THEORETICAL AND APPLIED MECHANICS
Volume 48 (2021) Issue 1, 29–37 DOI: https://doi.org/10.2298/TAM200615001S

LONGITUDINAL WAVES IN AN ELASTIC ROD CAUSED BY SUDDEN DAMAGE TO THE FOUNDATION

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Abstract. We study the problem of propagating longitudinal waves in an elastic rod connected to a locally damaged foundation through a thin elastic layer. The motion of the rigid foundation blocks is considered predetermined. We formulated the initial-boundary problem for the Klein–Gordon equation with a discontinuous right-hand side. The nonstationary fields of displacements, velocities, and deformations were investigated by the Laplace integral transformation method. Examples of sudden divergence of fragments of the foundation by a given value and their mutual separation at a constant speed are considered.

1. Introduction

The popularity of rod systems in modern technology forces researchers to develop adequate models for analyzing the behavior of rods under static and dynamic loads. The problems of interaction of rods with the environment deserve special attention. Demands for these studies are constantly arising, for example, in such areas as drilling wells [1–3], the operation of extended objects in areas of tectonic faults [4–8], construction on piles [9], mechanics of fiber composites [10], etc.

Currently, a number of publications on the dynamics of rods with complicated conditions (often nonlinear ones) on the side surface are known. In particular, the propagation of shock waves in rods with dry external friction was investigated in [11–14], and the dynamics of a rod with viscous and viscous-plastic external resistance was studied in [15] and [9, 16]. Cases of dynamic interaction of an elastic rod with a rigid-plastic matrix taking into account the effects of hardening and softening are considered in [17, 18].

These publications usually deal with semi-infinite or finite rods under the force or kinematic perturbation given at their end. At the same time, in the presence

2010 *Mathematics Subject Classification:* Primary 74H05; Secondary 74K10.
Key words and phrases: elastic rod, thin layer, damaged foundation, dynamics, longitudinal wave.

of damage to the foundation, there are problems of transmitting kinematic perturbations discontinuous in space from the foundation to the rod through its lateral surface. Examples of application of this approach to the problems of statics and stationary dynamics of the pipeline on a block foundation are given in [19–21].

The purpose of this paper is to study the nonstationary process of propagation of longitudinal waves caused by a given motion of fragments of the damaged foundation. To achieve this purpose, the initial-boundary value problem of the dynamics of the rod connected through a thin elastic layer with a damaged foundation is formulated. The analytical solution constructed by the Laplace integral transform method is illustrated by two examples of discontinuous displacements of the foundation.

2. Statement of the problem

Let the infinite elastic rod with a constant cross-section be connected to the absolutely rigid foundation through an elastic layer (Figure 1). At the initial point in time, the system is at rest. Subsequently, due to local damage, the foundation blocks diverge (i.e. part) with a given law of displacement. We intend to investigate the non-stationary process of rod deformation caused by the movement of the damaged foundation.

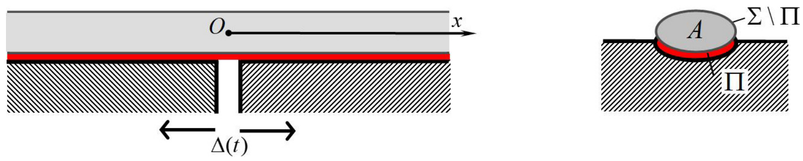

FIGURE 1. Scheme of the problem.

To describe the dynamics of the rod, we use the one-dimensional Bernoulli theory based on the classical hypothesis of a plane rigid cross section [22]. The behavior of a thin elastic layer is subordinated to Winkler's traditional model of the proportionality of stresses and relative displacements. Energy dissipation in the layer is not taken into account. Infinitely distant ends of the rod are free from external loads.

Let the abscissa axis pass through the center of mass of the cross-section of the rod. Then the dynamics equation together with Winkler's hypothesis will be:

$$(2.1)\quad EA\frac{\partial^2 u}{\partial X^2} + \int_\Sigma \tau ds = \rho A \frac{\partial^2 u}{\partial T^2}, \quad \tau = \begin{cases} -k(u-u^0), & s \in \Pi \\ 0, & s \in \Sigma\backslash\Pi \end{cases}, \quad X \in \mathbb{R},\ T > 0.$$

Here, X, T are the coordinate [m] and the time [s], $u(X,T)$ is the desired axial displacement of the rod [m], $u^0(X,T)$ is the given displacement of the foundation [m], $\tau(X,T)$ is the tangential stress on the lateral surface [Pa], A is the cross-sectional area [m^2], Σ is the cross-sectional contour, $\Pi \subseteq \Sigma$ is the part of the contour connected with the foundation, $\Sigma\backslash\Pi$ is the free part of the contour, E and ρ are Young's modulus [Pa] and the density [kg/m^3] of the rod material, and k is the coefficient of rigidity of the layer [Pa/m].

Assuming the hypothesis of plane rod sections (u, u^0 do not depend on s), we transform the contour integral:

$$\int_\Sigma \tau ds = -\int_\Pi k(u-u^0)ds = -k(u-u^0)\int_\Pi ds = -k\Pi(u-u^0).$$

Then instead of the relations in Eq. (2.1) we obtain:

$$EA\frac{\partial^2 u}{\partial X^2} - k\Pi(u-u^0) = \rho A\frac{\partial^2 u}{\partial T^2}, \quad X\in\mathbb{R},\ T>0. \tag{2.2}$$

It should be noted that in Eq. (2.2) Π is now the length of the contour Π [m].

Equations (2.2) should be solved under the following initial and boundary conditions:

$$u(X,0)=0, \quad \frac{\partial u}{\partial T}(X,0)=0, \qquad X\in\mathbb{R}; \tag{2.3}$$

$$EA\frac{\partial u}{\partial X}(\pm\infty,T)=0, \quad T>0.$$

Let us introduce the dimensionless variables $x = X/L$, $t = cT/L$, where $c = \sqrt{E/\rho}$ is the speed of propagation of the longitudinal wave [m/s], and $L = \sqrt{EA/(k\Pi)}$ is the characteristic linear dimension of the problem [m].

The initial-boundary value problem for the Klein–Gordon equation is an analogue of Eqs. (2.2), (2.3) in dimensionless coordinates:

$$u'' - u - \ddot{u} = -u^0, \quad x\in\mathbb{R},\ t>0; \tag{2.4}$$

$$u(x,0)=0, \quad \dot{u}(x,0)=0, \qquad x\in\mathbb{R};$$

$$u'(\pm\infty,t)=0, \quad t>0.$$

Here, the dash and dot indicate the partial derivatives with respect to dimensionless coordinate and time, respectively.

3. Analytical solution

Further, we consider a local damage to the foundation as its displacement discontinuity at the origin of coordinates:

$$u^0(x,t) = \tfrac{1}{2}\Delta(t)\,\mathrm{sgn}\,x,$$

where $\Delta(t)$ is the given function of time.

In this case, we construct the analytical solution for the problem in Eq. (2.4) by the method of integral Laplace transform over time [23, 24].

Let us denote the L-transform of the function $f(t)$ as follows:

$$f_L(s) = \int_0^\infty \exp(-st)f(t)dt,$$

where s is the transformation parameter.

At zero initial data we have $(\ddot{u})_L = s^2u_L$, so, in the image space, the problem in Eq. (2.4) is transformed into a boundary value problem for an ordinary differential equation with a discontinuous right-hand side:

$$u_L'' - (1+s^2)u_L = -\tfrac{1}{2}\Delta_L\,\mathrm{sgn}\,x, \quad x\in\mathbb{R}; \tag{3.1}$$

$$u'_L(\pm\infty) = 0.$$

The continuous piecewise differentiable solution for the problem in Eq. (3.1) is as follows:

$$u_L(x) = \frac{\Delta_L}{2}\frac{1-\exp\left(-\sqrt{1+s^2}|x|\right)}{1+s^2}\operatorname{sgn} x. \tag{3.2}$$

The desired displacement field is the original of the function in Eq. (3.2):

$$u(x,t) = \tfrac{1}{2}\Delta(t) * \left\{ \sin t - J_0(t) * \left(J_0\left(\sqrt{t^2-x^2}\right)H(t-|x|)\right)\right\}\operatorname{sgn} x. \tag{3.3}$$

Here, the symbol $(*)$ indicates the operation of convolution of the functions over time:

$$f * g = \int_0^t f(t-\tau)g(\tau)d\tau,$$

$J_0(t)$ is a zero-order Bessel function of the first kind [25], and $H(t)$ is the Heaviside function.

We have obtained the result in the form of Eq. (3.3) by taking into account the chains of the relations [23, 24]:

$$\frac{1-\exp\left(-\sqrt{1+s^2}|x|\right)}{1+s^2} = \frac{1}{1+s^2} - \frac{1}{\sqrt{1+s^2}}\frac{\exp\left(-\sqrt{1+s^2}|x|\right)}{\sqrt{1+s^2}};$$

$$(\sin t)_L = \frac{1}{1+s^2}, \quad (J_0(t))_L = \frac{1}{\sqrt{1+s^2}},$$

$$\left(J_0\left(\sqrt{t^2-x^2}\right)H(t-|x|)\right)_L = \frac{\exp\left(-\sqrt{1+s^2}|x|\right)}{\sqrt{1+s^2}}; \quad (f*g)_L = f_L g_L.$$

The fields of velocities and deformations, in turn, are found by the time and coordinate derivation expression given by Eq. (3.3):

$$\dot u(x,t) = \tfrac{1}{2}\dot\Delta(t) * \left\{\sin t - J_0(t) * \left(J_0\left(\sqrt{t^2-x^2}\right)H(t-|x|)\right)\right\}\operatorname{sgn} x, \tag{3.4}$$

$$u'(x,t) = \tfrac{1}{2}\Delta(t) * \left\{J_0\left(\sqrt{t^2-x^2}\right)H(t-|x|)\right\}.$$

4. Analysis and discussion

Further, let us consider some examples of calculation.

4.1. Example 1. Let us assume that $\Delta(t) = \Delta_0 H(t)$. This means that the two semi-boundless blocks of the foundation suddenly parted by the value Δ_0.

Then from Eqs. (3.3), (3.4) it follows that:

$$u(x,t) = \frac{\Delta_0}{2}\left\{1-\cos t - \int_{|x|}^{t}\int_{|x|}^{\tau} J_0(\tau-\eta)J_0\left(\sqrt{\eta^2-x^2}\right)d\eta\, d\tau H(t-|x|)\right\}\operatorname{sgn} x, \tag{4.1}$$

$$\dot u(x,t) = \frac{\Delta_0}{2}\left\{\sin t - \int_{|x|}^{t} J_0(t-\tau)J_0\left(\sqrt{\tau^2-x^2}\right)d\tau H(t-|x|)\right\}\operatorname{sgn} x,$$

$$u'(x,t) = \frac{\Delta_0}{2}\int_{|x|}^{t} J_0\left(\sqrt{\tau^2-x^2}\right)d\tau H(t-|x|).$$

Some results from Eq. (4.1), tabulated at $\Delta_0 = 1$ using the Mathcad software, are shown in Figures 2 and 3. The space and plane variants of plots are presented.

The analysis of these results shows that the space-time area of motion $(x, t) \in \mathbb{R} \times \mathbb{R}_+$ is divided by the characteristics $t = |x|$ into two zones: elastically undisturbed $t < |x|$ and elastically disturbed $t > |x|$. This can also be clearly seen in the plane graphs.

In the elastically undisturbed zone, before the arrival of the wave from the place of the foundation rupture, the rod parts $x > t$ and $x < -t$ perform a harmonic motion in antiphase as some absolutely solid bodies:

$$u(x,t) = \frac{\Delta_0}{2}(1 - \cos t)\,\mathrm{sgn}\,x, \quad \dot{u}(x,t) = \frac{\Delta_0}{2}\sin t\,\mathrm{sgn}\,x, \qquad u'(x,t) = 0.$$

When the perturbation arrives, the greatest deformation of the rod is observed at the origin and then the motions of the rod attenuate with increasing time; when $t \to \infty$, its state slowly moves to the position of static equilibrium, which is given in [19, 21]:

$$u(x,\infty) = \frac{\Delta_0}{2}(1 - \exp(-|x|))\,\mathrm{sgn}\,x, \quad \dot{u}(x,\infty) = 0, \quad u'(x,\infty) = \frac{\Delta_0}{2}\exp(-|x|).$$

In this example, the characteristics are the lines of weak rupture of acceleration and gradient of axial deformation. The velocity, displacement, and deformation during the transition through the characteristics change continuously.

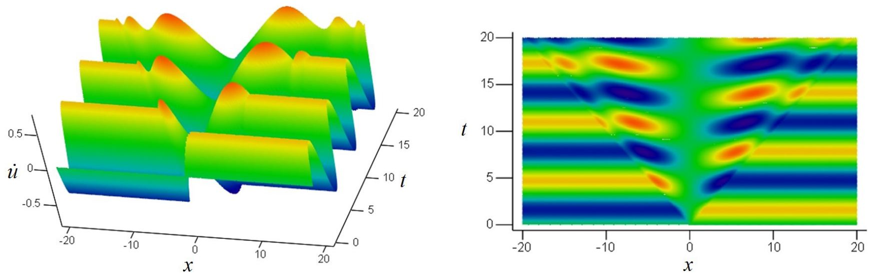

FIGURE 2. Distribution of the velocity in the rod in case $\Delta(t) = \Delta_0 H(t)$, $\Delta_0 = 1$.

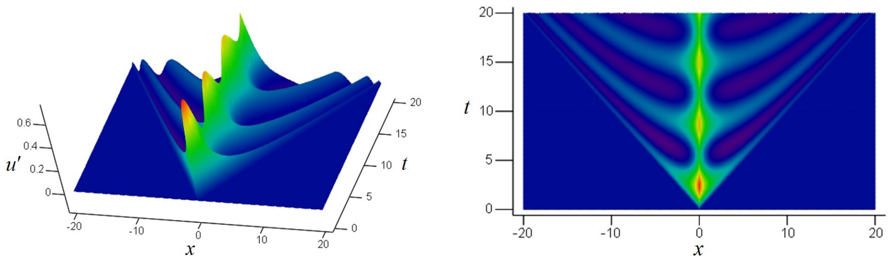

FIGURE 3. Deformation field in the rod in case $\Delta(t) = \Delta_0 H(t)$, $\Delta_0 = 1$.

4.2. Example 2. Let us assume that $\Delta(t) = \Delta_0 t H(t)$. This means that in real time the fragments of the damaged foundation diverge in different directions at a constant speed $V_0 = c\Delta_0/L$. Thus, the rod is involved in the movement too.

Therefore, according to Eqs. (3.3), (3.4),

$$u(x,t) = \frac{\Delta_0}{2}\Big\{t - \sin t - \int_{|x|}^{t} (t\tau) \int_{|x|}^{\tau} J_0(\tau-\eta) J_0\big(\sqrt{\eta^2 - x^2}\big) d\eta d\tau \, H(t-|x|)\Big\} \operatorname{sgn} x, \tag{4.2}$$

$$\dot{u}(x,t) = \frac{\Delta_0}{2}\Big\{1 - \cos t - \int_{|x|}^{t}\int_{|x|}^{\tau} J_0(\tau-\eta) J_0\big(\sqrt{\eta^2 - x^2}\big) d\eta \, d\tau H(t-|x|)\Big\} \operatorname{sgn} x,$$

$$u'(x,t) = \frac{\Delta_0}{2}\int_{|x|}^{t} (t-\tau) J_0\big(\sqrt{\tau^2 - x^2}\big) d\tau H(t-|x|).$$

Figures 4 and 5 show the individual results from Eq. (4.2) when $\Delta_0 = 1$.

As can be seen from these graphs, in the elastically undisturbed zone, the rod parts also move like some solid bodies. The value of the speed does not fluctuate around zero, as in the previous example, but near a given fixed value $\Delta_0 \operatorname{sgn} x$. Over time, the rod deformation in the elastically disturbed domain greatly increases.

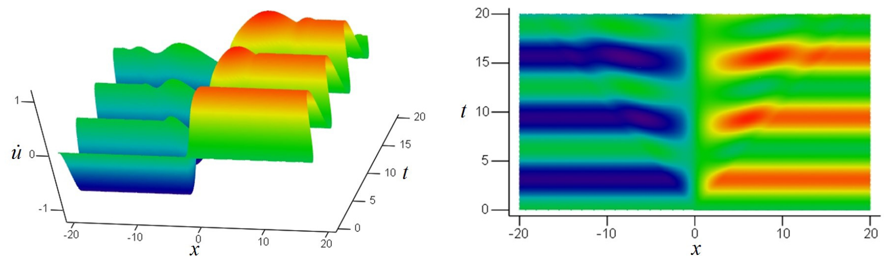

FIGURE 4. Distribution of the velocity in the rod in case $\Delta(t) = \Delta_0 t H(t)$, $\Delta_0 = 1$.

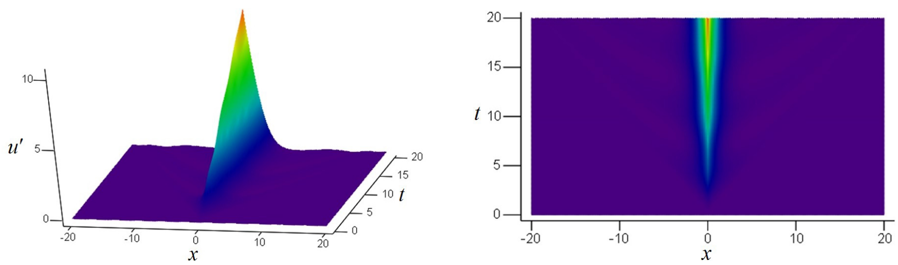

FIGURE 5. Deformation field in the rod in case$\Delta(t) = \Delta_0 t H(t)$, $\Delta_0 = 1$.

5. Conclusions

Using analytical means, we have proposed an effective model that allows investigating the nature of a nonstationary wave field in an elastic rod perturbed by the motion of a damaged rigid foundation, which is transmitted to the rod through a thin elastic layer.

It is established that in the elastically undisturbed zone the rod parts move in antiphase as some absolutely rigid bodies, similarly to single-mass oscillators. In the elastically disturbed zone, the wave fields are expressed through the convolution integral with Bessel functions and have an oscillatory character with an amplitude damping in time and coordinate.

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ЛОНГИТУДИНАЛНИ ТАЛАСИ У ЕЛАСТИЧНОМ ШТАПУ УЗРОКОВАНИ ИЗНЕНАДНИМ ОШТЕЋЕЊЕМ ОСНОВЕ

Резиме. Проучавамо проблем ширења лонгитудиналних таласа у еластичном штапу повезаном танким еластичним слојем на локално оштећену основу. Кретање крутих блокова основе сматра се унапред одређеним. Формулисан је проблем почетног граничног услова за Клајн-Гордонову једначину са дисконтинуираном десном страном. Лапласовом интегралном трансформацијом испитивана су нестационарна поља померања, брзина и деформација. Разматрани су примери изненадног разилажења фрагмената основе за задату вредност константе брзине њиховог међусобног раздвајања.

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(Received 15.06.2020.)
(Revised 18.12.2020.)
(Available online 18.03.2021.)

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