2-1} - \operatorname{arccos}h(w)\big)}{(w^2-1)^{3/2}}, & \text{for } w > 1 \\ \dfrac{2}{3}, & \text{for } w = 1 \end{cases}$$

and

$$\xi^* = (w^2-1)\mu^*(3k^* + 4\mu^*).$$

Other equivalent expressions of the component of Hill's tensor for the case of spheroidal inclusion can be found in [27]. Now, taking the limit at $p \to \infty$, instantaneous Hill's tensor $\mathbb{P}_0$ can be determined. It has a similar form to $\mathbb{P}^*$ given by equation (B.1), where the components P^*_{ijkl} are replaced by the instantaneous components $P^{(0)}_{ijkl}$. The latter can be determined by equation (B.2) where the apparent bulk and shear moduli k^* and μ^* are replaced by the instantaneous bulk and shear moduli k_0 and μ_0. In the same way, long term Hill's tensor $\mathbb{P}_\infty$ can be determined by equations (B.1) and (B.2) using the long term bulk and shear moduli k_∞ and μ_∞.

To determine viscous Hill's tensor $\mathbb{P}_v$, it is necessary to consider the limit at $p \to 0$ of the derivative dP^*_{ijkl}/dp:

$$P^{(v)}_{ijkl} = \sum_{i=1}^{n} P^{(vi)}_{ijkl} = \lim_{p\to 0} \frac{dP^*_{ijkl}}{dp}$$

that yields:

$$\begin{aligned}
P^{(vi)}_{1111} &= \frac{3(6w^2 - 5h - 4w^2h)k_{vi} + (6w^2 - 29h + 20w^2h)\mu_{vi}}{16\xi_\infty} - \psi_i P^{(\infty)}_{1111} \\
P^{(vi)}_{1122} &= \frac{3(2w^2 + h - 4w^2h)k_{vi} + (2w^2 + h - 4w^2h)\mu_{vi}}{16\xi_\infty} - \psi_i P^{(\infty)}_{1122} \\
P^{(vi)}_{1133} &= \frac{(-2w^2 + h + 2w^2h)(3k_{vi} + \mu_{vi})}{4\xi_\infty} - \psi_i P^{(\infty)}_{1133} \\
P^{(vi)}_{3333} &= \frac{3(2w^2 - h - 2w^2h)k_{vi} + (-6 + 8w^2 + 5h - 8w^2h)\mu_{vi}}{2\xi_\infty} - \psi_i P^{(\infty)}_{3333} \\
P^{(vi)}_{3131} &= \frac{3(-2 - 2w^2 + 3h + 3w^2h)k_{vi} + 2(-4 + 2w^2 + 3h)\mu_{vi}}{8\xi_\infty} - \psi_i P^{(\infty)}_{3131}
\end{aligned}$$

where

$$\xi_\infty = (w^2-1)\mu_\infty(3k_\infty + 4\mu_\infty)$$

and

$$\sum_{i=1}^{n} \psi_i = \lim_{p\to 0}\left(\frac{1}{\xi^*}\frac{d\xi^*}{dp}\right) = \frac{\mu_v}{\mu_\infty} + \frac{3k_v + 4\mu_v}{3k_\infty + 4\mu_\infty}$$

with

$$k_v = \sum_{i=1}^{n} k_{vi}; \qquad \mu_v = \sum_{i=1}^{n} \mu_{vi}. \tag{B.3}$$

That corresponds to:

$$\psi_i = \frac{\mu_{vi}}{\mu_\infty} + \frac{3k_{vi} + 4\mu_{vi}}{3k_\infty + 4\mu_\infty}.$$

By introducing viscoelastic Hill's tensors, calculated using equations (B.2) to (B.3), in equations (2.10) and (2.11) to calculate the viscoelastic concentration tensors that are then introduced in equations (2.3), (2.8) and (2.9), we can obtain the rheological effective viscoelastic properties of the material. It is worth mentioning that the tensorial calculations related to transversely isotropic viscoelastic Hill's tensors can be easily handled in the Walpole base (see Appendix D).

Appendix C. Relaxation and Creep moduli

The relaxation bulk modulus can be derived considering a constant macroscopic isotropic strain boundary condition $\mathbf{E} = E_0\mathbf{1}$, i.e. the displacement condition on the boundary of the REV is $\underline{u}(\underline{z}) = E_0\mathbf{1}.\underline{z}$, where $\mathbf{1}$ is the second order identity tensor. Consequently, the macroscopic stress tensor is isotropic: $\mathbf{\Sigma} = \Sigma\mathbf{1}$. The macroscopic stress strain relation can be expressed as:

$$\Sigma = 3k_\infty^{\text{hom}} E_0 + \sum_{i=1}^{n} \Sigma_i \tag{C.1}$$

$$\dot{\Sigma}_i + \frac{k_i^{\text{hom}}}{k_{vi}^{\text{hom}}} \Sigma_i = 0, \quad \forall i = 1 \div n \tag{C.2}$$

Solving equation (C.2) for Σ_i, with the initial condition $\Sigma_i(0) = 3k_{ei}^{\text{hom}} E_0$, and introducing the result in equation (C.1), we can obtain:

$$\Sigma = 3E_0 \left[k_\infty^{\text{hom}} + \sum_{i=1}^{n} k_{ei}^{\text{hom}} \exp\left(-t k_{ei}^{\text{hom}} / k_{vi}^{\text{hom}}\right) \right]$$

where t is time that has the same unit of the characteristic time defined by: $\tau_{ki} = k_{vi}^{\text{hom}} / k_{ei}^{\text{hom}}$.

Then the relaxation bulk modulus, which is defined by $k_r^{\text{hom}} = \Sigma/(3E_0)$, is:

$$k_r^{\text{hom}}(t) = k_\infty^{\text{hom}} + \sum_{i=1}^{n} k_{ei}^{\text{hom}} \exp\left(-\frac{t}{\tau_{ki}}\right).$$

Similarly, the relaxation shear modulus can be obtained by considering a constant macroscopic pure shear strain condition as follows:

$$\mu_r^{\text{hom}}(t) = \mu_\infty^{\text{hom}} + \sum_{i=1}^{n} \mu_{ei}^{\text{hom}} \exp\left(-\frac{t}{\tau_{\mu i}}\right).$$

Finally, the relaxation Poisson ratio is related to the relaxation bulk and shear moduli by the classical formula of continuum mechanics:

$$\nu_r^{\mathrm{hom}}(t) = \frac{3k_r^{\mathrm{hom}}(t) - 2\mu_r^{\mathrm{hom}}(t)}{6k_r^{\mathrm{hom}}(t) + 2\mu_r^{\mathrm{hom}}(t)}. \tag{C.3}$$

The creep bulk modulus can be derived considering a constant macroscopic isotropic stress condition $\mathbf{\Sigma} = \Sigma_0 \mathbf{1}$. The related isotropic macroscopic strain tensor is $\mathbf{E} = E\mathbf{1}$. With such a boundary condition, the stress-strain relationship is:

$$\Sigma_0 = 3k_\infty^{\mathrm{hom}} E + \left(\sum_{i=1}^{n} 3k_{vi}^{\mathrm{hom}} \right) \dot{E}. \tag{C.4}$$

Solving equation (C.4) for E, with the initial condition $E(0) = \Sigma_0/(3k_0^{\mathrm{hom}})$, we can obtain:

$$E = \frac{\Sigma_0}{3} \left[\frac{1}{k_\infty^{\mathrm{hom}}} + \frac{1}{k_0^{\mathrm{hom}}} \left(1 - \frac{k_0^{\mathrm{hom}}}{k_\infty^{\mathrm{hom}}} \right) \exp\left(-\frac{t k_\infty^{\mathrm{hom}}}{\sum_{i=1}^{n} k_{vi}^{\mathrm{hom}}} \right) \right]$$

where $k_0^{\mathrm{hom}} = k_\infty^{\mathrm{hom}} + \sum_{i=1}^{n} k_{ei}^{\mathrm{hom}}$ is the instantaneous homogeneous elastic bulk modulus.

The creep bulk modulus, which is defined by $k_c^{\mathrm{hom}} = \Sigma_0/(3E)$, is:

$$k_c^{\mathrm{hom}}(t) = \left[\frac{1}{k_\infty^{\mathrm{hom}}} + \frac{1}{k_0^{\mathrm{hom}}} \left(1 - \frac{k_0^{\mathrm{hom}}}{k_\infty^{\mathrm{hom}}} \right) \exp\left(-\frac{t k_\infty^{\mathrm{hom}}}{\sum_{i=1}^{n} k_{vi}^{\mathrm{hom}}} \right) \right]^{-1}.$$

Similarly, the creep shear modulus can be obtained by considering a constant macroscopic pure shear stress condition such as:

$$\mu_c^{\mathrm{hom}}(t) = \left[\frac{1}{\mu_\infty^{\mathrm{hom}}} + \frac{1}{\mu_0^{\mathrm{hom}}} \left(1 - \frac{\mu_0^{\mathrm{hom}}}{\mu_\infty^{\mathrm{hom}}} \right) \exp\left(-\frac{t \mu_\infty^{\mathrm{hom}}}{\sum_{i=1}^{n} \mu_{vi}^{\mathrm{hom}}} \right) \right]^{-1}.$$

Similarly to equation (C.3), the creep Poisson ratio is related to the creep bulk and shear moduli by:

$$\nu_c^{\mathrm{hom}}(t) = \frac{3k_c^{\mathrm{hom}}(t) - 2\mu_c^{\mathrm{hom}}(t)}{6k_c^{\mathrm{hom}}(t) + 2\mu_c^{\mathrm{hom}}(t)}.$$

Appendix D. Tensorial calculations in the Walpole base

In the Walpole base, transversely isotropic Hill's tensor defined by equation (B.1) can be decomposed to:

$$\mathbb{P} = \sum_{i=1}^{6} P_i \mathbb{E}_i = \{P_1, P_2, P_3, P_4, P_5, P_6\}$$

where $\mathbb{E}_1$ to $\mathbb{E}_6$ are six directional tensors of the Walpole base. The detailed expression of the latter can be found in the handbook [11]. The parameters P_1 to P_6 are related to the components P_{ijkl} by:

$$P_1 = P_{1111} + P_{1122}; \quad P_2 = P_{3333}; \quad P_1 = P_{1111} - P_{1122};$$
$$P_4 = 2P_{3131}; \quad P_5 = P_6 = P_{1133}.$$

Note that Hill's tensor depends on five independent parameters because of the condition $P_5 = P_6$. However, expanding it to six components is convenient for tensorial calculation. For example, the production of two certain tensors $\mathbb{a} = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ and $\mathbb{b} = \{b_1, b_2, b_3, b_4, b_5, b_6\}$ can be easily realized by:

$$\mathbb{a} : \mathbb{b} = \{a_1b_1 + 2a_6b_5, a_2b_2 + 2a_5b_6, a_3b_3, a_4b_4, a_5b_1 + a_2b_5, a_6b_2 + a_1b_6\}$$

The inversion of a tensor can be also easily computed by:

$$\mathbb{a}^{-1} = \left\{ \frac{a_2}{a}, \frac{a_1}{a}, \frac{1}{a_3}, \frac{1}{a_4}, -\frac{a_5}{a}, -\frac{a_6}{a} \right\}$$

with

$$a = a_1a_2 - 2a_5a_6.$$

The average overall the directions of $\mathbb{a}$ is an isotropic tensor that can be decomposed into spherical and deviatoric parts such as:

$$\langle \mathbb{a} \rangle = a_J \mathbb{J} + a_K \mathbb{K}$$

where the notation $\langle . \rangle$ denotes an average overall the directions; $\mathbb{J}$ and $\mathbb{K}$ the spherical and deviatoric parts of the fourth order identity tensor. The parameters a_J and a_K are related to the components in the Walpole base by:

$$a_J = \frac{1}{3}(2a_1 + a_2 + 2a_5 + 2a_6)$$
$$a_K = \frac{2}{15}\left(\frac{1}{2}a_1 + a_2 + 3a_3 + 3a_4 - a_5 - a_6\right)$$

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АНАЛИТИЧКА РЕШЕЊА ЗА ЕФЕКТИВНЕ ВИСКОЕЛАСТИЧНЕ КАРАКТЕРИСТИКЕ КОМПОЗИТНИХ МАТЕРИЈАЛА СА РАЗЛИЧИТИМ ОБЛИЦИМА ИНКЛУЗИЈЕ

Резиме. Циљ овог рада је да моделује ефекат различитих облика инклузије на хомогенизована вискоеластична својства композитних материјала израђених од вискоеластичне матрице и инклузивних честица. Вискоеластично понашање матричне фазе моделирано је уопштеном Маквелловом реологијом. Ефективна својства се прво добијају комбиновањем теорије хомогенизације еластичности и принципа кореспонденције. Затим се ефективна реолошка својства у простор0у времена експлицитно изводе без употребе инверзне комплексне Лаплас-Карсонове трансформације (LC). Решења у затвореном облику за ефективна својства реолошких вискоеластичних својстава смицања, модула релаксације и пузања, као и Пуасонов однос, добијају се за изотропни случај са случајно оријентисаном расподелом и различитим облицима инклузија: сферних, сплоштених и издужених инклузија. Развијени приступ се вреднује у односу на тачна решења добијена класичном инверзном LC методом. Примећено је да су хомогенизовани вискоеластични модули веома осетљиви на различите облике инклузија.

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