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EXISTENCE AND STABILITY RESULTS OF A NONLINEAR TIMOSHENKO EQUATION WITH DAMPING AND SOURCE TERMS

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Abstract. In this paper, we consider a nonlinear Timoshenko equation. First, we prove the local existence solution by the Faedo–Galerkin method, and, under suitable assumptions with positive initial energy, we prove that the local existence is global in time. Finally, the stability result is established based on Komornik's integral inequality.

1. Introduction

We consider the following boundary value problem:

$$\begin{cases} u_{tt}+\Delta^2 u-\Delta u+|u_t|^{m-2}u_t=|u|^{r-2}u, & (x,t)\in\Omega\times(0,T) \\ u(x,t)=0, & (x,t)\in\partial\Omega\times(0,T) \\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n, n\geqslant 1$ with the smooth boundary $\partial\Omega$, $m>2$ and $2<r<\infty$ $(n=1,2)$ or $2<r\leqslant 2\frac{n-1}{n-2}(n\geqslant 3)$.

The equation

$$u_{tt}+\Delta^2 u+g(u_t)=|u|^{r-2}u \tag{1.2}$$

is a special case of (1.1), which has been discussed by many authors. Timoshenko [13], a pioneer in the strength of materials, developed a theory in 1921, which was an amendment to the Euler beam theory. The modified theory is called the Timoshenko beam theory. For $g(u_t)=|u_t|^{p-2}u_t$, the global existence and blow-up results can be found in [8]. Messaoudi [11] studied the following equation

$$u_{tt}+\Delta^2 u+a|u_t|^{p-2}u_t=b|u|^{r-2}u. \tag{1.3}$$

He proved the local existence and blow-up of the solution. Also, Wu and Tsai [14] obtained the global existence and blow-up of the solution to the problem (1.3). Later, Chen and Zhou [2] studied the blow-up of the solution to the problem (1.3)

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for positive initial energy. Equation (1.1), without the fourth-order term $\Delta^2 u$, which is replaced by $-\Delta u$, can be written in the following form

$$u_{tt} - \Delta u + u_t = |u|^{r-2}u. \tag{1.4}$$

Many authors have studied the existence and blow-up in finite time of solutions for (1.4); see e.g., [**3**, **5**, **6**, **10**, **12**].

This paper is organized as follows. In Section 2, some notations, assumptions and preliminaries are introduced, the global existence of the solution is proved in Section 3 and the main results of this article are shown in Section 4.

2. Preliminaries

We begin this section with some notations and definitions. Denote by $\|.\|_p$ the $L^p(\Omega)$ norm of a Lebesgue function $u \in L^p(\Omega)$ endowed with the norm

$$\|u\|_p^p = \int_\Omega |u(x)|^p dx.$$

We also consider the Sobolev space equipped with the scalar product

$$(u, v)_{H^2(\Omega)} = (u, v) + (\Delta u, \Delta v).$$

We define the subspace of $H^2(\Omega)$, denoted by $H_0^2(\Omega)$, as the closure of $C_0^\infty(\Omega)$ in the strong topology of $H^2(\Omega)$. This space $H_0^2(\Omega)$ endowed with the norm induced by the scalar product

$$(u, v)_{H_0^2(\Omega)} = (\Delta u, \Delta v)$$

is a Hilbert space.

Lemma 2.1 (Hölder's Inequality). *Suppose that $p, q, s \geqslant 1$ are measurable functions defined on Ω such that*

$$\frac{1}{s} = \frac{1}{p} + \frac{1}{q}.$$

If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^s(\Omega)$, with

$$\|uv\|_s \leqslant c\|u\|_p\|v\|_q.$$

Lemma 2.2 (Sobolev–Poincare inequality [**1**]). *Let p be a number with $2 \leqslant p < \infty$ $(n = 1, 2)$ or $2 \leqslant p \leqslant \frac{2n}{n-2}(n \geqslant 3)$. Then there is a constant $C_* = C_*(p, \Omega)$ such that*

$$\|u\|_p \leqslant C_*\|\nabla u\|_2, \quad \textit{for} \quad u \in H_0^1(\Omega).$$

Theorem 2.1. *Suppose that $(u_0, u_1) \in H_0^2(\Omega) \cap H^4(\Omega) \times L^2(\Omega)$. Then problem (1.1) has a unique weak local solution*

$$\begin{aligned} u &\in L^\infty((0, T),\ H_0^2(\Omega) \cap H^4(\Omega)), \\ u_t &\in L^\infty((0, T),\ L^2(\Omega)) \cap L^m(\Omega \times (0, T)), \\ u_{tt} &\in L^\infty((0, T),\ L^2(\Omega)). \end{aligned}$$

In order to state and prove our result, we define the potential energy functional and Nehari's functional in the following way

$$E(t) = E(u(t)) = \frac{1}{2}(\|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2 + \|\Delta u(t)\|_2^2) - \frac{1}{r}\|u(t)\|_r^r, \tag{2.1}$$

$$I(t) = I(u(t)) = \|\nabla u(t)\|_2^2 + \|\Delta u(t)\|_2^2 - \|u(t)\|_r^r, \tag{2.2}$$

$$J(t) = J(u(t)) = \frac{1}{2}(\|\nabla u(t)\|_2^2 + \|\Delta u(t)\|_2^2) - \frac{1}{r}\|u(t)\|_r^r. \tag{2.3}$$

Lemma 2.3. *Under the assumptions of Theorem 2.1, we have*

$$E(t) \leqslant E(0).$$

Proof. We multiply the first equation of (1.1) by u_t and, integrating over the domain Ω, we get

$$\frac{d}{dt}\Big(\frac{1}{2}(\|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2 + \|\Delta u(t)\|_2^2) - \frac{1}{r}\|u(t)\|_r^r\Big) = -\|u_t(t)\|_m^m,$$

and then

$$E'(t) = -\|u_t(t)\|_m^m \leqslant 0. \tag{2.4}$$

Integrating (2.4) over $(0,t)$, we obtain

$$E(t) \leqslant E(0). \qquad \square$$

Lemma 2.4. *Assume that* $(u_0,u_1) \in H_0^2(\Omega)\cap H^4(\Omega)\times L^2(\Omega)$, $E(0) > 0$, $I(0) > 0$ *and*

$$\theta_1 + \theta_2 < 1, \tag{2.5}$$

where

$$\theta_1 := \alpha c_{1,*}^r\Big(\frac{2r}{r-2}E(0)\Big)^{\frac{r-2}{2}}, \quad \theta_2 := (1-\alpha)c_{2,*}^r\Big(\frac{2r}{r-2}E(0)\Big)^{\frac{r-2}{2}},$$

with $0 < \alpha < 1$, *and* $c_{1,*}$ *and* $c_{2,*}$ *are the best embedding constants of* $H_0^1(\Omega) \hookrightarrow L^r(\Omega)$ *and* $H_0^2(\Omega) \hookrightarrow L^r(\Omega)$ *respectively. Then* $I(t) > 0$, *for all* $t \in [0,T]$.

Proof. By continuity, there exists T_*, such that

$$I(t) \geqslant 0, \quad \text{for all} \quad t \in [0,T_*]. \tag{2.6}$$

Now, we have for all $t \in [0,T_*]$:

$$\begin{aligned} J(t) = J(u(t)) &= \frac{1}{2}(\|\nabla u(t)\|_2^2 + \|\Delta u(t)\|_2^2) - \frac{1}{r}\|u(t)\|_r^r \\ &\geqslant \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2}\|\Delta u(t)\|_2^2 - \frac{1}{r}(\|\nabla u(t)\|_2^2 + \|\Delta u(t)\|_2^2 - I(t)) \\ &\geqslant \frac{r-2}{2r}(\|\nabla u(t)\|_2^2 + \|\Delta u(t)\|_2^2) + \frac{1}{r}I(t), \end{aligned}$$

and, using (2.6), we obtain

$$\|\nabla u(t)\|_2^2 + \|\Delta u(t)\|_2^2 \leqslant \frac{2r}{r-2}J(t), \quad \text{for all} \quad t \in [0,T_*]. \tag{2.7}$$

By Lemma 2.3, we get

$$\|\nabla u(t)\|_2^2 + \|\Delta u(t)\|_2^2 \leqslant \frac{2r}{r-2}E(t) \leqslant \frac{2r}{r-2}E(0). \tag{2.8}$$

On the other hand, we have

$$\|u(t)\|_r^r = \alpha\|u(t)\|_r^r + (1-\alpha)\|u(t)\|_r^r.$$

By the embedding of $H_0^1(\Omega) \hookrightarrow L^r(\Omega)$ and $H_0^2(\Omega) \hookrightarrow L^r(\Omega)$, we obtain

$$\begin{aligned}
\|u(t)\|_r^r &\leqslant \alpha c_{1,*}^r\|\nabla u(t)\|_2^r + (1-\alpha)c_{2,*}^r\|\Delta u(t)\|_2^r \\
&\leqslant \alpha c_{1,*}^r\|\nabla u(t)\|_2^{r-2} \times \|\nabla u(t)\|_2^2 \\
&\quad + (1-\alpha)c_{2,*}^r\|\Delta u(t)\|_2^{r-2} \times \|\Delta u(t)\|_2^2 \\
&\leqslant \alpha c_{1,*}^r\Big(\frac{2r}{r-2}E(0)\Big)^{\frac{r-2}{2}} \times \|\nabla u(t)\|_2^2 \\
&\quad + (1-\alpha)c_{2,*}^r\Big(\frac{2r}{r-2}E(0)\Big)^{\frac{r-2}{2}} \times \|\Delta u(t)\|_2^2.
\end{aligned}$$

Then, we get

$$\|u(t)\|_r^r \leqslant \theta_1\|\nabla u(t)\|_2^2 + \theta_2\|\Delta u(t)\|_2^2, \quad \text{for all} \quad t \in [0, T_*]. \tag{2.9}$$

Since $\theta_1 + \theta_2 < 1$, then

$$\|u(t)\|_r^r < \|\nabla u(t)\|_2^2 + \|\Delta u(t)\|_2^2, \quad \text{for all} \quad t \in [0, T_*]. \tag{2.10}$$

This implies that

$$I(t) > 0, \quad \text{for all} \quad t \in [0, T_*].$$

By repeating the above procedure, we can extend T_* to T. □

3. Local existence

In this section we are going to obtain the existence of the local solution to the problem (1.1). We will use Faedo–Galerkin's method approximation. Let $\{v_l\}_{l=1}^{\infty}$ be a basis of $H_0^2(\Omega)$ which constructs a complete orthonormal system in $L^2(\Omega)$. Denote by $V_k = \operatorname{span}\{v_1, v_2, \dots, v_k\}$ the subspace generated by the first k vectors of the basis $\{v_l\}_{l=1}^{\infty}$. Applying the normalization, we have $\|v_l\| = 1$ for any given integer k.

Consider the approximation solution

$$u_k(t) = \sum_{l=1}^{k} u_{lk}(t)v_l,$$

where u_k is the solution to the following Cauchy problem

$$\begin{aligned}
(u_k''(t), v_l) + (\Delta^2 u_k(t), v_l) - (\Delta u_k(t), v_l) + (|u_k'(t)|^m u_k'(t), v_l) \\
= (|u_k(t)|^{r-2}u_k(t), v_l), \quad l = 1, 2, \dots, k,
\end{aligned} \tag{3.1}$$

$$u_k(0) = u_{0k} = \sum_{i=1}^{k}(u_k(0), v_l)v_l \to u_0 \;\text{ in }\; H_0^2(\Omega) \cap H^4(\Omega), \tag{3.2}$$

$$u'_k(0) = u_{1k} = \sum_{l=1}^{k} (u'_k(0), v_l)v_l \to u_1 \quad \text{in} \quad L^2(\Omega). \tag{3.3}$$

Note that we can solve the system (3.1)–(3.3) by Picard's iteration method in ordinary differential equations. Hence, there exists a solution in $[0, T_*]$ for some $T_* > 0$ and we can extend this solution to the whole interval $[0, T]$ for any given $T > 0$ by making use of the prior estimates below.

STEP 1. (The prior estimate) Multiplying equation (3.1) by $u'_{lk}(t)$ and summing over l from 1 to k, we get

$$\frac{d}{dt}\Big(\frac{1}{2}\|u'_k\|_2^2 + \frac{1}{2}\|\nabla u_k\|_2^2 + \frac{1}{2}\|\Delta u_k\|_2^2 - \frac{1}{r}\|u_k\|_r^r\Big) = -\|u'_k(t)\|_m^m. \tag{3.4}$$

Then

$$E'(u_k(t)) = -\|u'_k(t)\|_m^m \leqslant 0.$$

Integrating (3.4) over $(0, t)$, we obtain the estimate

$$\frac{1}{2}\|u'_k\|_2^2 + \frac{1}{2}\|\nabla u_k\|_2^2 + \frac{1}{2}\|\Delta u_k\|_2^2 - \frac{1}{r}\|u_k\|_r^r + \int_0^t \|u'_k(s)\|_m^m ds \leqslant E(0). \tag{3.5}$$

Then, from (2.10), the inequality (3.5) becomes

$$\begin{aligned}\frac{1}{2}\sup_{t\in(0,T)} \|u'_k\|_2^2 &+ \frac{r-2}{2r}\sup_{t\in(0,T)} \|\nabla u_k\|_2^2 \\ &+ \frac{r-2}{2r}\sup_{t\in(0,T)} \|\Delta u_k\|_2^2 + \int_0^t \|u'_k(s)\|_m^m ds \leqslant E(0).\end{aligned} \tag{3.6}$$

From (3.6), we conclude that

$$\begin{cases}\{u_k\} \text{ is uniformly bounded in } L^\infty([0,T], H_0^2(\Omega)),\\ \{u'_k\} \text{ is uniformly bounded in } L^\infty(0,t,L^2(\Omega)) \cap L^m(\Omega\times[0,T]).\end{cases} \tag{3.7}$$

Since $\{u'_k\}$ is uniformly bounded in $L^m(\Omega\times[0,T])$, then $\{|u'_k|^{m-2}u'_k\}$ is bounded in $L^{\frac{m}{m-1}}(\Omega\times[0,T])$; hence, up to a subsequence, $|u'_k|^{m-2}u'_k \rightharpoonup \Phi$ weakly in $L^{\frac{m}{m-1}}(\Omega\times[0,T])$.

We have to show that $\Phi = |u'|^{m-2}u'$.

Furthemore, it follows from Lemma 2.3 and (3.7) that

$$\{|u_k|^{r-2}u_k\} \text{ is uniformly bounded in } L^\infty([0,T], L^2(\Omega)). \tag{3.8}$$

By (3.7) and (3.8), we infer that there exists a subsequence of u_k (still denoted by the same symbol) and a function u such that

$$\begin{cases}u_k \rightharpoonup u \text{ weakly star in } L^\infty([0,T], H_0^2(\Omega)),\\ u'_k \rightharpoonup u' \text{ weakly star in } L^\infty([0,T], L^2(\Omega)) \text{ and weakly in } L^m(\Omega\times[0,T]),\\ |u_k|^{r-2}u_k \rightharpoonup \Psi \text{ weakly in } L^\infty([0,T], L^2(\Omega)).\end{cases} \tag{3.9}$$

By the Aubin–Lions theorem [7], we conclude from (3.9) that

$$u_k \rightharpoonup u \text{ strongly in } C([0,T], H_0^2(\Omega)),$$

which implies

$$u_k \rightharpoonup u \text{ everywhere in } [0,T]\times\Omega. \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$|u_k|^{r-2}u_k \rightharpoonup |u|^{r-2}u \text{ weakly in } L^\infty([0,T],L^2(\Omega)). \tag{3.11}$$

Now, we would like to get more estimates. In doing so, differentiating (3.1) with respect to t, we get

$$\begin{aligned}(u_k'''(t),v_l)+(\Delta^2 u_k'(t),v_l)\\ -(\Delta u_k'(t),v_l)+((m-1)|u_k'(t)|^{m-2}u_k''(t),v_l)\\ =((r-1)|u_k(t)|^{r-2}u_k'(t),v_l),\quad l=1,2,\dots,k.\end{aligned} \tag{3.12}$$

Next, multiplying the equation (3.12) by $u_{lk}''(t)$ and summing over l from 1 to k, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega |u_k''|^2dx+\int_\Omega|\Delta u_k'|^2dx+\int_\Omega|\nabla u_k'|^2dx\Big)\\ +\int_\Omega(m-1)|u_k'|^{m-2}u_k''^2dx=\int_\Omega(r-1)|u_k|^{r-2}u_k'u_k''dx.\end{aligned} \tag{3.13}$$

It follows from Hölder's inequality that

$$\Big|\int_\Omega (r-1)|u_k|^{r-2}u_k'u_k''dx\Big|\leqslant (r-1)\|u_k\|_{2(r-1)}^{r-2}\|u_k'\|_{2(r-1)}\|u_k''\|_2. \tag{3.14}$$

We have $u_k\in L^\infty([0,T),H_0^2(\Omega))$, and then

$$\int_\Omega |u_k|^{2r-2}dx<+\infty,$$

since $2(r-1)\leqslant 2\frac{n}{n-2}$.

The inequality (3.14) becomes

$$\Big|\int_\Omega (r-1)|u_k|^{r-2}u_k'u_k''dx\Big|\leqslant c_1\|u_k'\|_{2(r-1)}\|u_k''\|_2. \tag{3.15}$$

It follows from Poincare's inequality and Young's inequality that

$$\Big|\int_\Omega (r-1)|u_k|^{r-2}u_k'u_k''dx\Big|\leqslant c_\delta\|\nabla u_k'\|_2^2+\delta\|u_k''\|_2^2. \tag{3.16}$$

Substituting (3.16) into (3.13) and integrating over $(0,t)$ for all $t\in[0,T]$, we obtain

$$\begin{aligned}\int_\Omega |u_k''|^2dx+\int_\Omega|\Delta u_k'|^2dx+\int_\Omega|\nabla u_k'|^2dx\\ \leqslant \|u_k''(0)\|_2^2+\|\Delta u_k'(0)\|_2^2+\|\nabla u_k'(0)\|_2^2+c_2\int_0^t(\|\nabla u_k'\|_2^2+\|u_k''\|_2^2)ds.\end{aligned} \tag{3.17}$$

It follows from (3.2), (3.3) and the fact that $\|\nabla u_k'(0)\|_2^2\leqslant c_3\|\Delta u_k'(0)\|_2^2$ that

$$\|\nabla u_k'(0)\|_2^2+\|\Delta u_k'(0)\|_2^2\leqslant c_4, \tag{3.18}$$

where c_4 is a positive constant independent of k.

Multiplying both sides of (3.1) by $u''_{lk}(t)$, and then summing over l from 1 to k and putting $t = 0$, we get

$$\|u''_k(0)\|_2^2 + (\Delta^2 u_k(0), u''_k(0)) - (\Delta u_k(0), u''_k(0)) + (|u'_k(0)|^{m-2}u'_k(0), u''_k(0)) = (|u_k(0)|^{r-2}u_k(0), u''_k(0)).$$

It follows from Young's inequality, (3.2), and (3.3) that

$$\|u''_k(0)\|_2 \leqslant c_5, \tag{3.19}$$

where c_5 is a positive constant independent of k.

By (3.18) and (3.19), (3.17) becomes

$$\int_\Omega |u''_k|^2 dx + \int_\Omega |\Delta u'_k|^2 dx + \int_\Omega |\nabla u'_k|^2 dx \leqslant c_6 + c_7 \int_0^t (\|u''_k\|_2^2 + \|\Delta u'_k\|_2^2 + \|\nabla u'_k\|_2^2) ds. \tag{3.20}$$

We get from (3.20) and Gronwall's lemma that

$$\|u''_k\|_2^2 + \|\Delta u'_k\|_2^2 + \|\nabla u'_k\|_2^2 \leqslant c_8, \tag{3.21}$$

for all $t \in [0, T]$, and c_8 is a positive constant independent of k.

We conclude from (3.21) that

$$\begin{cases} \{u'_k\} \text{ is uniformly bounded in } L^\infty([0,T], H_0^2(\Omega)), \\ \{u''_k\} \text{ is uniformly bounded in } L^\infty([0,T], L^2(\Omega)). \end{cases} \tag{3.22}$$

Similarly, we have

$$\begin{cases} u'_k \rightharpoonup u' \text{ weakly star in } L^\infty([0,T], H_0^2(\Omega)), \\ u''_k \rightharpoonup u'' \text{ weakly star in } L^\infty([0,T], L^2(\Omega)). \end{cases} \tag{3.23}$$

Step 2. Setting up $k \longrightarrow \infty$ and passing to the limit in (3.1), we obtain

$$(u''(t), v_l) + (\Delta^2 u(t), v_l) - (\Delta u(t), v_l) + (|u'(t)|^{m-2}u'(t), v_l) = (|u(t)|^{r-2}u(t), v_l), \quad l = 1, 2, \dots, k. \tag{3.24}$$

Since $\{v_l\}_{l=1}^\infty$ is a basis of $H_0^2(\Omega)$, we deduce that u satisfies the equation (1.1). From (3.9), (3.23), and Lemma 3.1.7 in [15] with $B = H_0^2(\Omega)$ and $B = L^2(\Omega)$, respectively, we infer that

$$\begin{cases} u_k(0) \rightharpoonup u(0) \text{ weakly in } H_0^2(\Omega), \\ u'_k(0) \rightharpoonup u'(0) \text{ weakly in } L^2(\Omega). \end{cases} \tag{3.25}$$

We get from (3.2), (3.3), and (3.25) that $u(0) = u_0$, $u'(0) = u_1$.

Thus, the proof of existence is complete.

Step 3. (Uniqueness of the solution) Now it remains to prove uniqueness. Let u^1, u^2 be two solutions in the class described in the statement of this theorem, and $w = u^1 - u^2$.

Then w satisfies

$$w_{tt} + \Delta^2 w - \Delta w + |u_t^1|^{m-2}u_t^1 - |u_t^2|^{m-2}u_t^2 = |u^1|^{r-2}u^1 - |u^2|^{r-2}u^2, \tag{3.26}$$

and

$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x).$$

Multiplying (3.26) by w_t and integrating after that with respect to x, we get

$$\frac{1}{2}\int_\Omega |w_t|^2 dx + \frac{1}{2}\int_\Omega |\Delta w|^2 dx + \frac{1}{2}\int_\Omega |\nabla w|^2 dx + \int_0^t \int_\Omega (|u_t^1|^{m-2}u_t^1 - |u_t^2|^{m-2}u_t^2) w_t dx\, ds = \int_0^t \int_\Omega (|u^1|^{r-2}u^1 - |u^2|^{r-2}u^2) w_t dx\, ds.$$

We use the inequality

$$(|a|^{m-2}a - |b|^{m-2}b)(a-b) \geqslant 0,$$

for all $a, b \in \mathbb{R}$ and almost everywhere $x \in \Omega$.

This implies

$$\|w_t\|_2^2 dx + \|\Delta w\|_2^2 + \|\nabla w\|_2^2 \leqslant C \int_0^t \int_\Omega (|u^1|^{r-2}u^1 - |u^2|^{r-2}u^2) w_t dx\, ds. \tag{3.27}$$

By repeating the estimate as in [9], we arrive at

$$\int_\Omega |w_t|^2 dx + \|\Delta w\|_2^2 + \|\nabla w\|_2^2 \leqslant C \int_0^t \left(\int_\Omega |w_t|^2 dx + \|\nabla w\|_2^2 \right) ds. \tag{3.28}$$

Then

$$\int_\Omega |w_t|^2 dx + \|\Delta w\|_2^2 + \|\nabla w\|_2^2 \leqslant C \int_0^t \left(\int_\Omega |w_t|^2 dx + \|\Delta w\|_2^2 + \|\nabla w\|_2^2 \right) ds. \tag{3.29}$$

Gronwall's inequality yields

$$\|w_t\|_2^2 + \|\Delta w\|_2^2 + \|\nabla w\|_2^2 = 0.$$

Thus, $w = 0$. This shows the uniqueness. □

4. Global existence and stability solution

In this section our main result is based on Komornik's inequality [4].

Now, we state our main result:

THEOREM 4.1. *Under the assumptions of Lemma 2.4, the local solution for (1.1) is global.*

PROOF. We have

$$\begin{aligned} E(u(t)) &= \frac{1}{2}(\|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2 + \|\Delta u(t)\|_2^2) - \frac{1}{r}\|u(t)\|_r^r \\ &\geqslant \frac{1}{2}\|u_t(t)\|_2^2 + \frac{r-2}{2r}\|\nabla u(t)\|_2^2 + \frac{r-2}{2r}\|\Delta u(t)\|_2^2, \end{aligned}$$

so that

$$\|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2 + \|\Delta u(t)\|_2^2 \leqslant CE(t). \tag{4.1}$$

By Lemma 2.3, we obtain

$$\|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2 + \|\Delta u(t)\|_2^2 \leqslant CE(0). \tag{4.2}$$

This implies that the local solution is global in time. □

Lemma 4.1. *Suppose that the assumptions of Lemma 2.4 hold. Then there exists a positive constant c such that*

$$\int_\Omega |u(t)|^m dx \leqslant cE(t).$$

Proof. We have

$$\int_\Omega |u(t)|^m dx = \alpha\|u(t)\|_m^m + (1-\alpha)\|u(t)\|_m^m.$$

By the embedding of $H_0^1(\Omega) \hookrightarrow L^m(\Omega)$ and $H_0^2(\Omega) \hookrightarrow L^m(\Omega)$, we obtain

$$\begin{aligned}\int_\Omega |u(t)|^m dx &\leqslant \alpha\lambda_{1,*}^m\|\nabla u(t)\|_2^m + (1-\alpha)\lambda_{2,*}^m\|\Delta u(t)\|_2^m \\ &\leqslant \alpha\lambda_{1,*}^m\|\nabla u(t)\|_2^{m-2} \times \|\nabla u(t)\|_2^2 + (1-\alpha)\lambda_{2,*}^m\|\Delta u(t)\|_2^{m-2} \times \|\Delta u(t)\|_2^2 \\ &\leqslant c_1\|\nabla u(t)\|_2^2 + c_2\|\Delta u(t)\|_2^2.\end{aligned}$$

By using (2.8), we obtain

$$\int_\Omega |u(t)|^m dx \leqslant cE(t).$$ □

Theorem 4.2. *Let the assumptions of Theorem 2.1 hold. Then there exists the positive constant $C > 0$, such that*

$$E(t) \leqslant \frac{C}{(1+t)^{\frac{2}{m-2}}}, \quad \textit{for all} \ \ t \geqslant 0.$$

Proof. Multiplying the first equation from (1.1) by $u(t)E^{\frac{m-2}{2}}(t)$ and integrating over $\Omega \times (S,T)$ $(S < T)$, we obtain

$$\int_S^T \int_\Omega u(t)E^{\frac{m-2}{2}}(t)[u_{tt}(t) + \Delta^2 u - \Delta u + |u_t|^{m-2}u_t]dx\,dt = \int_S^T \int_\Omega E^{\frac{m-2}{2}}(t)|u(t)|^r dx\,dt,$$

so that

$$\begin{aligned}\int_S^T E^{\frac{m-2}{2}}(t) \int_\Omega [(u(t)u_t(t))_t - |u_t(t)|^2 + |\Delta u(t)|^2 + |\nabla u(t)|^2 + u(t)|u_t|^{m-2}u_t]dx\,dt \\ = \int_S^T E^{\frac{m-2}{2}}(t) \int_\Omega |u(t)|^r dx\,dt\end{aligned}$$

We add and subtract the term

$$\int_S^T E^{\frac{m-2}{2}}(t)[\theta_1\|\nabla u(t)\|_2^2 + \theta_2\|\Delta u(t)\|_2^2 + (2+\theta_1+\theta_2)\|u_t(t)\|_2^2]dt$$

and use (3.9) to get

$$(1-\theta_1)\int_S^T E^{\frac{m-2}{2}}(t)(\|\nabla u(t)\|_2^2+\|u_t(t)\|_2^2)dt \tag{4.3}$$
$$+(1-\theta_2)\int_S^T E^{\frac{m-2}{2}}(t)(\|\Delta u(t)\|_2^2+\|u_t(t)\|_2^2)dt$$
$$+\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega (u(t)u_t(t))_t dx\, dt-(3-\theta_1-\theta_2)\int_S^T E^{\frac{m-2}{2}}(t)\|u_t(t)\|_2^2 dt$$
$$+\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega u(t)|u_t|^{m-2}u_t(t)dx\, dt$$
$$=-\int_S^T E^{\frac{m-2}{2}}(t)(\theta_1\|\nabla u(t)\|_2^2+\theta_2\|\Delta u(t)\|_2^2-\|u(t)\|_r^r)dt\leqslant 0.$$

It is clear that

$$\gamma\int_S^T E^{\frac{m-2}{2}}(t)\Big(\frac{1}{2}\|\nabla u(t)\|_2^2+\frac{1}{2}\|\Delta u(t)\|_2^2+\frac{1}{2}\|u_t(t)\|_2^2-\frac{1}{r}\|u(t)\|_r^r\Big)dt \tag{4.4}$$
$$\leqslant(1-\theta_1)\int_S^T E^{\frac{m-2}{2}}(t)\Big(\frac{1}{2}\|\nabla u(t)\|_2^2+\frac{1}{2}\|u_t(t)\|_2^2\Big)dt$$
$$+(1-\theta_2)\int_S^T E^{\frac{m-2}{2}}(t)\Big(\frac{1}{2}\|\Delta u(t)\|_2^2+\frac{1}{2}\|u_t(t)\|_2^2\Big)dt,$$

where $\gamma=\mathrm{Min}((1-\theta_1),(1-\theta_2))$. By (4.2), (4.3), and definition of $E(t)$, we get

$$\gamma\int_S^T E^{\frac{m}{2}}(t)dt\leqslant-\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega (u(t)u_t(t))_t dx\, dt \tag{4.5}$$
$$-\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega u(t)u_t(t)dx\, dt$$
$$+(3-\theta_1-\theta_2)\int_S^T E^{\frac{m-2}{2}}(t)\|u_t(t)\|_2^2 dt.$$

Using the definition of $E(t)$ and the following expression

$$\frac{d}{dt}\Big(E^{\frac{m-2}{2}}(t)\int_\Omega u(t)u_t(t)dx\Big)=E^{\frac{m-2}{2}}(t)\int_\Omega (u(t)u_t(t))_t dx$$
$$+\frac{m-2}{2}\int_S^T E^{\frac{m-2}{2}-1}(t)\frac{d}{dt}E(t)\int_\Omega u(t)u_t(t)dx\, dt,$$

inequality (4.5) becomes

$$\gamma\int_S^T E^{\frac{m}{2}}(t)dt\leqslant-\int_S^T \frac{d}{dt}\Big(E^{\frac{m-2}{2}}(t)\int_\Omega u(t)u_t(t)dx\Big)dt \tag{4.6}$$
$$-\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega u(t)|u_t(t)|^{m-2}u_t(t)dx\, dt$$
$$+\frac{m-2}{2}\int_S^T E^{\frac{m-2}{2}-1}(t)\frac{d}{dt}E(t)\int_\Omega u(t)u_t(t)dx\, dt$$

$$+ (3-\theta_1-\theta_2)\int_S^T E^{\frac{m-2}{2}}(t)\|u_t(t)\|_2^2 dt.$$

We estimate the terms on the right-hand side of (4.6) as follows.

For the first term, we have

$$\begin{aligned}(4.7)\quad & -\int_S^T \frac{d}{dt}\Big(E^{\frac{m-2}{2}}(t)\int_\Omega u(t)u_t(t)dx\Big)dx\,dt \\ &\leqslant \Big|E^{\frac{m-2}{2}}(t)\int_\Omega u(S)u_t(S)dx - E^{\frac{m-2}{2}}(t)\int_\Omega u(T)u_t(T)dx\Big| \\ &\leqslant E^{\frac{m-2}{2}}(t)\Big|\int_\Omega u(x,S)u_t(x,S)dx\Big| + E^{\frac{m-2}{2}}(t)\Big|\int_\Omega u(x,T)u_t(x,T)dx\Big| \\ &\leqslant cE^{\frac{m}{2}}(S)+cE^{\frac{m}{2}}(T)\leqslant cE^{\frac{m-2}{2}}(0)E(S)\leqslant cE(S).\end{aligned}$$

For the second term, we use the following Young inequality:

$$XY \leqslant \frac{\varepsilon}{\lambda_1}X^{\lambda_1}+\frac{1}{\lambda_2\varepsilon^{\frac{\lambda_2}{\lambda_1}}}Y^{\lambda_2},\quad X,Y\geqslant 0,\ \varepsilon>0 \text{ and } \frac{1}{\lambda_1}+\frac{1}{\lambda_2}=1,$$

with $\lambda_1=m$, $\lambda_2=\frac{m}{m-1}$.

By Lemma 2.3 and Lemma 4.1, we have

$$\begin{aligned}(4.8)\quad & -\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega u(t)|u_t(t)^{m-2}u_t(t)dx\,dt \\ &\leqslant \int_S^T E^{\frac{m-2}{2}}(t)(\varepsilon\|u(t)\|_m^m + c_\varepsilon\|u_t(t)\|_m^m)dt \\ &\leqslant \varepsilon c\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega |u(t)|^m dx\,dt + c_\varepsilon\int_S^T E^{\frac{m-2}{2}}(t)(-E'(t))dt \\ &\leqslant \varepsilon c\int_S^T E^{\frac{m}{2}}(t)dt + c_\varepsilon E(S).\end{aligned}$$

Using Young's, Poincare's inequalities, and (4.1), we obtain

$$\begin{aligned}(4.9)\quad & \frac{m-2}{2}\int_S^T E^{\frac{m-2}{2}-1}(t)\frac{d}{dt}E(t)\int_\Omega u(t)u_t(t)dx\,dt \\ &\leqslant \frac{m-2}{2}\int_S^T E^{\frac{m-2}{2}-1}(t)(-E'(t))\int_\Omega\Big(\frac12|u(t)|^2+\frac12|u_t(t)|^2\Big)dx\,dt \\ &\leqslant c\int_S^T E^{\frac{m-2}{2}}(t)(-E'(t))dt \leqslant cE^{\frac{m}{2}}(S)-E^{\frac{m}{2}}(T) \\ &\leqslant cE^{\frac{m}{2}-1}(0)E(S)\leqslant cE(S).\end{aligned}$$

For the last term of (4.6), by the embedding of $L^m(\Omega)\hookrightarrow L^2(\Omega)$, we have

$$(3-\theta_1-\theta_2)\int_S^T E^{\frac{m-2}{2}}(t)\|u_t(t)\|_2^2 dt \leqslant (3-\theta_1-\theta_2)\int_S^T E^{\frac{m-2}{2}}(t)(\|u_t(t)\|_m^m)^{\frac{2}{m}}dt$$

$$\leqslant (3-\theta_1-\theta_2)\int_S^T E^{\frac{m-2}{2}}(t)(-E'(t))^{\frac{2}{m}}dt.$$

We use the Young's inequality, with $\lambda_1 = \frac{m}{m-2}$, $\lambda_2 = \frac{m}{2}$, we obtain

$$\int_S^T E^{\frac{m-2}{2}}(t)(-E'(t))^{\frac{2}{m}}dt \leqslant \varepsilon c\int_S^T E^{\frac{m}{2}}(t)dt + c_\varepsilon\int_S^T(-E'(t))dt.$$

This implies

$$(3-\theta_1-\theta_2)\int_S^T E^{\frac{m-2}{2}}(t)\|u_t(t)\|_2^2dt \leqslant \varepsilon c\int_S^T E^{\frac{m}{2}}(t)dt + c_\varepsilon E(S). \tag{4.10}$$

By inserting (4.7), (4.8), (4.9) and (4.10) in (4.6), we arrive at

$$\gamma\int_S^T E^{\frac{m}{2}}(t)dt \leqslant \varepsilon c\int_S^T E^{\frac{m}{2}}(t)dt + c_\varepsilon E(S).$$

We choose ε small enough such that

$$\int_S^T E^{\frac{m}{2}}(t)dt \leqslant cE(S).$$

By taking $T\to\infty$, we get

$$\int_S^\infty E^{\frac{m}{2}}(t)dt \leqslant cE(S).$$

Using Komornik's integral inequality, we obtain the result. □

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ЕГЗИСТЕНИЦИЈА И СТАБИЛНОСТ РЕШЕЊА НЕЛИНЕАРНЕ ТИМОШЕНКОВЕ ЈЕДНАЧИНЕ СА ЧЛАНОВИМА ПРИГУШЕЊА И ПРИНУДЕ

Резиме. У овом раду разматрамо нелинеарну Тимошенкову једначину. Прво доказујемо решење локалне егзистенције Фаедо-Галеркиновом методом и, под одговарајућим претпоставкама са позитивном почетном енергијом, доказујемо да је локална егзистенција глобална у времену. Коначно, добијен је резултат из стабилности решења на основу Коморникове интегралне неједнакости.

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