

## ASYMPTOTIC STABILITY OF A VISCOELASTIC PROBLEM WITH TIME-VARYING DELAY IN BOUNDARY FEEDBACK

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**ABSTRACT.** In this paper, we investigate a nonlinear viscoelastic equation. By assuming time-varying delay feedback acting on the boundary, under certain assumptions on the given data, the general decay estimates for the energy are established by introducing suitable Lyapunov functionals. This model improves earlier ones in the literature in which only the dissipative term in the feedback condition is considered.

### 1. Introduction

The nonlinear viscoelastic wave equation with time-varying delay in the boundary feedback is written in the form of partial integro-differential equations

$$(1.1) \quad u_{tt}(x, t) - \Delta u(x, t) + \int_0^t g(t-s)\Delta u(x, s)ds = 0 \quad \text{in } \Omega \times (0, +\infty),$$

$$(1.2) \quad u = 0 \quad \text{on } \Gamma_1 \times (0, +\infty),$$

$$(1.3) \quad \frac{\partial u}{\partial \nu}(x, t) - \int_0^t g(t-s)\frac{\partial}{\partial \nu}u(x, s)ds \\ + \mu_1 h_1(u_t(x, t)) + \mu_2 h_2(u_t(x, t - \tau(t))) = 0 \quad \text{on } \Gamma_0 \times (0, +\infty),$$

$$(1.4) \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega,$$

$$(1.5) \quad u_t(x, t - \tau(0)) = j_0(x, t - \tau(0)), \quad \text{in } \Gamma_0 \times (0, \tau(0)),$$

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where

$$\left\{ \begin{array}{l} u_{tt}(x, t) - \Delta u(x, t) + \int_0^t g(t-s)\Delta u(x, s)ds = 0 \text{ in } \Omega \times (0, +\infty) \text{ represents the} \\ \text{non-linear viscoelastic wave equation;} \\ u = 0 \text{ on } \Gamma_1 \times (0, +\infty) \text{ designates the Dirichlet boundary condition on the} \\ \text{portion of } \Gamma; \\ \frac{\partial u}{\partial \nu}(x, t) - \int_0^t g(t-s)\frac{\partial}{\partial \nu}u(x, s)ds + \mu_1 h_1(u_t(x, t)) + \mu_2 h_2(u_t(x, t-\tau(t))) = 0 \text{ on} \\ \Gamma_0 \times (0, +\infty) \text{ indicates the non-linear external feedback dynamical boundary;} \\ h_1(u_t(x, t)) \text{ represents the linear localized frictional damping;} \\ h_2(u_t(x, t-\tau(t))) \text{ denotes the time-varying delay term,} \end{array} \right.$$

where  $\Omega$  is a bounded domain of  $\mathbb{R}^n$  ( $n \geq 1$ ) with a sufficiently smooth boundary  $\Gamma = \Gamma_0 \cup \Gamma_1$  of class  $C^2$ . Here  $\Gamma_0$  and  $\Gamma_1$  are closed and disjoint, with  $\Gamma_0 \neq \emptyset$ ,  $\nu$  is the outward normal to  $\Gamma$ ,  $g(t)$  is a positive function that represents the kernel of the memory term,  $h_1, h_2$  are some functions which will be specified later (see Section 2),  $\mu_1, \mu_2$  are fixed real constants, with  $\mu_1 > 0, \mu_2 \neq 0, \tau(t) > 0$  represents the time delay, and  $u_0, u_1, j_0$  are given functions belonging to suitable spaces.

This model appears in viscoelasticity (see [26]). In the case of velocity-dependent material density (i.e.,  $\rho = 0$ ) as well as the presence of  $\mu_2 = 0$  and the absence of the memory effect (i.e.,  $h_1 = 0$ ), (1.1)–(1.5) reduces to the wave equation. There is a vast literature on the global existence and uniform stabilization of the wave equations. We refer the readers to [27].

Our equation can be regarded as a spring of an elastic constant  $E$  model for a viscoelastic material in the presence of a delay response in the boundary. Such a model can be represented by a Volterra equation connecting stress and strain:

$$\sigma(x, t) = E_{\text{inst,relax}}\varepsilon(x, t) - \int_0^t g(t-s)\varepsilon(x, s)ds,$$

where  $t$  is time,  $\sigma(t)$  is stress,  $\varepsilon(t)$  is strain,  $E_{\text{inst,relax}}$  are instantaneous elastic moduli for relaxation, and,  $g(t)$  is the relaxation function.

For small deformations, the strain is proportional to the deformation gradient  $\varepsilon = \kappa \nabla u$ . Hence, by substituting

$$u_{tt}(x, t) = \text{div } \sigma(x, t),$$

in the motion equation, we obtain our equation (1.1). We refer the reader to [56] for application of the model in biological tissues.

Such boundary conditions (1.3) are usually called dynamic boundary conditions. They are not only important from the theoretical point of view but also arise in several physical applications. For instance, in one space dimension, in the absence of delay ( $\mu_2 = 0$ ), when  $\Omega = (0, L)$ , problem (1.1)–(1.5), can be seen as a model of the dynamic evolution of motion for a body made of viscoelastic material, with a long memory, which occupies the interval  $[0, L]$  and such that one of its ends is clamped while the other is free and subject to the action of nonlinear dissipation (see [1, 31, 43] and the references therein). The analysis of stabilization when the damping is effective only on a subset of the domain  $\Omega$  is much more subtle than

that on the whole domain. Such problems have been extensively investigated in the context of wave equations and the literature on the subject is quite impressive (see for instance [5, 6], and the references therein). The same problems have also been addressed for plate equations. But the corresponding results are not many and we refer the reader to [7–9].

Problem (1.1)–(1.5) has been studied by many authors in the absence of the viscoelastic term (i.e. if  $g = 0$ ), when  $h_2 \equiv 1$ , and  $h_1 = 1$ . For instance, Nicaise et al. [38] considered the wave system with a linear boundary damping term and a boundary time-varying delay

$$(1.6) \quad \begin{cases} u_{tt} - \Delta u = 0 & (x, t) \in \Omega \times (0, \infty), \\ u = 0 & (x, t) \in \Gamma_0 \times (0, +\infty), \\ \frac{\partial u}{\partial \nu}(x, t) = -\mu_1 u_t(x, t) - \mu_2 u_t(x, t - \tau(t)) & (x, t) \in \Gamma_1 \times (0, +\infty), \end{cases}$$

where  $\tau(t)$  is a positive bounded function. The authors examined system (1.6) and proved under the assumption  $\mu_2 < \mu_1$  that the energy is exponentially stable. In the other situation, they constructed a sequence of delays for which the corresponding solution is unstable. The main approach used there is an observability inequality together with a Carleman estimate. In the presence of the viscoelastic term (i.e. if  $g \neq 0$ ), and time-varying delay, Zhang et al. [25] investigated the global existence and asymptotic behavior of a nonlinear viscoelastic equation with interior time-varying delay and nonlinear dissipative boundary feedback:

$$\begin{aligned} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + \mu_2 u_t(x, t - \tau(t)) &= 0 \quad x \in \Omega, \quad t > 0, \\ u(x, t) &= 0, \quad \text{on } \Gamma_0 \times (0, \infty), \\ \frac{\partial u}{\partial \nu} + \mu_1 h_1(u_t(x, t)) &= 0, \quad \text{on } \Gamma_1 \times [0, \infty), \\ u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \\ u_t(x, t - \tau(t)) &= f_0(x, t), \quad x \in \Omega, \quad -\tau(0) \leq t \leq 0, \end{aligned}$$

where  $\Omega$  is a bounded domain of  $\mathbb{R}^n$  ( $n \geq 1$ ) with a smooth boundary  $\partial\Omega$  of  $C^2$ , and  $\tau(t) > 0$  represents the time-varying delay effect and the initial data. This type of equation usually arises in the theory of viscoelasticity. It is well known that viscoelastic materials have memory effects, which is due to the mechanical response influenced by the history of the materials themselves. From the mathematical point of view, their memory effects are modeled by integrodifferential equations. Hence, equations related to the behavior of the solutions for the PDE system have attracted considerable attention in recent years. We can refer to recent work in [17, 18, 58]. The dynamic boundary conditions are not only important from the theoretical point of view, but also arise in numerous practical problems. Recently, some authors have studied the existence and decay of solutions for a wave equation with dynamic boundary conditions [19, 20]. Time-delay system is a class of systems in which the evolution tendency depends not only on the current state but also on a certain or occurrence. It often appears in mechanical applications, electronic circuit, biological, chemical, and economical phenomena because

these phenomena depend not only on the present state but also on the history of the system in a more complicated way, and are produced by various factors, such as the variable measurement of the system and slow chemical reaction process. Time-delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc. The existence of pure time lag, regardless of whether it is present in the control or/and the state, these hereditary effects are sometimes unavoidable, and they may turn a well-behaved system into a wild one, and may also cause undesirable system transient response, or even instability; see [21, 22]. The stability issue of the systems with delay is, therefore, of theoretical and practical importance. In recent years, differential equations with time delay effects have become an active area of research; see for example [23, 24] and the references therein. To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary. During the last three decades, the problem of stability analysis of time-delay systems has received considerable attention and many papers dealing with this problem have appeared. In the literature, various stability analysis techniques have been utilized to derive stability criteria for asymptotic stability of time-delay systems by many researchers. Time delay is the property of a physical system by which the response to an applied force is delayed in its effect (see [2]). Whenever material, information, or energy is physically transmitted from one place to another, there is a delay associated with the transmission. In recent years, PDEs with time delay effects have become a popular field of research and arise in many practical real-world problems (see, for example, Abdallah et al. [3], Suh and Bien [4], as well as [13, 15, 34]). There is an extensive literature on stabilization of the wave equation with a delay in the internal/boundary feedback (see, for instance, [13, 34] and the references therein). However, to the best of the author's knowledge, only a few papers [14, 16, 36] address the stabilization of the plate equation with delay effects. To stabilize a hyperbolic system involving delay terms, additional control terms are necessary (see Nicaise and Pignotti [34], Xu et al. [55]). Nicaise and Pignotti [14] obtained stability results for the Euler–Bernoulli plate equation with time delay regarding the plate equation as a particular case of more general systems. It is worth mentioning the work of Cavalcanti et al. [32] concerning the following problem

$$(1.7) \quad \begin{cases} y_{tt} - \Delta y + \int_0^t h(t-\tau)\Delta y(\tau)d\tau = 0 & \text{in } \Omega \times (0, \infty), \\ y = 0 & \text{on } \Gamma_1 \times (0, \infty), \\ \frac{\partial y}{\partial \nu} - \int_0^t h(t-\tau)\frac{\partial y}{\partial \nu}(\tau)d\tau + g(y_t) = 0 & \text{on } \Gamma_0 \times (0, \infty), \\ y(x, 0) = y^0(x), \quad y_t(x, 0) = y^1(x) & \text{in } \Omega, \end{cases}$$

in which the uniform stability of (1.7) has been proved, with a more general assumption on the relaxation function  $h$  and no growth assumption on the boundary frictional damping function  $g$ . They established explicit decay rate results, depending on  $h$  and  $g$ , for some special cases and implicitly for the general case. It should be mentioned that problem (1.7) was later improved by many authors, who proved the uniform stability and explicit decay rate results for some special cases, see [32]. Considering the time-varying delay term  $\mu_2 h_2(u_t(x, t-\tau(t)))$  in boundary feedback,

the problem is different from those in the existing literature. Time delays arise in many physical, chemical, biological, thermal, and economical phenomena because these phenomena depend not only on the present state but also on the previous history of the system in a more complicated way (see, for example, [37, 42, 44]). In recent years, systems with time delay effects have become an active area of research, see for example [45, 52] and the references therein. In [53], the authors showed that a small delay in boundary control is a source of instability. To stabilize a hyperbolic system involving input delay, additional conditions or control terms have been used, see [21, 33, 34, 55]. For instance, considering a wave equation with a delay of the form

$$(1.8) \quad u_{tt}(x, t) - \Delta u(x, t) + \mu_0 \sigma(t) h_1(u_t(x, t)) + \mu_1 \sigma(t) h_2(u_t(x, t - \tau(t))) = 0,$$

Nicaise and Pignotti [34] proved that the energy of the problem is exponentially stable when  $\sigma(t) = 1$ ,  $\mu_0, \mu_1 > 0$ ,  $\tau(t) = \tau(\text{constant})$ , and  $h_1(s) = h_2(s) = s$ . On the other hand, the case of time-varying delay in the wave equation in one-dimensional space has been studied recently by Nicaise et al. in [35], where the authors proved an exponential stability result under the condition  $0 < \mu_2 < \sqrt{1-d}\mu_1$ , where the function  $\tau(t)$  satisfies  $\tau'(t) \leq d, \forall t > 0$ , for the constant  $d < 1$ . In [38], Nicaise, Pignotti, and Valein extended the above result to higher-dimensional space and established an exponential decay. In [54], the authors studied a nonlinear viscoelastic wave equation with strong damping, time-varying delay, and a dynamical boundary condition. They proved a general decay result of the energy, from which the usual exponential and polynomial decay rates are only special cases. For the related problems, we also refer to [46–49]. In this paper, inspired by these results, we investigate a viscoelastic problem with time-varying delay appearing in the control term in (1.1)–(1.5) by dropping the restriction  $\mu_1 > 0$  and establishing suitable Lyapunov functionals, and we prove a general decay estimate for the energy, which depends on the behavior of  $h_1$ . The paper is organized as follows. In Section 2, we give some assumptions that will be needed for our work and state the main results. We establish the general decay result of the energy in Section 3.

## 2. Preliminaries and main results

In this section, we present some material that we shall use in order to present our results. We use the notation

$$(u, v) = \int_{\Omega} u(x, t)v(x, t)dx, \text{ and } (u, v)_{\Gamma_0} = \int_{\Gamma_0} u(x)v(x)d\Gamma,$$

and we mean by  $\|\cdot\|_2$  the  $L^2(\Omega)$  norm, and by  $\|\cdot\|_{2, \Gamma_0}$  the  $L^2(\Gamma_0)$  norm. Also we denote by

$$H_{\Gamma_1}^1(\Omega) = \{u \in H^1(\Omega), u = 0 \text{ on } \Gamma_1\},$$

the closed subspace of  $H^1(\Omega)$  equipped with the norm equivalent to the usual norm in  $H^1(\Omega)$ . The Poincaré inequality holds on  $H_{\Gamma_1}^1(\Omega)$ , i.e. there exists a constant  $C_*$  such that:

$$\forall u \in H_{\Gamma_1}^1(\Omega), \|u(t)\|_2 \leq C_* \|\nabla u(t)\|_2,$$

and there exists a constant  $\bar{C}_* > 0$  such that

$$\|u\|_{2,\Gamma_0} \leq \bar{C}_* \|\nabla u\|_2, \text{ for all } u \in H_{\Gamma_1}^1(\Omega).$$

For studying the problem (1.1)–(1.5), we will need the following assumptions.

**(H1) Hypothesis on  $g$ :**  $g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a bounded  $C^1$  function satisfying

$$(2.1) \quad g(0) = g_0 > 0, \quad l_0 = \int_0^\infty g(s)ds < \infty, \quad 1 - \int_0^\infty g(s)ds = l > 0, \quad \forall t \in \mathbb{R}^+,$$

and there exists a nonincreasing differentiable function  $\gamma: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  with

$$(2.2) \quad \gamma(t) > 0, \quad g'(t) \leq -\gamma(t)g(t), \quad \text{for all } t \geq 0.$$

**(H2) Hypothesis on  $h_1$ :**  $h_1: \mathbb{R} \rightarrow \mathbb{R}$  is a non-decreasing function of the class  $C(\mathbb{R})$  such that there exist positive constants  $r < 1$ ,  $\alpha_1$ ,  $\alpha_2$  satisfying

$$(2.3) \quad \alpha_1|s| \leq |h_1(s)| \leq \alpha_2|s| \quad \text{for } |s| \geq r.$$

Moreover, assume that there exists a convex increasing function  $H_1: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  of class  $C^1(\mathbb{R}_+) \cap C^2((0, \infty))$  satisfying

$$(2.4) \quad H_1(0) = 0;$$

$$(2.5) \quad H_1 \text{ is linear on } (0, r], \text{ or } H_1'(0) = 0 \text{ and } H_1''(t) > 0 \text{ on } (0, r].$$

$$(2.6) \quad h_1^2(s) \leq H_1^{-1}(sh_1(s)) \text{ for } |s| \leq r.$$

**(H3) Hypothesis on  $h_2$ :**  $h_2: \mathbb{R} \rightarrow \mathbb{R}$  is an odd non-decreasing function of the class  $C^1(\mathbb{R})$  such that there exist positive constants  $\alpha_i$ ,  $i = 3, 4, 5$  satisfying

$$(2.7) \quad |h_2'(s)| \leq \alpha_3 \quad \text{for } s \in \mathbb{R},$$

$$(2.8) \quad \alpha_4 sh_2(s) \leq H_2(s) \leq \alpha_5 sh_1(s) \quad \text{for } s \in \mathbb{R},$$

where  $H_2(s) = \int_0^s h_2(t)dt$ .

**(H4) Hypothesis on  $\tau(\cdot)$ :** For the time-varying delay  $\tau$ , we assume as in [38] that  $\tau \in W^{2,\infty}([0, T])$ ,  $\forall T > 0$  and there exist positive constants  $\tau_0$ ,  $\tau_1$  and  $d$  satisfying

$$(2.9) \quad 0 < \tau_0 \leq \tau(t) \leq \tau_1, \quad \tau'(t) \leq d < 1, \quad \forall t > 0.$$

**(H5) Hypothesis on  $\mu_1$ ,  $\mu_2$ :** The weight of dissipation and the delay satisfy

$$(2.10) \quad 0 < |\mu_2| < \frac{\alpha_4(1-d)}{\alpha_5(1-\alpha_4d)}\mu_1.$$

REMARK 2.1. The condition (H2) was originally introduced by Lasiecka and Tataru [57].

In order to deal with the delay feedback term, motivated by [36], we introduce the following new dependent variable,

$$z(x, \rho, t) := z(\rho, t) = u_t(x, t - \rho\tau(t)), \quad (x, \rho, t) \in \Gamma_0 \times (0, 1) \times (0, \infty).$$

By computation we have

$$\tau(t)z_t + (1 - \tau'(t)\rho)z_\rho = 0, \quad \text{in } \Gamma_0 \times (0, 1) \times (0, \infty).$$

Therefore, problem (1.1)–(1.5) can be transformed into

$$(2.11) \quad \frac{\partial^2 u}{\partial t^2} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = 0 \quad \text{in } \Omega \times (0, +\infty),$$

$$(2.12) \quad u = 0, \quad \text{on } \Gamma_1 \times (0, +\infty),$$

$$(2.13) \quad \frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\frac{\partial}{\partial \nu}u(s)ds + \mu_1 h_1(u_t(x, t)) \\ + \mu_2 h_2(z(x, 1, t)) = 0 \quad , \text{ on } \Gamma_0 \times (0, +\infty),$$

$$(2.14) \quad \tau(t)z_t + z_\rho(1 - \tau'(t)\rho) = 0, \quad \text{in } \Gamma_0 \times (0, 1) \times (0, \infty),$$

$$(2.15) \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega,$$

$$(2.16) \quad z_t(x, \rho, 0) = j_0(x, -\rho\tau(0)), \quad \text{in } \Gamma_0 \times (0, 1).$$

First, we begin with the well-posedness result for problem (1.1)–(1.5), which can be obtained by using the Faedo-Galerkin method and combining the ideas from [31, 41] with the ones from [39] (see also [40, 42]).

**THEOREM 2.1.** *Assume that (2.1) and (H1) – (H3) hold. Then for the given  $(u_0, u_1, j_0) \in H_{\Gamma_1}^1(\Omega) \cap H^1(\Omega) \times H_{\Gamma_1}^1(\Omega) \times L^2(\Gamma_0, H^1(0, 1))$ , there exist  $T > 0$  and a unique weak solution  $(u(t), z(t))$  to problem (1.1)–(1.5) such that*

$$u(t) \in C([0, T] : H_{\Gamma_1}^1(\Omega)), \\ u_t(t) \in C([0, T] : L^2(\Omega)) \cap C([0, T] : H_{\Gamma_1}^1(\Omega)), \\ u_{tt}(t) \in C([0, T] : L^2(\Omega)), \\ z \in C([0, T] : L^2(\Gamma_0, H^1(0, 1))).$$

It is of interest, for proving our main result, to know the following lemma.

**LEMMA 2.1** (Jensen's Inequality). *If  $H$  is a convex function on  $[a, b]$ ,  $h: D \rightarrow [a, b]$  and  $q$  are integrable functions on  $D$ ,  $q(x) \geq 0$  and  $\int_D q(x)dx = Q > 0$ , then*

$$H\left(\frac{1}{Q} \int_D h(x)q(x)dx\right) \leq \frac{1}{Q} \int_D H(h(x))q(x)dx.$$

### 3. General decay of the energy

In this section, we shall investigate the asymptotic behavior of the energy function  $E$ . For this, we construct a Lyapunov functional  $L$  equivalent to  $E$ , with which we can show the desired result given by Theorem 3.1. We define the modified energy functional  $E$  associated with problem (2.11)–(2.16) by

$$E(t) = \frac{1}{2} \|u_t(t)\|_2^2 + \frac{1}{2} \left( 1 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 \\ + \frac{\xi}{2} \int_{\Gamma_0} \int_{t-\tau(t)}^t e^{\lambda(s-t)} H_2(u_t(x, s)) ds d\Gamma + \frac{1}{2} (g \circ \nabla u)(t),$$

where

$$(g \circ \nabla u)(t) = \int_0^t g(t-s) \int_{\Omega} |\nabla u(s) - \nabla u(t)|^2 dx ds,$$

and  $\lambda, \zeta$  are positive constants satisfying

$$(3.1) \quad \frac{2|\mu_2|(1-\alpha_4)}{\alpha_4(1-d)} < \xi < \frac{2(\mu_1 - |\mu_2|\alpha_5)}{\alpha_5},$$

$$(3.2) \quad \lambda < \frac{1}{\tau_1} \left| \ln \frac{2\mu_2(1-\alpha_4)}{\xi\alpha_4(1-d)} \right|.$$

Note that (3.1) makes sense thanks to (2.10).

The purpose of this paper is to prove the following result.

**THEOREM 3.1.** *Under the conditions of Theorem 2.1, there exist positive constants  $\omega, \kappa, t_0$ , and  $\epsilon_0$  such that the energy for problem (2.11)–(2.16) satisfies*

$$(3.3) \quad E(t) \leq \kappa H^{-1} \left\{ \omega \left( 1 + \int_{t_0}^t \gamma(s) ds \right) \right\} \quad \text{for } t \geq t_0,$$

with

$$(3.4) \quad H(t) = \int_t^1 \frac{1}{H_0(s)} ds,$$

and

$$(3.5) \quad H_0(t) = \begin{cases} t & \text{if } H_1 \text{ is linear on } [0, r], \\ tH_1'(\epsilon_0 t) & \text{if } H_1'(0) = 0 \text{ and } H_1''(t) > 0 \text{ on } (0, r], \end{cases}$$

The following three lemmas are essential to prove the main result given in Theorem 2.1.

**LEMMA 3.1.** *Let  $(u, z)$  be the solution to (2.11)–(2.16). Then, for some two positive constants  $\beta_1$  and  $\beta_2$ , we have*

$$(3.6) \quad E'(t) \leq -\beta_1 \int_{\Gamma_0} H_1(u_t) u_t d\Gamma - \beta_2 \int_{\Gamma_0} h_2(z(1, t)) z(1, t) d\Gamma \\ - \frac{\lambda \xi}{2} \int_{\Gamma_0} \int_{t-\tau(t)}^t e^{\lambda(s-t)} H_2(u_t(x, s)) ds d\Gamma + \frac{1}{2} (g' \circ \nabla(u))(t)$$



$$-\frac{1}{2}g(t)\|\nabla u(t)\|_2^2.$$

PROOF. Multiplying Eq. (2.11) by  $u_t$ , integrating over  $\Omega$ , multiplying equation Eq. (2.14) by  $\zeta z e^{-\lambda\tau(t)\rho}$ , integrating the result over  $(0, 1) \times \Gamma_0$  with respect to  $\rho$  and  $x$ , using integration by parts and adding them up, we obtain

$$(3.7) \quad \begin{aligned} E'(t) = & -\mu_1 \int_{\Gamma_0} h_1(u_t)u_t d\Gamma - \mu_2 \int_{\Gamma_0} h_2(z(1, t))u_t dx \\ & - \frac{\lambda\xi}{2} \int_{\Gamma_0} \int_{t-\tau(t)}^t e^{\lambda(s-t)} H_2(u_t(x, s)) ds d\Gamma + \frac{\xi}{2} \int_{\Gamma_0} H_2(u_t(x, t)) d\Gamma \\ & - \frac{\xi}{2} \int_{\Gamma_0} e^{-\lambda\tau(t)} (1 - \tau'(t)) H_2(u_t(x, t - \tau(t))) d\Gamma \\ & + \frac{1}{2}(g' \circ \nabla(u))(t) - \frac{1}{2}\|\nabla u(t)\|_2^2 g(t). \end{aligned}$$

From (2.8) and (2.9), using  $z(1, t) = u_t(t - \tau(t))$ , we can see that

$$(3.8) \quad \begin{aligned} & -\frac{\xi}{2} \int_{\Gamma_0} e^{-\lambda\tau(t)} (1 - \tau'(t)) H_2(u_t(x, t - \tau(t))) d\Gamma + \frac{\xi}{2} \int_{\Gamma_0} H_2(u_t(x, t)) d\Gamma \\ & \leq -\frac{\xi\alpha_4}{2} e^{-\lambda\tau_1} (1 - \tau'(t)) \int_{\Gamma_0} h_2(z(x, 1, t))z(x, 1, t) d\Gamma \\ & \quad + \frac{\xi\alpha_5}{2} \int_{\Gamma_0} h_1(u_t(x, t))u_t(x, t) d\Gamma \\ & \leq -\frac{\xi\alpha_4}{2} e^{-\lambda\tau_1} (1 - d) \int_{\Gamma_0} h_2(z(1, t))z(1, t) d\Gamma + \frac{\xi\alpha_5}{2} \int_{\Gamma_0} h_1(u_t)u_t d\Gamma. \end{aligned}$$

To estimate the second term on the right hand side of (3.7), let  $G^*$  be the conjugate function of the convex function  $G$  defined by  $G^*(s) = \sup_{t \geq 0} (st - G(t))$ ; Then  $G^*$  is Legendre transform of  $G$  which is given by (see Arnold [50, pp. 61–62])

$$(3.9) \quad G^*(s) = s(G')^{-1}(s) - G((G')^{-1}(s)) \quad \forall s \geq 0,$$

and satisfies the inequality

$$(3.10) \quad st \leq G^*(s) + G(t) \quad \text{for } s, t \geq 0.$$

Taking the definition of  $H_2$  into account and (3.9), we get

$$(3.11) \quad H_2^*(s) = sh_2^{-1}(s) - H_2(h_2^{-1}(s)) \quad \text{for } s \geq 0.$$

Using (3.11), we can easily check that

$$(3.12) \quad \begin{aligned} -\mu_2 \int_{\Gamma_0} h_2(z(1, t))u_t d\Gamma \leq & |\mu_2| \int_{\Gamma_0} (h_2(z(x, 1, t))z(x, 1, t) \\ & - H_2(z(x, 1, t)) + H_2(u_t(x, t))) d\Gamma, \end{aligned}$$

which, together with (2.8), leads to

$$(3.13) \quad -\mu_2 \int_{\Gamma_0} h_2(z(1, t))u_t d\Gamma \leq |\mu_2|(1 - \alpha_4) \int_{\Gamma_0} h_2(z(1, t))z(1, t) d\Gamma$$

$$+ |\mu_2| \alpha_5 \int_{\Gamma_0} h_1(u_t) u_t d\Gamma.$$

Substituting (3.8) and (3.13) into (3.7) yields

$$\begin{aligned} E'(t) = & - \left( \mu_1 - \frac{\xi \alpha_5}{2} - |\mu_2| \alpha_5 \right) \int_{\Gamma_0} h_1(u_t) u_t d\Gamma \\ & - \left( \frac{\xi \alpha_4}{2} e^{-\lambda \tau_1} (1-d) - |\mu_2| (1-\alpha_4) \right) \int_{\Gamma_0} h_2(z(1,t)) z(1,t) d\Gamma \\ & + \frac{1}{2} (g' \circ \nabla(u))(t) - \frac{1}{2} \|\nabla u(t)\|_2^2 g(t) \\ & - \frac{\lambda \xi}{2} \int_{\Gamma_0} \int_{t-\tau(t)}^t e^{\lambda(s-t)} H_2(u_t(x,s)) ds d\Gamma. \end{aligned}$$

Putting  $\beta_1 = \mu_1 - \frac{\xi \alpha_5}{2} - |\mu_2| \alpha_5 > 0$  and  $\beta_2 = \frac{\xi \alpha_4}{2} e^{-\lambda \tau_1} (1-d) - |\mu_2| (1-\alpha_4) > 0$ , we complete the proof of Lemma 3.1.  $\square$

Next, let us define the perturbed energy by

$$(3.14) \quad L(t) = ME(t) + \epsilon \Psi(t) + \Phi(t) + \mathcal{E}(t),$$

where  $M$  is a positive constant to be chosen later, and

$$\begin{aligned} \Psi(t) &= \int_{\Omega} u_t(t) u(t) dx, \\ \Phi(t) &= - \int_0^t \int_{\Omega} g(t-s) u(t) - u(s) u_t(t) dx ds, \\ \mathcal{E}(t) &= \int_{\Gamma_0} \int_{t-\tau(t)}^t e^{(s-t)} H_2(u_t(x,s)) ds d\Gamma. \end{aligned}$$

The functional  $L$  is equivalent to the energy function  $E$  by the following lemma.

LEMMA 3.2. *For  $M > 0$  large enough, there exist two positive constants  $C_1$  and  $C_2$  such that*

$$C_1 E(t) \leq L(t) \leq C_2 E(t), \quad t \geq 0.$$

PROOF. Integrating by parts, using Young's inequality and Poincaré's Theorem, we have

$$\begin{aligned} |\Psi(t)| &\leq \frac{1}{2} \|u_t\|^2 + \frac{1}{2} C_* \|\nabla u\|^2 \\ &\leq \frac{1}{2} \|u_t\|^2 + \frac{C_*}{2l} \left( 1 - \int_0^t g(s) ds \right) \|\nabla u\|^2 \leq cE(t), \\ |\Phi(t)| &\leq \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \left( \int_0^t g(t-s) \|u(t) - u(s)\| ds \right)^2 \\ &= \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \left( \int_0^t g(t-s) \|u(t) - u(s)\| ds \right)^2 \\ &\leq \frac{1}{2} \|u_t\|^2 + \frac{1-l}{2} C_* (g \circ \nabla(u)) \leq cE(t), \end{aligned}$$

and

$$|\mathcal{E}(t)| \leq cE(t).$$

Choosing  $M > 0$  large enough, we obtain

$$|L(t) - ME(t)| \leq cE(t),$$

and the proof of Lemma 3.2 is concluded.  $\square$

LEMMA 3.3. *There exist positive constants  $C_3, C_4, C_5$  and  $t_0 > 0$  such that*

$$(3.15) \quad L'(t) \leq -C_3E(t) + C_4\|h_1(u_t)\|_{\Gamma_0}^2 + C_5(g \circ \nabla u)(t), \quad t \geq t_0.$$

PROOF. Using problem (2.11)–(2.16), we have

$$(3.16) \quad \begin{aligned} \psi'(t) &= \int_{\Omega} u_{tt}u \, dx + \int_{\Omega} u_t^2 \, dx \\ &= \int_{\Omega} \left( \Delta u - \int_0^t g(t-s)\Delta u(s) \, ds - \int_0^t g(t-s)\Delta u \right) u \, dx \\ &\quad + \int_{\Gamma_0} \{-\mu_1 h_1(u_t) - \mu_2 h_2(u_t(t-\tau(t)))\} u \, d\Gamma + \int_{\Omega} u_t^2 \, dx \\ &= -\|\nabla u\|_2^2 + \int_{\Omega} \int_0^t g(t-s)\nabla u(s) \, ds \nabla u \, dx - \mu_1 \int_{\Gamma_0} h_1(u_t)u \, d\Gamma \\ &\quad - \mu_2 \int_{\Gamma_0} h_2(u_t(t-\tau(t)))u \, d\Gamma + \int_{\Omega} u_t^2 \, dx \\ &\quad + \int_{\Omega} \int_0^t g(t-s)\nabla u(s) \, ds \nabla u(t) \, dx. \end{aligned}$$

By using Hölder's inequality and Young's inequality, the second term on the right-hand side of (3.16) is estimated as follows

$$(3.17) \quad \begin{aligned} &\int_{\Omega} \int_0^t g(t-s)\nabla u(s) \, ds \nabla u \, dx \\ &\leq \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} \left| \int_0^t g(t-s)\nabla u(s) \, ds \right|^2 \, dx \right)^{\frac{1}{2}} \\ &\leq \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} \int_0^t g(s) \, ds \int_0^t g(t-s)|\nabla u(s)|^2 \, ds \, dx \right)^{\frac{1}{2}} \\ &\leq \left( \int_{\Omega} |\nabla u|^2 \, dx \int_0^t g(s) \, ds \right)^{\frac{1}{2}} \left( \int_{\Omega} \int_0^t g(t-s)|\nabla u(s)|^2 \, ds \, dx \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx \int_0^t g(s) \, ds + \frac{1}{2} \int_{\Omega} \int_0^t g(t-s)|\nabla u(s)|^2 \, ds \, dx \\ &\leq \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx \int_0^t g(s) \, ds + \frac{1}{2} \int_{\Omega} \int_0^t g(t-s)|\nabla u(s) - \nabla u(t) + \nabla u(t)|^2 \, ds \, dx. \end{aligned}$$

We use Young's inequality, and (H1) to obtain for every  $\eta > 0$

$$\begin{aligned}
(3.18) \quad & \frac{1}{2} \int_{\Omega} \int_0^t g(t-s) [\nabla u(s) - \nabla u(t) + \nabla u(t)]^2 ds dx \\
& \leq \frac{1}{2} \int_{\Omega} \int_0^t g(t-s) ((\nabla u(s) - \nabla u(t))^2 + 2|\nabla u(s) - \nabla u(t)||\nabla u| + |\nabla u|^2) ds dx \\
& = \frac{1}{2} \int_{\Omega} \int_0^t g(t-s) |\nabla u(s) - \nabla u(t)|^2 ds dx + \frac{1}{2} \int_a \int_0^t g(t-s) |\nabla u|^2 ds dx \\
& \quad + \int_{\Omega} \int_0^t g(t-s) |\nabla u(s) - \nabla u(t)||\nabla u| ds dx \\
& \leq \frac{1}{2} (g \circ \nabla u)(t) + \frac{1}{2} \int_0^t g(s) ds \int_{\Omega} |\nabla u|^2 dx \\
& \quad + \frac{\eta}{2} \int_0^t g(s) ds \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2\eta} (g \circ \nabla u)(t) \\
& \leq \frac{1}{2} (1+\eta) \int_0^t g(s) ds \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \left(1 + \frac{1}{\eta}\right) (g \circ \nabla u)(t) \\
& \leq (1+\eta) \frac{(1-l)}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \left(1 + \frac{1}{\eta}\right) (g \circ \nabla u)(t).
\end{aligned}$$

Combining (3.17) and (3.18), we get

$$\begin{aligned}
& \int_{\Omega} \int_0^t g(t-s) \nabla u(s) ds \nabla u dx \\
& \leq \frac{(1-l)}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{(1-l)}{2} (1+\eta) \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \left(1 + \frac{1}{\eta}\right) (g \circ \nabla u)(t) \\
& = (2+\eta) \frac{(1-l)}{2} \|\nabla u\|^2 + \frac{1}{2} \left(1 + \frac{1}{\eta}\right) (g \circ \nabla u)(t).
\end{aligned}$$

By taking  $\eta = \frac{l}{1-l}$ , we infer that

$$(3.19) \quad \int_{\Omega} \int_0^t g(t-s) \nabla u(s) ds \nabla u dx \leq \left(1 - \frac{l}{2}\right) \|\nabla u\|_2^2 + \frac{1}{2l} (g \circ \nabla u)(t).$$

For the third and fourth terms, Young's inequality gives

$$\begin{aligned}
(3.20) \quad & \mu_1 \int_{\Gamma_0} h_1(u_t) u d\Gamma + \mu_2 \int_{\Gamma_0} h_2(u_t(t - \tau(t))) u d\Gamma \\
& \leq \mu_1 \int_{\Gamma_0} |h_1(u_t)| |u| dx + \mu_2 \int_{\Gamma_0} |h_2(u_t(t - \tau(t)))| |u| dx \\
& \leq \mu_1 \|u\|_{\Gamma_0} \|h_1(u_t)\|_{\Gamma_0} + \mu_2 \|u\|_{\Gamma_0} \|h_2(u_t(t - \tau(t)))\|_{\Gamma_0} \\
& \leq \eta \|u\|_{\Gamma_0}^2 + \frac{\mu_1^2}{4\eta} \|h_1(u_t)\|_{\Gamma_0}^2 + \eta \|u\|_{\Gamma_0}^2 + \frac{\mu_2^2}{4\eta} \|h_2(u_t(t - \tau(t)))\|_{\Gamma_0}^2 \\
& \leq \eta C_*^2 \|\nabla u\|^2 + \frac{\mu_1^2}{4\eta} \|h_1(u_t)\|_{\Gamma_0}^2 + \eta C_*^2 \|\nabla u\|^2 + \frac{\mu_2^2}{4\eta} \|h_2(u_t(t - \tau(t)))\|_{\Gamma_0}^2
\end{aligned}$$

$$= 2\eta C_*^2 \|\nabla u\|^2 + \frac{\mu_1^2}{4\eta} \|h_1(u_t)\|_{\Gamma_0}^2 + \frac{\mu_2^2}{4\eta} \|h_2(z(1, t))\|_{\Gamma_0}^2.$$

Substituting these estimates into (3.16), we get

$$(3.21) \quad \begin{aligned} \psi'(t) &\leq -\|\nabla u\|_2^2 + \frac{2-l}{2} \|\nabla u\|^2 + \frac{1}{2l} (g \circ \nabla u)(t) + \|u_t\|^2 \\ &\quad + 2\eta C_*^2 \|\nabla u\|^2 + \frac{\mu_1^2}{4\eta} \|h_1(u_t)\|_{\Gamma_0}^2 + \frac{\mu_2^2}{4\eta} \|h_2(z(1, t))\|_{\Gamma_0}^2 \\ &\leq \|u_t\|^2 + \frac{\mu_1^2}{4\eta} \|h_1(u_t)\|_{\Gamma_0}^2 + \frac{\mu_2^2}{4\eta} \|h_2(z(1, t))\|_{\Gamma_0}^2 \\ &\quad + \frac{1}{2l} (g \circ \nabla u)(t) - \left(\frac{l}{2} - 2\eta C_*^2\right) \|\nabla u\|^2. \end{aligned}$$

Besides,

$$(3.22) \quad \begin{aligned} \phi'(t) &= -\left(\int_0^t g(s) ds\right) \|u_t(t)\|^2 - \int_0^t g'(t-s)(u(t) - u(s), u_t(t)) ds \\ &\quad + \int_0^t g(t-s)(\nabla u(t) - \nabla u(s), \nabla u(t)) ds \\ &\quad - \int_0^t g(t-s) \left(\nabla u(t) - \nabla u(s), \int_0^t g(t-s) \nabla u(s) ds\right) ds \\ &\quad + \int_0^t g(t-s) \int_{\Gamma_0} (u(t) - u(s)) (\mu_1 h_1(u_t(t)) + \mu_2 h_2(z(1, t))) d\Gamma ds \\ &:= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

We are now going to estimate the  $I_j (j = 1, 2, 3, 4, 5)$  terms in (3.22). Applying Young's inequality and employing a usual computation we have for every  $\eta > 0$

$$\begin{aligned} |I_2| &\leq \eta \|u_t(t)\|^2 - \frac{g(0)}{4\eta} C_*(g' \circ \nabla u)(t) \\ |I_3| &\leq \eta \|\nabla u(t)\|^2 + \frac{(1-l)}{4\eta} (g \circ \nabla u)(t) \\ |I_4| &= \int_{\Omega} \left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s)) ds\right) \left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s) - \nabla u(t)) ds\right) dx \\ &\leq \int_{\Omega} \left(\int_0^t g(t-s) |\nabla u(t) - \nabla u(s)| ds\right) \left(\int_0^t g(t-s) (|\nabla u(t) - \nabla u(s)| + |\nabla u(t)|) ds\right) dx \\ &\leq \eta \int_{\Omega} \left(\int_0^t g(t-s) (|\nabla u(t) - \nabla u(s)|^2 + |\nabla u(t)|) ds\right)^2 dx \\ &\quad + \frac{1}{4\eta} \int_{\Omega} \left(\int_0^t g(t-s) |\nabla u(t) - \nabla u(s)| ds\right)^2 dx \\ &\leq \left(2\eta + \frac{1}{4\eta}\right) \int_{\Omega} \left(\int_0^t g(t-s) |\nabla u(t) - \nabla u(s)| ds\right)^2 dx \\ &\quad + 2\eta \int_{\Omega} \left(\int_0^t g(t-s) |\nabla u(t)| ds\right)^2 dx \end{aligned}$$

$$\leq \left(2\eta + \frac{1}{4\eta}\right)(1-l)(g \circ \nabla u)(t) + 2\eta(1-l)^2 \|\nabla u(t)\|^2,$$

and

$$\begin{aligned} |I_5| &\leq \eta\mu_1 \|h_1(u_t(t))\|_{\Gamma_0}^2 + \eta|\mu_2| \|h_2(z(1,t))\|_{\Gamma_0}^2 \\ &\quad + \left\{ \frac{\mu_1(1-l)\bar{C}_*}{4\eta} + \frac{|\mu_2|(1-l)\bar{C}_*}{4\eta} \right\} (g \circ \nabla u)(t). \end{aligned}$$

Plugging these estimates into (3.22), we get

$$\begin{aligned} (3.23) \quad \phi'(t) &\leq - \left( \int_0^t g(s) ds - \eta \right) \|u_t\|^2 - \frac{g(0)}{4\eta} C_* (g' \circ \nabla u)(t) \\ &\quad + \eta \{1 + 2(1-l)^2\} \|\nabla u(t)\|^2 \\ &\quad + \left\{ \frac{1(1-l)}{4\eta} + \left(2\eta + \frac{1}{4\eta}\right)(1-l) + \frac{(\mu_1 + |\mu_2|)(1-l)\bar{C}_*}{4\eta} \right\} (g \circ \nabla u)(t) \\ &\quad + \eta\mu_1 \|h_1(u_t(t))\|_{\Gamma_0}^2 + \eta|\mu_2| \|h_2(z(1,t))\|_{\Gamma_0}^2. \end{aligned}$$

Moreover, analogous to (3.8), we can see that

$$(3.24) \quad \mathcal{E}'(t) \leq -\mathcal{E}(t) - \alpha_4 e^{-\tau_1} (1-d) \int_{\Gamma_0} h_2(z(1,t)) z(1,t) d\Gamma + \alpha_5 \int_{\Gamma_0} h_1(u_t) u_t d\Gamma.$$

From (3.14), (3.6), (3.21), (3.23), and (3.24), we have

$$\begin{aligned} (3.25) \quad L'(t) &\leq - \left( \int_0^t g(s) ds - \eta - \epsilon \right) \|u_t\|^2 + \left( \frac{M}{2} - \frac{g(0)}{4\eta} C_* \right) (g' \circ \nabla u)(t) \\ &\quad + [\eta \{1 + 2(1-l)^2\} - \epsilon \{1 - (1+\eta)(1-l) - 2\eta\}] \|\nabla u(t)\|^2 \\ &\quad + \left( C + \frac{\epsilon}{4\eta} \right) (g \circ \nabla u)(t) + \left( \eta\mu_1 + \frac{\epsilon\mu_1^2}{4\eta} \bar{C}_* \right) \|h_1(u_t)\|_{\Gamma_0}^2 \\ &\quad + \left( \eta|\mu_2| + \frac{\epsilon\mu_2^2}{4\eta} \bar{C}_* \right) \|h_2(z(1,t))\|_{\Gamma_0}^2 - (M\beta_1 - \alpha_5) \int_{\Gamma_0} h_1(u_t) u_t d\Gamma \\ &\quad - (M\beta_2 + \alpha_4(1-d)e^{-\tau_1}) \int_{\Gamma_0} h_2(z(1,t)) z(1,t) d\Gamma - \mathcal{E}(t). \end{aligned}$$

Making use of (2.7), we find

$$\|h_2(z(1,t))\|_{\Gamma_0}^2 \leq \alpha_3 \int_{\Gamma_0} h_2(z(1,t)) z(1,t) d\Gamma.$$

Owing to (2.9), it can be seen that

$$-\mathcal{E}(t) \leq -\tau(t) \int_{\Gamma_0} \int_0^1 e^{-\tau(t)} H_2 z(x, \rho, t) d\rho d\Gamma \leq -e^{-\tau_1 \rho} \int_{\Gamma_0} \int_0^1 H_2(z(x, \rho, t)) d\rho d\Gamma.$$

Since  $g$  is positive and continuous, then there exists  $t_0 > 0$  such that

$$\int_0^t g(s) d\tau \geq \int_0^{t_0} g(s) ds := g_0 \quad \text{for all } t \geq t_0.$$

Applying these to (3.25), we get

$$\begin{aligned}
L'(t) &\leq - (g_0 - \eta - \epsilon) \|u_t\|^2 + \left( \frac{M}{2} - \frac{g(0)}{4\eta} C_* \right) (g' \circ \nabla u)(t) \\
&\quad - [\epsilon \{ (1 - (1 + \eta)(1 - l) - 2\eta) - \eta \{ 1 + 2(1 - l)^2 \} \}] \|\nabla u(t)\|^2 \\
&\quad + \left( c + \frac{\epsilon}{4\eta} \right) (g \circ \nabla u)(t) + \left( \eta \mu_1 + \frac{\epsilon \mu_1^2}{4\eta} \bar{C}_* \right) \|h_1(u_t(t))\|_{\Gamma_0}^2 \\
&\quad - e^{-\tau_1 \rho} \int_{\Gamma_0} \int_0^1 H_2(z(x, \rho, t)) d\rho d\Gamma - (M\beta_1 - \alpha_5) \int_{\Gamma_0} h_1(u_t) u_t d\Gamma \\
&\quad - \left\{ M\beta_2 + \alpha_4(1 - d)e^{-\tau_1} - \alpha_3 \left( \eta |\mu_2| + \frac{\epsilon \mu_2^2}{4\eta} \bar{C}_* \right) \right\} \int_{\Gamma_0} h_2(z(1, t)) z(1, t) d\Gamma
\end{aligned}$$

for all  $t \geq t_0$ .

At this point, we choose  $\epsilon > 0$  small enough such that  $g_0 - \epsilon > 0$ , and then we pick  $\eta > 0$  sufficiently small such that

$$\begin{aligned}
1 - (1 + \eta)(1 - l) - 2\eta &> 0, \\
\epsilon \{ 1 - (1 + \eta)(1 - l) - 2\eta \} - \eta \{ 1 + 2(1 - l)^2 \} &> 0, \\
g_0 - \epsilon - \eta &> 0, \\
\epsilon - \eta &> 0.
\end{aligned}$$

Then we choose  $M > 0$  so large that

$$\begin{aligned}
\frac{M}{2} - \frac{g(0)}{4\eta} C_* &> 0, \quad M\beta_1 - \alpha_5 > 0, \\
M\beta_2 + \alpha_4(1 - d)e^{-\tau_1} - \alpha_3 \left( \eta |\mu_2| + \frac{\epsilon \mu_2^2}{4\eta} \bar{C}_* \right) &> 0,
\end{aligned}$$

and we complete the proof.  $\square$

With this preparation, we are ready to prove Theorem 3.1.

PROOF OF THEOREM 3.1. Multiplying (3.15) by  $\gamma(t)$ , it follows from (2.2) and (3.6) that

$$\begin{aligned}
(3.26) \quad \gamma(t)L'(t) &\leq -C_3\gamma(t)E(t) + C_4\gamma(t)\|h_1(u_t(t))\|_{\Gamma_0}^2 + C_5\gamma(t)(g \circ \nabla u)(t) \\
&\leq -C_3\gamma(t)E(t) + C_4\gamma(t)\|h_1(u_t(t))\|_{\Gamma_0}^2 - C_5(g' \circ \nabla u)(t) \\
&\leq -C_3\gamma(t)E(t) + C_4\gamma(t)\|h_1(u_t(t))\|_{\Gamma_0}^2 - 2C_5E'(t) \quad \text{for } t \geq t_0.
\end{aligned}$$

Now, we define

$$\mathcal{L}(t) = \gamma(t)L(t) + 2C_5E(t).$$

As  $\gamma$  is nonincreasing, we can see from (3.26) that

$$\begin{aligned}
(3.27) \quad \mathcal{L}'(t) &\leq \gamma'(t)L(t) - C_3\gamma(t)E(t) + C_4\gamma(t)\|h_1(u_t(t))\|_{\Gamma_0}^2 \\
&\leq -C_3\gamma(t)E(t) + C_4\gamma(t)\|h_1(u_t(t))\|_{\Gamma_0}^2 \quad \text{for } t \geq t_0.
\end{aligned}$$

In order to obtain the desired results, we needed to estimate the term  $\gamma(t)\|h_1(u_t(t))\|_{\Gamma_0}^2$  in (3.27). For this, the arguments of [51] are adapted.

Let

$$\Gamma_0^1 = \{x \in \Gamma_0 : |u_t| > r\} \quad \text{and} \quad \Gamma_0^2 = \{x \in \Gamma_0 : |u_t| \leq r\}.$$

For  $\delta_1 = \frac{\alpha_2 \gamma(0)}{\beta_1}$ , (2.3) and (3.6) imply that

$$(3.28) \quad \gamma(t) \int_{\Gamma_0^1} |h_1(u_t)|^2 d\Gamma \leq \alpha_2 \gamma(0) \int_{\Gamma_0^1} u_t h_1(u_t) d\Gamma \leq -\delta_1 E'(t).$$

Two cases are distinguished

CASE 1:  $H_1$  is linear on  $[0, r]$ . Then according to (2.3) and (2.6), we can easily check that there exist  $\delta_2 > 0$  and  $\delta_3 > 0$  such that

$$\delta_2 |s| \leq h_1(s) \leq \delta_3 |s| \quad \text{for all } |s| \leq r,$$

and, thus, for  $\delta_4 = \frac{\alpha_3 \gamma(0)}{\beta_1}$ ,

$$(3.29) \quad \begin{aligned} \gamma(t) \int_{\Gamma_0^2} |h_1(u_t)|^2 d\Gamma &\leq \delta_3 \gamma(t) \int_{\Gamma_0^2} u_t h_1(u_t) d\Gamma \\ &\leq \delta_3 \gamma(0) \int_{\Gamma_0^2} u_t h_1(u_t) d\Gamma \leq -\delta_4 E'(t). \end{aligned}$$

By substituting (3.28) and (3.29) into (3.27), we get

$$(3.30) \quad (\mathcal{L}(t) + \delta E(t))' \leq -C_3 \gamma(t) E(t) = -C_7 \gamma(t) H_0 \left( \frac{E(t)}{E(0)} \right) \quad \text{for } t \geq t_0,$$

where  $\delta = C_4(\delta_1 + \delta_4)$ ,  $C_7 = C_3 E(0)$  and  $H_0$  is the function given in (3.5).

CASE 2:  $H_1(0) = 0$  and  $H_1' > 0$  on  $(a, r]$ . From (2.6) and (3.6) it follows that

$$\begin{aligned} \gamma(t) \int_{\Gamma_0^2} h_1(u_t)^2 d\Gamma &\leq \gamma(t) \int_{\Gamma_0^2} H_1^{-1}(u_t h_1(u_t)) d\Gamma \\ &\leq \gamma(t) |\Gamma_0^2| H_1^{-1} \left( \frac{1}{|\Gamma_0^2|} \int_{\Gamma_0^2} u_t h_1(u_t) d\Gamma \right) \\ &\leq \gamma(t) |\Gamma_0^2| H_1^{-1} \left( -\frac{E'(t)}{\beta_1 |\Gamma_0^2|} \right), \end{aligned}$$

where Jensen's inequality 2.1 for a concave function, with  $D = \Gamma_0^2$ ,  $q(x) = 1$ ,  $H = H_1$  and  $f(x) = H_1^{-1}(u_t(x) h_1(u_t(x)))$  in the second inequality is used.

Adapting this and (3.28) to (3.27), for  $\tilde{\delta} = C_4 \delta_1$  and  $C_6 = C_4 |\Gamma_0^2|$ , we get

$$(3.31) \quad (\mathcal{L}(t) + \tilde{\delta} E(t))' \leq -C_3 \gamma(t) E(t) + C_6 \gamma(t) H_1^{-1} \left( -\frac{E'(t)}{\beta_1 |\Gamma_0^2|} \right).$$

For  $0 < \epsilon_0 < r$  and  $c_0 > 0$ , inequalities (3.31), (3.10), together with (3.9), lead to

$$(3.32) \quad \begin{aligned} &\left\{ H_1' \left( \epsilon_0 \frac{E(t)}{E(0)} \right) (\mathcal{L}(t) + \tilde{\delta} E(t)) + c_0 E(t) \right\}' \\ &= \epsilon_0 \frac{E'(t)}{E(0)} H_1'' \left( \epsilon_0 \frac{E(t)}{E(0)} \right) (\mathcal{L}(t) + \tilde{\delta} E(t)) + H_1' \left( \epsilon_0 \frac{E(t)}{E(0)} \right) (\mathcal{L}(t) + \tilde{\delta} E(t))' + c_0 E'(t) \end{aligned}$$



$$\begin{aligned}
&\leq -C_3\gamma(t)H_1'\left(\epsilon_0\frac{E(t)}{E(0)}\right)E(t) + C_6\gamma(t)H_1'\left(\epsilon_0\frac{E(t)}{E(0)}\right)H_1^{-1}\left(-\frac{E'(t)}{\beta_1|\Gamma_0^2|}\right) + c_0E'(t) \\
(3.32) \quad &\leq -C_3\gamma(t)H_1'\left(\epsilon_0\frac{E(t)}{E(0)}\right)E(t) + C_6\gamma(t)H_1^*\left(H_1'\left(\epsilon_0\frac{E(t)}{E(0)}\right)\right) - \frac{C_6\gamma(t)}{\beta_1|\Gamma_0^2|}E'(t) + c_0E'(t) \\
&= -C_3\gamma(t)H_1'\left(\epsilon_0\frac{E(t)}{E(0)}\right)E(t) + C_6\gamma(t)H_1'\left(\epsilon_0\frac{E(t)}{E(0)}\right)\epsilon_0\frac{E(t)}{E(0)} \\
&\quad - C_6\gamma(t)H_1\left(\epsilon_0\frac{E(t)}{E(0)}\right) - \frac{C_6\gamma(t)}{\beta_1|\Gamma_0^2|}E'(t) + c_0E'(t) \\
&\leq -(C_3E(0) - C_6\epsilon_0)\gamma(t)H_1'\left(\epsilon_0\frac{E(t)}{E(0)}\right)\frac{E(t)}{E(0)} + \left(c_0 - \frac{C_6\gamma(0)}{\beta_1|\Gamma_0^2|}\right)E'(t).
\end{aligned}$$

Taking  $\epsilon_0$  sufficiently small such that  $C_3E(0) - C_6\epsilon_0 > 0$  and choosing  $c_0 > 0$  suitably such that  $c_0 - \frac{C_6\gamma(0)}{\beta_1|\Gamma_0^2|} > 0$ , we obtain

$$\begin{aligned}
(3.33) \quad &\left\{H_1'\left(\epsilon_0\frac{E(t)}{E(0)}\right)(\mathcal{L}(t) + \tilde{\delta}E(t)) + c_0E(t)\right\}' \leq -C_8\gamma(t)H_1'\left(\epsilon_0\frac{E(t)}{E(0)}\right)\frac{E(t)}{E(0)} \\
&= -C_8\gamma(t)H_0\left(\frac{E(t)}{E(0)}\right) \quad \text{for } t \geq t_0,
\end{aligned}$$

where  $C_8 = C_3E(0) - C_6\epsilon_0$  is a positive constant.

Now, let

$$\tilde{\mathcal{L}}(t) = \begin{cases} \mathcal{L}(t) + \delta E(t) & \text{if } H_1 \text{ is linear on } [0, r], \\ H_1'\left(\epsilon_0\frac{E(t)}{E(0)}\right)(\mathcal{L}(t) + \tilde{\delta}E(t)) + C_0E(t) & \text{if } H_1'(0) = 0 \text{ and } H_1'' > 0 \text{ on } (0, r]. \end{cases}$$

Then from (3.30) and (3.33) we can see that

$$(3.34) \quad \widetilde{\mathcal{L}}'(t) \leq -C_9\gamma(t)H_0\left(\frac{E(t)}{E(0)}\right) \quad \text{for } t \geq t_0,$$

where  $C_9 = \min\{C_7, C_8\}$ . Since  $\tilde{\mathcal{L}}(t)$  is equivalent to  $E(t)$ , there exist two positive constants  $\alpha_3$  and  $\alpha_4$  such that

$$(3.35) \quad \alpha_3\widetilde{\mathcal{L}}(t) \leq E(t) \leq \alpha_4\widetilde{\mathcal{L}}(t).$$

Let us define

$$(3.36) \quad \mathcal{J}(t) = \alpha_3\frac{\widetilde{\mathcal{L}}(t)}{E(0)}.$$

It is to be noted that

$$(3.37) \quad \mathcal{J}(t) \leq \frac{E(t)}{E(0)} < 1 \quad (\text{see (3.35)}).$$

From (3.36), (3.34), (3.37) and the fact that  $H_0$  is increasing, we arrive at

$$(3.38) \quad \mathcal{J}'(t) \leq -\frac{\alpha_3C_9}{E(0)}\gamma(t)H_0\left(\frac{E(t)}{E(0)}\right) \leq -C_{10}\gamma(t)H_0(\varepsilon(t)),$$

where  $C_{10} = \frac{\alpha_3C_9}{E(0)}$ .

Integrating this over  $(t_0, t)$  and using  $H'(t) = -\frac{1}{H_0(t)}$  (see (3.4)), we observe that

$$H(\mathcal{J}(t)) - H(\mathcal{J}(t_0)) \geq C_{10} \int_{t_0}^t \gamma(s) ds.$$

Since  $H^{-1}$  is decreasing, we infer

$$\mathcal{J}(t) \leq H^{-1} \left( H(\mathcal{J}(0)) + C_{10} \int_{t_0}^t \gamma(s) ds \right) \quad \text{for } t \geq t_0.$$

This completes the proof from the equivalent relation of  $\mathcal{J}$  and  $E$ .  $\square$

#### 4. Conclusion

It is well known that viscoelastic materials have memory effects, which is due to the mechanical response influenced by the history of the materials themselves. As these materials have a wide application in the natural sciences, their dynamics are interesting and of great importance. Also, the dynamic boundary conditions are not only important from the theoretical point of view but also arise in numerous practical problems and the acoustic boundary conditions are related to noise control and suppression in practical applications. Moreover, time delay so often arises in many physical, chemical, biological, and economic phenomena because these phenomena depend not only on the present state but also on the history of the system in a more complicated way. Motivated by the approaches from [24, 34, 35] with the ones from [10–12, 28] and [29, 30], under the assumption  $|\mu_2| < \mu_1 \sqrt{1-d}$ , the general decay estimates for the solution energy are established. In achieving our goal, the energy method combined with the choice of a suitable Lyapunov functional  $L$  equivalent to the associate energy  $E$  is employed.

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**АСИМПТОТСКА СТАБИЛНОСТ ВИСКООЕЛАСТИЧНОГ  
ПРОБЛЕМА СА ПРОМЕНЉИВИМ ВРЕМЕНОМ  
КАШЊЕЊА ПОВРАТНЕ СПРЕГЕ НА ГРАНИЦИ**

РЕЗИМЕ. У овом раду истражујемо нелинеарну вискоеластичну једначину. Претпостављањем променљивог времена кашњења повратне спреге која делује на граници, под одређеним претпоставкама на датим подацима, утврђују се опште процене промене енергије увођењем одговарајућих Лапуновљевих функционала. Овај модел побољшава раније моделе доступне у литератури у којима се узима у обзир само дисипативни члан у повратној спреси.

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