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DEMCHENKO'S NONHOLONOMIC CASE OF A GYROSCOPIC BALL ROLLING WITHOUT SLIDING OVER A SPHERE AFTER HIS 1923 BELGRADE DOCTORAL THESIS

Vladimir Dragović, Borislav Gajić, and Božidar Jovanović

Dedicated to Anton Bilimović and his scientific school on the occasions of the 50^{th} anniversaries of the deaths of Anton Bilimović and Vasilije Demchenko.

Abstract. We present an integrable nonholonomic case of rolling without sliding of a gyroscopic ball over a sphere. This case was introduced and studied in detail by Vasilije Demchenko in his 1923 doctoral dissertation defended at the University of Belgrade, with Anton Bilimović as the advisor. These results are absolutely unknown to modern researchers. The study is based on the C. Neumann coordinates and the Voronec principle. By using the involved technique of elliptic functions, a detailed study of motion is performed. Several special classes of trajectories are distinguished, including regular and pseudo-regular precessions. The so-called remarkable trajectories, introduced by Paul Painlevé and Anton Bilimović, are described in the present case. The historical context of the results as well as their place in contemporary mechanics are outlined.

Contents

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Key words and phrases: nonholonimic dynamics, rolling without sliding, C. Neumann coordinates, elliptic functions, elliptic integrals, Voronec principle, regular and pseudo-regular precessions, remarkable trajectories.

1. Introduction

We present an integrable nonholonomic case of rolling without sliding of a gyroscopic ball over a sphere. This case was introduced and studied in detail by Vasilije Demchenko in his 1923 doctoral dissertation defended at the University of Belgrade, with Anton Bilimović as the advisor. These results are absolutely unknown to modern researchers. The study is based on the C. Neumann coordinates and the Voronec principle. By using the involved technique of elliptic functions, a detailed study of motion is performed. Several special classes of trajectories are distinguished, including regular and pseudo-regular precessions. The so-called remarkable trajectories, introduced by Paul Painlevé and Anton Bilimović, are described in the present case. The historical context of the results as well as their place in contemporary science are outlined.

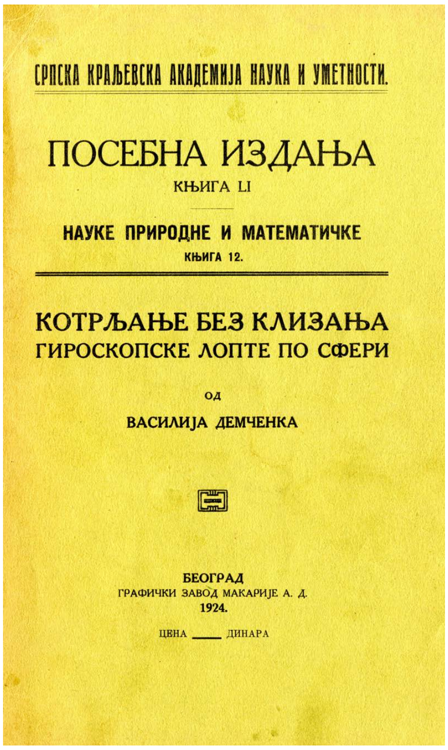
Anton Bilimović (1879–1970) was an outstanding representative of the Russian scientific elite who ended up in Belgrade as a result of the turmoil induced by the revolution in Russia. As an already established scientist and a former Rector of Malorosiisk University in Odessa, Bilimović made a tremendous contribution to the further development and organization of mathematics and mechanics in Belgrade, Serbia, and Yugoslavia. A detailed biography of Academician Bilimović can be found in [**41**, **42**]. A comprehensive study of Anton Bilimović's scientific school in Serbia till the mid-1970's was given in [**60**]. It is presented there as an integral part of G. K. Suslov's school. Namely, Bilimović's advisor was Peter Vasilievich Voronec (1871–1923), a distinguished pupil of Gavril Konstantinovich Suslov (1857–1935). Also, [**60**] describes the results of Serbian students of the Bilimović school: Tatomir Andjelić (1903–1993), Rastko Stojanović (1926–1972), Veljko Vujičić (1929–2020), Božidar Vujanović (1930–2014), and Djordje Djukić (1943–2019). In his editorial concluding remarks in [**60**], Andjelić also listed works of Viacheslav Zhardecki (1896–1962) and Djordje Mušicki (1921–2018) as parts of the school.

Bilimović was not an isolated example of a Russian scientist who came to Belgrade at that time. Along with him, a notable scientist and his former scientific advisor, P. V. Voronec also came to Belgrade. According to one of the most romantic Belgrade urban stories, Peter and his son, Konstantin Voronec (1902–1974), met for the last time as recruited soldiers of different units of "Whites". Feeling that the civil war was not developing in favor of the "Whites" and that it was coming to an end, they agreed to meet in Belgrade, on the condition that the one who came first would wait for the other one every day at noon in front of the National Theatre. Peter, the father, reached Belgrade the first. Waiting for his son day after day, he lost patience and decided to go back and search for Konstantin. He requested Bilimović to take over his duty, and wait for Konstantin in Belgrade. Peter and Konstantin did not manage to find each other. Konstantin crossed Romania on foot, and came to Belgrade to be met and taken care of by Bilimović. At the same time, Peter ended up in Ukraine, got ill and died in 1923. The communications between the two countries were so poor at that time that Konstantin learned about his father's fate only after he had come to Paris in the 1930's. A comprehensive description of this dramatic period of the Voronec family was given in [**87**].

Along with the son, Bilimović also took care of a student of Peter Voronec, Vasilije Gregorevich Demchenko (1898–1972). In 1923, Demchenko prepared and defended the doctoral dissertation:

> *Rolling without sliding of a gyroscopic ball over a sphere*, University of Belgrade, 1923, pp. 94. (in Serbian)

The dissertation was published as a separate book in Belgrade in 1924, see [**35**]. The work was motivated by and based on the results of his teachers, Peter Voronec and Anton Bilimović, see [**93**, **96**] and [**13**]. The dissertation was accepted for the final doctoral examination on the meeting of the Faculty of Philosophy of the University of Belgrade, on November 15, 1923, based on the report of the members of the examination committee: Anton Bilimović, Mihailo Petrović (1868–1943), and Milutin Milanković (1879–1952). Petrović was a founding father of modern Serbian mathematics, see for example [**43**, **44**, **83**]. Milanković is best known for developing his mathematical theories of climate, which, in particular, produced one of the most significant theories relating Earth motions and long-term climate change, nowadays called Milanković cycles [**79**] and [**80**].


СРПСКА КРАЉЕВСКА АКАДЕМИЈА НАУКА И УМЕТНОСТИ.

ПОСЕБНА ИЗДАЊА

КЊИГА LI

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КЊИГА 12.

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Figure 1. The cover page of [**35**].

The dissertation consists of a Preface, six chapters and a Résumé in French. In the last sentence of the Preface, Demchenko expresses the deepest gratitude to his teachers, Professors P. Voronec and A. Bilimović.

Demchenko classifies into three categories the material related to nonholonomic problems of rolling of a surface over another surface accumulated by that moment: rolling of a ball over a surface, rolling of a surface over a plane, and rolling of a surface over a sphere. The case studied in the dissertation relates to the last category. It is interesting since it reduces to elliptic quadratures, as was indicated in [**96**]. In a sense, this work is also a continuation of works of Bobilev and Zhukovsky, [**17**] and [**105**], who considered special cases of rolling of a gyroscopic ball over the plane.

2. Demchenko's results: a gyroscopic ball rolling without sliding over a sphere

2.1. The dissertation chapter by chapter. The subject of the dissertation is at the interface of differential geometry of curves and surfaces, nonholonomic mechanics and the theory of elliptic functions and integrals. It is written in a clear and illuminating fashion, demonstrating the author's full expertise in each of these fields and mastery in their synergy. The study is detailed and very well rounded. The obtained results are complete, numerous, interesting, transparent and rigorous. The exposition is elegant, with a well-thought-our organization which connects various chapters and subchapters into a fully focused and convergent material. There is an excelent balance between the details and the global line of the presentation. Let us also observe that the dissertation is written in clean and smooth Serbian, with a few instances of constructions which could be seen as more natural in Russian than in Serbian.

Chapter 1, Kinematics of a rigid body rolling over a fixed surface, consists of three subchapters. Section 1.1 Motion over the surface of a Darboux trihedral; Section 1.2 Kinematic elements of a rolling rigid body, in Neumann coordinates; Section 1.3 The case of rolling without sliding.

Chapter 2, The equations of motion of a rigid body, in a moving frame with an arbitrary motion with respect to the rigid body, has four subchapters. Section 2.1 The equations of motion of a free rigid body in a moving frame; Section 2.2 The equations of motion of a non-free rigid body; Section 2.3 Applications to rolling without sliding; Section 2.4 Particular cases.

Chapter 3, Voronec Principle, has three subchapters. Section 3.1 A principle similar to Hamiltonian which is applicable to nonholonomic systems; Section 3.2 Application to rolling without sliding over a fixed surface; Section 3.3 Rolling of gyroscopic bodies.

Chapter 4, Reducing to quadratures, consists of the following subchapters: Section 4.1 Bobilev problem and its generalization; Section 4.2 Kinematic elements and expressions for the kinetic energy; 4.3 Differential equations of motion and first integrals; Section 4.4 Calculation of coordinates u and v; Section 4.5 Calculation of cyclic coordinates u_1, v_1, and ϑ; Section 4.6 A particular solution.

Chapter 5, Solution in the finite form, has the following subchapters: Section 5.1 Inversion of the elliptic integral. Discriminant; Section 5.2 Arguments

a_0, b_0, a, b; Section 5.21 Calculation of v; Section 5.22 Calculation of s, τ; Section 5.3 Arguments a_1 and b_1; Section 5.31 Calculation of v_1; Section 5.32 Calculation of ϑ; Section 5.4 Elliptic and mechanical constants; Section 5.5 Discussion of elliptic arguments; Section 5.51 Discussion of functions $\Phi, \Phi_1, \Phi', \Phi'_1$; Section 5.6 Discussion of the obtained formulae; Section 5.7 The general interpretation of motion; Section 5.8 Special cases of motion.

Chapter 6, The special cases of motion, is the last one. It consists of: Section 6.1 Constants s_0, n_0 and x'. The characteristic curve of degree 3; Section 6.2 Approximate calculation of motion; Section 6.3 Regular precession. Perturbation of motion; Section 6.4 Pseudo-regular precession; Section 6.5 Rolling of a ball over a sphere; Section 6.6 Stationary motion. Perturbed motion; Section 6.7 Remarkable trajectories.

The first three chapters are more general and introductory. The second part, consisting of chapters 4–6, is more specific, contains the original solution to the posed problem and a detailed analysis of the obtained solution. This second part occupies the major part of the text.

2.2. Rolling without slipping of a body over a surface in the Neumann variables. Chapter 1 introduces very convenient coordinates of C. Neumann, [**81**], in the study of rolling of one surface over another. Suppose that a body T bounded by its surface S is rolling over a surface S_1. Let $O_1\mathbf{x}_1\mathbf{y}_1\mathbf{z}_1$ be the coordinate frame fixed in space and let $O\mathbf{xyz}$ be the frame attached to the body with the same orientation[1]. Let

$$x_1 = x_1(u_1, v_1), \qquad y_1 = y_1(u_1, v_1), \qquad z_1 = z_1(u_1, v_1), \tag{2.1}$$

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v) \tag{2.2}$$

be the parameterizations of S_1 and S in the corresponding coordinates of the given frames, where u_1, v_1 are the Gauss coordinates defined along the principle curvature lines of S_1 and, similarly, u, v are the Gauss coordinates of S.

Let M be their point of contact and $\mathbf{n}_1$ and $\mathbf{n}$ the unit vectors normal to S_1 and S respectively, such that the frames $M\mathbf{u}_1\mathbf{v}_1\mathbf{n}_1$ and $M\mathbf{uvn}$ are positively oriented. Here $\mathbf{u}_1, \mathbf{v}_1, \mathbf{u}, \mathbf{v}$ are unit tangent vectors to the point of contact M of the coordinate lines u_1, v_1 of S_1 and the coordinate lines u, v of S, respectively. Then the *Neumann coordinates* of T are: the Gauss coordinates u, v, u_1, v_1 at M and the angle ϑ between $\mathbf{v}_1$ and $\mathbf{u}$.

Given the moving frames $O\mathbf{xyz}$, $M\mathbf{u}_1\mathbf{v}_1\mathbf{n}_1$, and $M\mathbf{uvn}$, the following angular velocities are defined:

ω: of $O\mathbf{xyz}$ with respect to $O_1\mathbf{x}_1\mathbf{y}_1\mathbf{z}_1$, i.e., the angular velocity of the body T,

ω_1: of $O\mathbf{xyz}$ with respect to $M\mathbf{uvn}$,

ω_2: of $M\mathbf{uvn}$ with respect to $M\mathbf{u}_1\mathbf{v}_1\mathbf{n}_1$,

ω_3: of $M\mathbf{u}_1\mathbf{v}_1\mathbf{n}_1$ with respect to $O_1\mathbf{x}_1\mathbf{y}_1\mathbf{z}_1$,

related by $\omega = \omega_1 + \omega_2 + \omega_3$.

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[1] As positive orientation Demchenko used what would nowadays be negative orientation. This is the reason the signs in several equations differ from the signs we used to have.

The nonholonomic constraint that the body T rolls without slipping over the surface S_1 is usually given by the condition that the current point of contact M considered in rest in the moving frame also has zero velocity in the space frame[2]. Here, the velocity of the point M is different from zero. Namely, Demchenko considers the point of contact M as a function of time in the coordinates $u_1(t), v_1(t)$, as a "trace" of the body T over S_1. The corresponding "trace" on S is given by the functions $u(t), v(t)$. Let $\mathfrak{v}^1$, $\mathfrak{v}$ be the vectors of absolute and relative velocities of M, i.e., the time derivatives of (2.1) and (2.2), respectively. Then they are related by the expression $\mathfrak{v}^1 = \mathfrak{v} + \mathfrak{w}$[3]. The condition that the body T rolls without slipping over the surface S_1 is then given by

$$\mathfrak{v}^1 = \mathfrak{v}. \tag{2.4}$$

The condition (2.4) is equivalent to (2.3).

Denote by $s, \tau, n, \mathfrak{w}_{\mathbf{u}}, \mathfrak{w}_{\mathbf{v}}, \mathfrak{w}_{\mathbf{n}}$ the projections of the angular velocity ω and $\mathfrak{w}$ on the axes $\mathbf{u}, \mathbf{v}, \mathbf{n}$. It is clear that $\mathfrak{w}_{\mathbf{n}} = 0$. In Section 1.2, the formulae (8–10) are derived which express $s, \tau, n, \mathfrak{w}_{\mathbf{u}}, \mathfrak{w}_{\mathbf{v}}$ as homogenous linear functions of the time derivatives of the Neumann coordinates $\dot{u}, \dot{v}, \dot{u}_1, \dot{v}_1, \dot{\vartheta}$, and vice versa. For further reference we will provide the equations:

$$\begin{aligned}
s &= -\frac{D''}{\sqrt{G}}\dot{v} - \frac{D_1''}{\sqrt{G_1}}\dot{v}_1 \sin\vartheta - \frac{D_1}{\sqrt{E_1}}\dot{u}_1 \cos\vartheta, \\
\tau &= \frac{D}{\sqrt{E}}\dot{u} - \frac{D_1}{\sqrt{E_1}}\dot{u}_1 \sin\vartheta + \frac{D_1''}{\sqrt{G_1}}\dot{v}_1 \cos\vartheta, \\
n &= -\dot{\vartheta} + \frac{1}{2\sqrt{E_1 G_1}}\Big(\frac{\partial E_1}{\partial v_1}\dot{u}_1 - \frac{\partial G_1}{\partial u_1}\dot{v}_1\Big) + \frac{1}{2\sqrt{EG}}\Big(\frac{\partial E}{\partial v}\dot{u} - \frac{\partial G}{\partial u}\dot{v}\Big),
\end{aligned} \tag{2.5}$$

for which Demchenko refers to **[95]**. Here and further E, F, G, D, D', D'' are the coefficients of the first and the second fundamental forms of S. Similarly, E_1, F_1, G_1, D_1, D_1', D_1'' are the coefficients of the first and the second fundamental forms of S_1. The choice of the Gauss coordinates gives $F = 0, D' = 0$ and, similarly, $F_1 = 0, D_1' = 0$.

The condition (2.4) gives the differential constraints $\mathfrak{w}_{\mathbf{u}} = 0$ and $\mathfrak{w}_{\mathbf{v}} = 0$ for rolling without sliding as expressed in (1) of Section 1.7:

$$\sqrt{E_1}\dot{u}_1 = -\sqrt{E}\dot{u}\sin\vartheta + \sqrt{G}\dot{v}\cos\vartheta, \quad \sqrt{G_1}\dot{v}_1 = \sqrt{E}\dot{u}\cos\vartheta + \sqrt{G}\dot{v}\sin\vartheta. \tag{2.6}$$

The equations of motion of rolling without slipping of the rigid body T over the surface S_1 are derived in two different equivalent ways. In Chapter 2 they are derived by using the general laws of mechanics, while in Chapter 3 they are derived by using the Voronec principle.

Let m be the mass of the body and $\mathbf{w}$ the velocity of the point O. It is assumed that the mass center of the body T is at the point O and that the axes $O\mathbf{x}$, $O\mathbf{y}$, and

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[2] In the usual vector notation with the standard orientation, the no-slipping condition is given in the form which is not used in the dissertation:

$$\omega \times \overrightarrow{OM} + \frac{d}{dt}\overrightarrow{O_1O} = 0. \tag{2.3}$$

[3] The velocity $\mathfrak{w}$ is usually expressed as the left hand side of (2.3).

$O\mathbf{z}$ are the principal axes of the body. Let p, q, r be the components of the angular velocity ω and A, B, C be the components of the inertia tensor, and $\mathbf{w_x}, \mathbf{w_y}, \mathbf{w_z}$ be the components of the velocity $\mathbf{w}$ in the moving frame $O\mathbf{xyz}$. Then the kinetic energy of the body is given by

$$T = \frac{m}{2}\big(\mathbf{w}_\mathbf{x}^2 + \mathbf{w}_\mathbf{y}^2 + \mathbf{w}_\mathbf{z}^2\big) + \frac{1}{2}\big(Ap^2 + Bq^q + Cr^2\big). \tag{2.7}$$

Denote the momentum of the body T by $\mathfrak{M}$ and the angular momentum with respect to the point M by $\mathbf{G}^{(M)}$. We have general laws of mechanics written in the fixed reference frame $O_1\mathbf{x}_1\mathbf{y}_1\mathbf{z}_1$:

$$\dot{\mathfrak{M}} = \mathbf{F}, \qquad \dot{\mathbf{G}}^{(M)} + [\mathfrak{v}^1, \mathfrak{M}] = \mathbf{L}^{(M)}, \tag{2.8}$$

where $[\cdot\,,\cdot]$ is the vector product[4], $\mathbf{F}$ is the sum of all forces, and $\mathbf{L}^{(M)}$ is the torque of all forces applied to the body T with respect to M. Note that the forces of reactions of constraints do not induce torque with respect to M.

From the equations (2.8) written in the moving frames, (2.6) and (2.5), by using quite interesting manipulations with different projections of the angular velocities $\omega, \omega_1, \omega_2, \omega_3$ and derivations of the kinetic energy T and the kinetic energy written in terms of $\mathfrak{m}_\mathbf{u}, \mathfrak{w}_\mathbf{v}, \mathfrak{m}_\mathbf{n}, s, \tau, n$,

$$\bar{T}(\mathfrak{m}_\mathbf{u}, \mathfrak{w}_\mathbf{v}, \mathfrak{m}_\mathbf{n}, s, \tau, n) = T(\mathbf{w}_\mathbf{x}, \mathbf{w}_\mathbf{y}, \mathbf{w}_\mathbf{z}, p, q, r), \tag{2.9}$$

the system of eight differential equations in eight unknown functions of time: $u, v, u_1, v_1, \vartheta, s, \tau, n$ (or, equivalently, $u, v, \vartheta, u_1, v_1, \dot{u}, \dot{v}, \dot{\vartheta}$) is derived.

2.3. The Voronec principle. In Chapter 3, Section 3.1, Demchenko recalls the derivation of the Voronec principle for nonholonomic systems [**93**]. Consider the nonholonomic system with the kinetic energy $T = T(t, q_s, \dot{q}_s)$ $(s = 1, \dots, n+k)$, the generalized forces Q_s that correspond to the coordinates q_s, and the time-dependent nonhomogeneous nonholonomic constraints

$$\dot{q}_{n+\nu} = \sum_{i=1}^{n} a_{\nu i}(q,t)\dot{q}_i + a_\nu(q,t) \qquad (\nu = 1, 2, \dots, k). \tag{2.10}$$

Let Θ be the kinetic energy T after imposing the constraints (2.10) and let K_ν be the partial derivative of the kinetic energy T with respect to $\dot{q}_\nu$ restricted to the constrained subspace defined by (2.10):

$$\Theta(t, q_1, \dots, q_{n+k}, \dot{q}_1, \dots, \dot{q}_n) = T(t, q_1, \dots, q_{n+k}, \dot{q}_1, \dots, \dot{q}_{n+k}),$$

$$K_\nu(t, q_1, \dots, q_{n+k}, \dot{q}_1, \dots, \dot{q}_n) = \frac{\partial T}{\partial \dot{q}_{n+\nu}}(t, q_1, \dots, q_{n+k}, \dot{q}_1, \dots, \dot{q}_{n+k}) \quad (\nu = 1, \dots, k).$$

Based on the Lagrange-d'Alembert principle, following Voronec [**93**], the equations of motion of the given noholonomic system are derived in the form without Lagrange multipliers:

$$\frac{d}{dt}\frac{\partial \Theta}{\partial \dot{q}_i} = \frac{\partial \Theta}{\partial q_i} + Q_i + \sum_{\nu=1}^{k} a_{\nu i}\Big(\frac{\partial \Theta}{\partial q_{n+\nu}} + Q_{n+\nu}\Big) \tag{2.11}$$

[4] Here, since $[\mathbf{x}, \mathbf{y}] = \mathbf{z}$, the sign differs from the usual one.

$$+\sum_{\nu=1}^{k} K_\nu\Big(\sum_{j=1}^{n} A_{ij}^{(\nu)}\dot q_j + A_j^{(\nu)}\Big) \quad (i=1,\dots,n),$$

where the components $A_{ij}^{(\nu)}$ and $A_i^{(\nu)}$ are functions of the time t and the coordinates $q_1,\dots,q_{n+k}$ given by

$$A_{ij}^{(\nu)} = \Big(\frac{\partial a_{\nu i}}{\partial q_j} + \sum_{\mu=1}^{k} a_{\mu j}\frac{\partial a_{\nu i}}{\partial q_{n+\mu}}\Big) - \Big(\frac{\partial a_{\nu j}}{\partial q_i} + \sum_{\mu=1}^{k} a_{\mu i}\frac{\partial a_{\nu j}}{\partial q_{n+\mu}}\Big),$$

$$A_i^{(\nu)} = \Big(\frac{\partial a_{\nu i}}{\partial t} + \sum_{\mu=1}^{k} a_{\mu}\frac{\partial a_{\nu i}}{\partial q_{n+\mu}}\Big) - \Big(\frac{\partial a_{\nu}}{\partial q_i} + \sum_{\mu=1}^{k} a_{\mu i}\frac{\partial a_{\nu}}{\partial q_{n+\mu}}\Big).$$

It is interesting that the equations can be written in a compact form by using a formal integral expression referred to as the *Voronec principle*:

$$\int_{t_1}^{t_2}\Big[\delta\Theta + \sum_{i=1}^{n+k} Q_i\delta q_i + \sum_{\nu=1}^{k} K_\nu\Big(\frac{d}{dt}\delta q_{n+\nu} - \delta\dot q_{n+\nu}\Big)\Big]dt = 0, \tag{2.12}$$

where virtual displacements $\delta q_1,\dots,\delta q_n$ are arbitrary and equal to zero at the endpoints of a trajectory $q(t)$ (for $t=t_1$ and $t=t_2$), while $\delta q_{n+1},\dots,\delta q_{n+k}$ are determined from the homogeneous constraints

$$\delta q_{n+\nu} = \sum_{i=1}^{n} a_{\nu i}\delta q_i \qquad (\nu = 1,2,\dots,k). \tag{2.13}$$

Here $\frac{d}{dt}\delta q_{n+\nu} - \delta\dot q_{n+\nu}$ are calculated according to the expressions (2.10), (2.13) and using the rule:

$$\frac{d}{dt}\delta q_i - \delta\dot q_i = 0 \qquad (i=1,2,\dots,n).$$

In the case where all the considered objects do not depend on the variables $q_{n+\nu}$, the system is known as the *Chaplygin system* and the equations (2.11) as the *Chaplygin equations*. This is the reason Bilimović used the notion Chaplygin–Voronec equations (see [**2**]).

2.4. The Voronec principle and rolling of a body over a surface. In Section 3.2, following [**96**], Demchenko applied the Voronec principle to the above problem of rolling without slipping of a body over a surface. The nonholonomic constraints are given by (2.6). One can choose $\dot u_1$ and $\dot v_1$ as dependent velocities. The corresponding generalized impulses K_1 and K_2 are defined as

$$K_1(u,v,\vartheta,u_1,v_1,\dot u,\dot v,\dot\vartheta) = \frac{\partial T}{\partial \dot u_1}, \quad K_2(u,v,\vartheta,u_1,v_1,\dot u,\dot v,\dot\vartheta) = \frac{\partial T}{\partial \dot v_1},$$

where $T = T(u,v,\vartheta,u_1,v_1,\dot u,\dot v,\dot\vartheta,\dot u_1,\dot v_1)$ is the kinetic energy (2.7) in the Neumann variables and the constraints (2.6) are imposed after the taking of partial derivatives.

From now on, Θ denotes the kinetic energy (2.9) as a function of the angular velocities s,τ,n taking into account the constraints $\mathfrak{w}_\mathbf{u}=0, \mathfrak{w}_\mathbf{v}=0, \mathfrak{w}_\mathbf{n}=0$, while

$\bar{\Theta}(u, v, \vartheta, u_1, v_1, \dot{u}, \dot{v}, \dot{\vartheta})$ denotes the kinetic energy $T(u, v, \vartheta, u_1, v_1, \dot{u}, \dot{v}, \dot{\vartheta}, \dot{u}_1, \dot{v}_1)$ after imposing the constraints (2.6). Then

$$\begin{aligned} K_1 &= M\sqrt{E_1}\big[(\epsilon s - \rho_u n)\cos\vartheta + (\epsilon\tau - \rho_v n)\sin\vartheta\big] \\ &\quad + \frac{1}{2\sqrt{E_1 G_1}}\frac{\partial E_1}{\partial v_1}\frac{\partial\Theta}{\partial n} - \frac{D_1}{\sqrt{E_1}}\Big(\frac{\partial\Theta}{\partial s}\cos\vartheta + \frac{\partial\Theta}{\partial\tau}\sin\vartheta\Big), \\ K_2 &= M\sqrt{G_1}\big[(\epsilon s - \rho_u n)\sin\vartheta - (\epsilon\tau - \rho_v n)\cos\vartheta\big] \\ &\quad + \frac{1}{2\sqrt{E_1 G_1}}\frac{\partial G_1}{\partial u_1}\frac{\partial\Theta}{\partial n} + \frac{D_1''}{\sqrt{G_1}}\Big(\frac{\partial\Theta}{\partial\tau}\cos\vartheta - \frac{\partial\Theta}{\partial s}\sin\vartheta\Big). \end{aligned}$$

Here ρ_u, ρ_v, ϵ are the coordinates of $\overrightarrow{OM}$ in the coordinate system $M\mathbf{uvn}$. In these expressions, s, τ, n should be expressed as functions of $\dot{u}, \dot{v}, \dot{\vartheta}$ by using (2.5) and the constraints (2.6).

Bearing in mind that virtual displacements satisfy

$$\begin{aligned} \sqrt{E_1}\delta u_1 &= -\sqrt{E}\delta u\sin\vartheta + \sqrt{G}\delta v\cos\vartheta, \\ \sqrt{G_1}\delta v_1 &= \sqrt{E}\delta u\cos\vartheta + \sqrt{G}\delta v\sin\vartheta, \end{aligned} \tag{2.14}$$

the coefficients in (2.12) of terms that contain K_1 and K_2 have the form

$$\begin{aligned} &\frac{1}{\sqrt{E_1}}\big[\sqrt{E}(n\delta u - \dot{u}n')\cos\vartheta + \sqrt{G}(n\delta v - \dot{v}n')\sin\vartheta\big], \\ &\frac{1}{\sqrt{G_1}}\big[\sqrt{E}(n\delta u - \dot{u}n')\sin\vartheta - \sqrt{G}(n\delta v - \dot{v}n')\cos\vartheta\big], \end{aligned}$$

where

$$n' = -\delta\vartheta + \frac{1}{2\sqrt{GE}}\Big(\frac{\partial E}{\partial v}\delta u - \frac{\partial G}{\partial u}\delta v\Big) + \frac{1}{2\sqrt{G_1 E_1}}\Big(\frac{\partial E_1}{\partial v_1}\delta u_1 - \frac{\partial G_1}{\partial u_1}\delta v_1\Big).$$

If one denotes

$$K_1' = \frac{K_1}{\sqrt{E_1}}\cos\vartheta + \frac{K_2}{\sqrt{G_1}}\sin\vartheta, \quad K_2' = \frac{K_1}{\sqrt{E_1}}\sin\vartheta - \frac{K_2}{\sqrt{G_1}}\cos\vartheta,$$

the Voronec principle (2.12) can be written in the form

$$\int_{t_1}^{t}\big[\delta\bar{\Theta} + \delta U + K_1'\sqrt{E}(n\delta u - \dot{u}n') + K_2'\sqrt{G}(n\delta v - \dot{v}n')\big]dt.$$

Here $U(u, v, \vartheta, u_1, v_1)$ is a *force function* (negative potential energy).

By using the expression for n', (2.14), and setting the terms that contain independent variations $\delta u, \delta v, \delta\vartheta$ to zero, one gets the equations of motion in the form

$$\begin{aligned} \frac{d}{dt}\frac{\bar{\Theta}}{\partial\dot{u}} - \frac{\partial(\bar{\Theta} + U)}{\partial u} &= \sqrt{E}\Big[-\frac{\partial(\bar{\Theta} + U)}{\partial u_1}\frac{\sin\vartheta}{\sqrt{E_1}} + \frac{\partial(\bar{\Theta} + U)}{\partial v_1}\frac{\cos\vartheta}{\sqrt{G_1}} - K_1'\dot{\vartheta}\Big] \\ &\quad - (\Delta_2 K_1' + \Delta_1 K_2')\sqrt{EG}\dot{v}, \\ \frac{d}{dt}\frac{\bar{\Theta}}{\partial\dot{v}} - \frac{\partial(\bar{\Theta} + U)}{\partial v} &= \sqrt{G}\Big[\frac{\partial(\bar{\Theta} + U)}{\partial u_1}\frac{\cos\vartheta}{\sqrt{E_1}} + \frac{\partial(\bar{\Theta} + U)}{\partial v_1}\frac{\sin\vartheta}{\sqrt{G_1}} - K_2'\dot{\vartheta}\Big] \end{aligned}$$

$$+ (\Delta_2 K_1' + \Delta_1 K_2')\sqrt{EG}\dot{u},$$

$$\frac{d}{dt}\frac{\bar{\Theta}}{\partial\dot{\vartheta}} - \frac{\partial(\bar{\Theta}+U)}{\partial\vartheta} = K_1'\sqrt{E}\dot{u} + K_2'\sqrt{G}\dot{v},$$

where

$$2\Delta_1 = \frac{1}{\sqrt{G}}\frac{\partial \ln E}{\partial v} - \frac{\sin\vartheta}{\sqrt{G_1}}\frac{\partial \ln E_1}{\partial v_1} - \frac{\cos\vartheta}{\sqrt{E_1}}\frac{\partial \ln G_1}{\partial u_1},$$

$$2\Delta_2 = \frac{1}{\sqrt{E}}\frac{\partial \ln G}{\partial u} - \frac{\sin\vartheta}{\sqrt{E_1}}\frac{\partial \ln G_1}{\partial u_1} - \frac{\cos\vartheta}{\sqrt{G_1}}\frac{\partial \ln E_1}{\partial v_1}.$$

These three differential equations of the second order, together with the two constraints (2.6) give a system of eight equations with eight unknown variables u, v, ϑ, $u_1, v_1, \dot{u}, \dot{v}, \dot{\vartheta}$.

Let us mention that on the occasion of the centennial of the seminal work of Voronec [**96**], Russian Journal of Nonlinear Dynamics published a Russian translation of the German original, prefaced with a short, but succinct text by A. S. Sumbatov. Sumbatov indicated that Voronec had been working on his principle for about 10 years. He also listed people who had successfully continued the work of Voronec: Ya. Shtaerman (1915), A. Bilimović (1916), and Yu. P. Bychkov (1965–67, 2004). Let us also mention the work of Bilimović [**14**], where he indicated the advantages of the Voronec equations with respect to other approaches to nonholonomic mechanics.

2.5. Rolling of a body with a gyroscope. The next step for Demchenko is to consider rolling of a body T with a gyroscope inside the body (Chapter 3, Section 3.3). He assumes that the axis of the gyroscope coincides with one of the principal axes of the body ($O\mathbf{z}$) and that the mass center of the body and of the gyroscope is the point O. It is also assumed that the forces applied to the gyroscope do not induce torque about the axis of the gyroscope. Thus, the gyroscope will rotate with a constant angular velocity $\tilde{\omega}$ around the axis $O\mathbf{z}$.

The kinetic energy of the system body + gyroscope takes the form (see (2.7) and (2.9))[5]

$$\tilde{T} = T + \frac{1}{2}\tilde{C}\tilde{\omega}^2 = \bar{T} + \frac{1}{2}\tilde{C}\tilde{\omega}^2,$$

where

$$T = \frac{m}{2}\big(\mathbf{w}_{\mathbf{x}}^2 + \mathbf{w}_{\mathbf{y}}^2 + \mathbf{w}_{\mathbf{z}}^2\big) + \frac{1}{2}\big(Ap^2 + Bq^2 + Cr^2\big),$$

p, q, r are the components of the angular velocity of the body T, A, B are $\mathbf{x}$ and $\mathbf{y}$ components of the inertia tensor of the system body + gyroscope, C and $\tilde{C}$ are the moments of inertia with respect to the axis $O\mathbf{z}$ of the body and the gyroscope in the body frame $O\mathbf{xyz}$, m is the mass of the system body + gyroscope, and $\bar{T}$ equals T written in the variables $\mathfrak{w}_{\mathbf{u}}, \mathfrak{w}_{\mathbf{v}}, \mathfrak{w}_{\mathbf{n}}, s, \tau, n$ (more details are given in Chapter 4, Section 4.2, see the equation (2.19) given below).

[5] In the presentation of the PhD thesis we completely followed the notation of the PhD thesis [**35**] except in denoting the mass, kinetic energies and the total angular momentum by m, T, $\bar{T}$, $\tilde{T}$ and $\tilde{\mathbf{G}}^{(M)}$, respectively.

The angular momentum of the system body + gyroscope with respect to the point M is $\tilde{\mathbf{G}}^{(M)} = \mathbf{G}^{(M)} + \kappa$, where $\mathbf{G}^{(M)}$ is the angular momentum of the system with the gyroscope when the $\mathbf{z}$-component $\tilde{\omega}$ of the angular velocity of the gyroscope is set to zero, and $\kappa = k\mathbf{z}$, $k = \tilde{C}\tilde{\omega}$. Due to the presence of the gyroscope, the second equation in (2.8) (also written in the fixed reference frame $O_1\mathbf{x}_1\mathbf{y}_1\mathbf{z}_1$) takes the form

$$\dot{\tilde{\mathbf{G}}}^{(M)} + [\mathfrak{v}, \mathfrak{M}] = \mathbf{L}^{(M)}, \tag{2.15}$$

where $\mathfrak{M}$ is the momentum of the system body + gyroscope and $\mathbf{L}^{(M)}$ is the torque of all forces. Here, the constraint (2.4) is imposed.

The projections of the total angular momentum $\tilde{\mathbf{G}}^{(M)}$ to the axis of $M\mathbf{uvn}$ are

$$\frac{\partial \bar{T}}{\partial s} + k\alpha'', \qquad \frac{\partial \bar{T}}{\partial \tau} + k\beta'', \qquad \frac{\partial \bar{T}}{\partial n} + k\gamma'',$$

where $\alpha'', \beta'', \gamma''$ are cosines of the angles between $\mathbf{z}$ and $\mathbf{u}, \mathbf{v}, \mathbf{n}$.

It is assumed that the forces in the system are potential and given by a force function U. Let s_1, τ_1, n_1 and $\mathfrak{v}_\mathbf{u}, \mathfrak{v}_\mathbf{v}, \mathfrak{v}_\mathbf{n}$ be the components of ω_1 and $\mathfrak{v}$ in the frame $M\mathbf{uvn}$. Using the law of change of angular momentum (2.15) and the kinematic equations $\frac{d\alpha''}{dt} = \tau_1\gamma'' - n_1\beta''$ (similar for β'' and γ''), the equations are written in the form:

$$\begin{aligned}
&\frac{d}{dt}\frac{\partial \Theta}{\partial s} + (\tau - \tau_1)\frac{\partial \Theta}{\partial n} - (n - n_1)\frac{\partial \Theta}{\partial \tau} + \mathfrak{v}_\mathbf{v}\frac{\partial \bar{T}}{\partial \mathfrak{m}_\mathbf{n}} - \mathfrak{v}_\mathbf{n}\frac{\partial \bar{T}}{\partial \mathfrak{m}_\mathbf{v}} = \frac{\partial \bar{\dot{U}}}{\partial s} + k(n\beta'' - \tau\gamma''),\\
&\frac{d}{dt}\frac{\partial \Theta}{\partial \tau} + (n - n_1)\frac{\partial \Theta}{\partial n} - (s - s_1)\frac{\partial \Theta}{\partial n} + \mathfrak{v}_\mathbf{n}\frac{\partial \bar{T}}{\partial \mathfrak{m}_\mathbf{u}} - \mathfrak{v}_\mathbf{u}\frac{\partial \bar{T}}{\partial \mathfrak{m}_\mathbf{n}} = \frac{\partial \bar{\dot{U}}}{\partial \tau} + k(s\gamma'' - n\alpha''),\\
&\frac{d}{dt}\frac{\partial \Theta}{\partial n} + (s - s_1)\frac{\partial \Theta}{\partial \tau} - (\tau - \tau_1)\frac{\partial \Theta}{\partial s} + \mathfrak{v}_\mathbf{u}\frac{\partial \bar{T}}{\partial \mathfrak{m}_\mathbf{v}} - \mathfrak{v}_\mathbf{v}\frac{\partial \bar{T}}{\partial \mathfrak{m}_\mathbf{u}} = \frac{\partial \bar{\dot{U}}}{\partial n} + k(\tau\alpha'' - s\beta''),
\end{aligned}$$

where $\frac{\partial \bar{\dot{U}}}{\partial s}, \frac{\partial \bar{\dot{U}}}{\partial \tau}, \frac{\partial \bar{\dot{U}}}{\partial n}$ are the derivatives of U along the vector fields that define the quasi-velocities s, τ, n.

The problem reduces to the integration of eight differential equations on eight unknown functions of time: $u, v, \vartheta, u_1, v_1, s, \tau, n$ (or, equivalently, $u, v, \vartheta, u_1, v_1, \dot{u}, \dot{v}, \dot{\vartheta}$). The explicit forms of all the mentioned variables and functions in terms of C. Neumann variables are given in Chapters 1 and 2.

The problem simplifies under the additional assumptions that the surface of the body is of revolution, that the axis of the gyroscope coincides with the axis of revolution and that the central ellipsoid of inertia is an ellipsoid of revolution with the axis of revolution coinciding with the axis of the gyroscope.

2.6. The Bobilev–Zhukovsky problem and its generalization. If one considers rolling *over the plane* and the gyroscopic body is a ball with the mass center coinciding with its geometric center, the problem can be resolved in quadratures. There are two cases where these quadratures are elliptic. These cases were studied by Bobilev [**17**] and Zhukovsky [**105**]. In the Bobilev case the central ellipsoid of inertia is *rotationally symmetric* and the gyroscope axis coincides with the axis of symmetry, while in the Zhukovsky case the additional condition is that *the moment of the ball with respect to the axis of the gyroscope is equal to the sum of the moments of the system ball + gyroscope with respect to the axes orthogonal to the axis of the gyroscope.* In Chapter 4 Demchenko used the same condition as Zhukovsky and considered rolling of a gyroscopic ball *over a sphere*.

Let a ball of radius R_2 with a rotational inertia ellipsoid roll without sliding over a fixed sphere of radius R_1. The ball contains a gyroscope whose axis is fixed with respect to the ball and coincides with the axis of symmetry of the inertia ellipsoid of the ball. It is assumed that the mass center of the moving ball, the mass center of the gyroscope and the geometric center of the moving ball are at the origin of the moving frame $O\mathbf{xyz}$ fixed to the ball such that $O\mathbf{z}$ is the axis of the gyroscope. Let A_1, A_1, C_1, A_2, A_2, C_2 denote the principal central moments of inertia of the ball and the gyroscope with respect to the frames $O\mathbf{xyz}$ attached to the ball and the frame $O\xi\eta\zeta$ rigidly connected to the gyroscope, such that $O\zeta = O\mathbf{z}$. In the given notation, the *Zhukovsky condition* reads

$$C_1 = A_1 + A_2. \tag{2.16}$$

The complements of a latitude and a longitude of the contact point M are chosen for the Gauss coordinates u, v on the moving ball and u_1, v_1 on the fixed sphere. The angle ϑ is as before the angle between u and v_1 coordinate lines. Let x, y, z and x_1, y_1, z_1 be the coordinates of the point M in the moving $O\mathbf{xyz}$ and the fixed frame $O_1\mathbf{x}_1\mathbf{y}_1\mathbf{z}_1$, respectively. One has

$$\begin{aligned} x &= R_2 \sin u \cos v & y &= R_2 \sin u \sin v & z &= R_2 \cos u \\ x_1 &= R_1 \sin u_1 \cos v_1 & y_1 &= R_1 \sin u_1 \sin v_1 & z_1 &= R_1 \cos u_1. \end{aligned}$$

Now, the nonholonomic constraints (2.6) read

$$\begin{aligned} \dot{u}_1 &= -\mu' \dot{u} \sin\vartheta + \mu' \dot{v} \cos\vartheta \sin u, \\ \dot{v}_1 \sin u_1 &= \mu' \dot{u} \cos\vartheta + \mu' \dot{v} \sin\vartheta \sin u, \end{aligned} \tag{2.17}$$

where $\mu' = \frac{R_2}{R_1}$.

Let, as above, s, τ, n denote the coordinates of angular velocity of the ball in the moving reference frame $M\mathbf{uvn}$. After substitution of the constraints (2.17) into (2.5), the expressions for s and τ are simplified:

$$s = \mu \sin u \dot{v}, \quad \tau = -\mu \dot{u}, \quad n = -\dot{\vartheta} - \cos u \dot{v} - \cos u_1 \dot{v}_1, \tag{2.18}$$

where

$$\mu = 1 + \frac{R_2}{R_1} = 1 + \mu'.$$

Let p_1, q_1, r_1 denote the projections of angular velocity of the ball to the axes of the frame $O\mathbf{xyz}$ and let p_2, q_2, r_2 denote the projections of angular velocity of the gyroscope to the axes of the frame $O\xi\eta\zeta$. It is assumed that the torque of active forces for the gyroscope axis is zero. Since the torque of reaction of constraints for the gyroscope axis is also zero, one concludes

$$C_2 r_2 = k = \text{const} \quad \text{and} \quad p_1^2 + q_1^2 = p_2^2 + q_2^2.$$

The kinetic energy of the system ball + gyroscope is then given by (see (2.7))

$$\begin{aligned}\tilde{T} &= \frac{1}{2}\left(A_1 p_1^2 + A_1 q_1^2 + C_1 r_1^2\right) + \frac{1}{2}\left(A_2 p_2^2 + A_2 q_2^2 + C_2 r_2^2\right) + \frac{1}{2} m\mathbf{w}^2 \\ &= \frac{1}{2}\left(A p_1^2 + A q_1^2 + C r_1^2\right) + \frac{1}{2} k r_2 + \frac{1}{2} m\mathbf{w}^2,\end{aligned} \tag{2.19}$$

where $\mathbf{M}$ is the mass of the system ball + gyroscope, $\mathbf{w}$ is the velocity of the point O, and $A = A_1 + A_2$, $C = C_1$.

Since $p_1^2 + q_1^2 + r_1^2 = s^2 + \tau^2 + n^2$ and for the ball we have the identity

$$\mathbf{w}^2 = R_2^2(s^2 + \tau^2),$$

from the Zhukovsky condition (2.16), the kinetic energy of the system expressed as a function of s, τ, n takes the form

$$\tilde{T} = \tfrac{1}{2}(P(s^2 + \tau^2) + An^2) + \tfrac{1}{2} k r_2, \tag{2.20}$$

where $P = I + A$ and $I = \mathbf{M}R_2^2$.

The equations of motion can be obtained from a general law of change of the angular momentum (2.15). Since in the considered system $[\mathfrak{v}, \mathfrak{M}] = 0$ and $\mathbf{L}^{(M)} = 0$, one concludes that the total angular momentum is constant in the fixed reference frame $O_1\mathbf{x}_1\mathbf{y}_1\mathbf{z}_1$:

$$\tilde{\mathbf{G}}^{(M)} = \text{const}.$$

Let Γ denote its magnitude. One can choose the axis $O_1\mathbf{z}_1$, such that $\tilde{\mathbf{G}}^{(M)} = \Gamma\mathbf{z}_1$. The cosines $\alpha_1'', \beta_1'', \gamma_1''$ of the angles between $\mathbf{z}_1$ and $\mathbf{u}, \mathbf{v}, \mathbf{n}$ in the Neumann variables are

$$\alpha_1'' = \sin u_1 \sin\vartheta, \quad \beta_1'' = -\sin u_1 \cos\vartheta, \quad \gamma_1'' = -\cos u_1.$$

Thus, the projections of $\tilde{\mathbf{G}}^{(M)}$ to the axes of the moving frame $M\mathbf{uvn}$ are given by

$$\begin{aligned}\Gamma \sin u_1 \sin\vartheta &= Ps - k\sin u \\ -\Gamma \sin u_1 \cos\vartheta &= P\tau \\ -\Gamma \cos u_1 &= An + k\cos u.\end{aligned} \tag{2.21}$$

It is interesting to mention that the equations of motion are obtained in the form

$$\begin{aligned}&P\dot{s} - \mu' A\, n\dot{u} - P\tau(n + \cos u\dot{v}) = k\mu\dot{u}\cos u, \\ &P\dot{\tau} - \mu' A\, n\sin u\dot{v} + P\, s(n + \cos u\dot{v}) = k(n\sin u + \mu\dot{v}\sin u\cos u), \\ &A\dot{n} = k\mu\sin u\dot{u},\end{aligned} \tag{2.22}$$

by derivation of (2.21) and the kinetic energy integral (2.20).

Finally, the problem reduces to the problem of solving the system of eight equations (2.22), (2.18), and (2.17) in the variables $u, v, \vartheta, u_1, v_1, s, \tau, n$.

2.7. Solving the system in terms of elliptic functions and elliptic integrals. Demchenko introduces a new variable $x = \cos u$ and derives an elliptic equation on x:

$$\left(\frac{dx}{dt}\right)^2 = X(x), \tag{2.23}$$

where $X(x)$ is a degree four polynomial in x. Namely, integrating the last of the equations (2.22), he obtains

$$An = -k\mu x + C_5 = -k\mu(x - x_0), \tag{2.24}$$

where $C_5 = k\mu x_0$ is a constant, as well as x_0. In order to get s, he eliminates τ from the first integrals, the area integral

$$P^2(s^2 + \tau^2) + A^2n^2 + k^2 - 2k(Ps\sin u - An\cos u) = \Gamma^2,$$

and the kinetic energy integral

$$P(s^2 + \tau^2) + An^2 = 2h. \tag{2.25}$$

He gets

$$b_2 s\sin u = k\mu(-b_0x^2 + 2b_1x_0x - \bar{\Gamma}), \tag{2.26}$$

where

$$b_0 = I\mu + 2A,\ b_1 = I\mu + A,\ b_2 = 2PA, \quad \text{and} \quad \bar{\Gamma} = \frac{IC_5^2 + A(\Gamma^2 - k^2) - 2hPA}{\mu k^2}.$$

Note that the inequalities

$$b_0 > b_1 > P = A + 1$$

are valid since $\mu > 1$. From the kinetic energy integral (2.25), it follows

$$b_2^2\tau^2 = -2b_2A^2n^2 + 2b_2Ah - b_2^2s^2.$$

By multiplying both sides by $\sin^2 u$ and by applying the formulae (2.24) and (2.26) one finally gets

$$b_2^2\tau^2\sin^2 u = \mu^2k^2X, \tag{2.27}$$

where

$$X = 2b_2(h' - x + x_0)(h' + x - x_0)(1 - x^2) - (-b_0x^2 + 2b_1x_0x - \bar{\Gamma})^2,$$

with

$$h' = \frac{\sqrt{2hA}}{\mu k}.$$

By substituting τ from the second of the equations (2.18) into (2.27), one comes to (2.23).

By using the first integrals of energy and area, Demchenko expresses the angular velocities s, τ, n and also u_1 and ϑ as functions of x. He needs two additional elliptic integrations

$$dv = \frac{\phi(x)dx}{(1 - x^2)\sqrt{X}}, \quad dv_1 = \frac{F(x)dx}{\theta(x)\sqrt{X}}, \tag{2.28}$$

where ϕ, F, θ are quadratic polynomials in x. The polynomial X can be presented in the form:

$$X(x) = (1 - x^2)\psi(x) - \phi(x)^2 = a_0(x - x^I)(x - x^{II})(x - x^{III})(x - x^{IV}),$$

where ψ is also a quadratic polynomial in x and a_0 is a negative constant.

Elliptic functions and addition theorems. Using the theory of elliptic functions heavily and skillfully, as presented in [**61**], Demchenko inverses the integrals (2.23) and (2.28). He uses the Weierstrass elliptic functions, $\wp(z)$, $\zeta(z)$, and $\sigma(z)$. The basic definitions and important identities can be found, for example, in [**1**], to list a source more modern than [**61**]. The addition theorem for elliptic functions, in particular for the Weierstrass function, played an important role.

THEOREM 2.1 (Addition theorem). *The Weierstrass function satisfies the following addition relation:*

$$\wp(u+\zeta)+\wp(u)+\wp(\zeta)=\frac{1}{4}\Big(\frac{\wp'(u)-\wp'(\zeta)}{\wp(u)-\wp(\zeta)}\Big)^2.$$

Some other typical identities are:

$$\wp'(\zeta)\frac{\wp'(u)-\wp'(\zeta)}{\wp(u)-\wp(\zeta)}=\wp''(\zeta)-2(\wp(u)-\wp(\zeta))(\wp(u+\zeta)-\wp(\zeta)),$$
$$\wp(u)-\wp(v)=-\frac{\sigma(u-v)\sigma(u+v)}{\sigma(u)^2\sigma(v)^2},$$
$$\frac{\wp'(u)}{\wp(u)-\wp(v)}=\zeta(u-v)+\zeta(u+v)-2\zeta(u).$$

Along with addition formulae for elliptic functions, Demchenko also used the Abel theorem for elliptic function, stating that the sum of zeros of an elliptic function equals the sum of poles (modulo the lattice which defines the underlying elliptic curve).

Inversion of elliptic integrals. In order to integrate the equation (2.23), Demchenko used an approach explained in [**61**], which is based on simultaneous parameterizations of the square of the polynomial X of degree four in x and the variable x in terms of elliptic functions of the same argument u. To that end, let us denote

$$2y=\frac{\wp'(u)-\wp'(\zeta)}{\wp(u)-\wp(\zeta)}.$$

Using the Addition Theorem 2.1 and formulae (2.23) and (2.28), one gets

$$y^2-3\wp(\zeta)=(\wp(u)-\wp(\zeta))+(\wp(u+\zeta)-\wp(\zeta)),$$

and

$$\wp''(\zeta)-2y\wp'(\zeta)=2(\wp(u)-\wp(\zeta))(\wp(u+\zeta)-\wp(\zeta)).$$

Let us introduce the polynomial Y of degree four in y as:

$$Y=(y-3\wp(\zeta))^2+2(2y\wp'(\zeta)-\wp''(\zeta).$$

From the above formula it follows that

$$Y=(\wp(u+\zeta)-\wp(u))^2.$$

Thus:

$$\sqrt{Y}=\wp(u+\zeta)-\wp(u)\quad\text{and}$$
$$Y=y^4-6y^2\wp(\zeta)+4y\wp'(\zeta)+9\wp(\zeta)-2\wp''(\zeta).$$

One can apply the above considerations to an arbitrary polynomial X of degree four in x:

$$X(x) = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4.$$

To eliminate the second term with x^3, one substitutes the variable $x = y + h$, i.e. $x = y - a_1/a_0$. One gets

$$\wp(\zeta) = \frac{{a_1}^2 - a_0a_2}{{a_0}^2}, \quad \wp'(\zeta) = \frac{{a_0}^2a_3 - 3a_0a_1a_2 + 2a_1^3}{a_0^3}$$

and

$$x = -\frac{a_1}{a_0} + \frac{1}{2}\frac{\wp'(u) - \wp'(\zeta)}{\wp(u) - \wp(\zeta)}, \quad \sqrt{X} = \sqrt{a_0}(-\wp(u+\zeta) + \wp(u)).$$

From Addition Theorem 2.1, it also follows that

$$-\wp(u+\zeta) + \wp(u) = \frac{1}{2}\frac{d}{du}\frac{\wp'(u) - \wp'(\zeta)}{\wp(u) - \wp(\zeta)},$$

and

$$\sqrt{X} = \sqrt{a_0}\frac{dx}{du}.$$

Finally,

$$\frac{u}{\sqrt{a_0}} = \int \frac{dx}{\sqrt{X}}.$$

2.8. Back to the Demchenko case. In general, the polynomial X can have zero, two, or four real roots. The first case would not produce any real motion and Demchenko did not consider it.

In the case of four real roots, ordered $x^I > x^{IV} > x^{III} > x^{II}$, the motion is possible for $x \in (x^I, x^{IV})$ or $x \in (x^{III}, x^{II})$. Without losing generality, Demchenko works with the first case: $x \in (x^I, x^{IV})$. In the case of two real roots, he again denotes them as $x^I > x^{IV}$. The trajectory of the point M on the mobile sphere goes between the parallels u^I and u^{IV}, which it touches alternately. The distance between two consecutive points of contact is constant. Demchenko distinguishes three cases:

1) The polynomial $\phi(x)$ has no roots in the interval (x^I, x^{IV}). The situation in this case is presented as curve A, see Figure 2.
2) The polynomial $\phi(x)$ has one root in the interval (x^I, x^{IV}). The situation in this case is presented as curve B, see Figure 3.
3) The polynomial $\phi(x)$ has two roots in the interval (x^I, x^{IV}). The situation in this case is presented as curves C, C_1 and C_2, see Figures 4 and 5.

The trajectories of the point M on the fixed sphere are similar, where the number of roots of the polynomial $F(x)$ in the interval (x^I, x^{IV}) now discriminates cases A, B, and C.

There are special cases of curves if x^I or x^{IV} coincides with one of the roots of the polynomial $\psi(x)$ or is equal to ± 1. If x^I or x^{IV} coincides with one of the roots of the polynomial $\psi(x)$, then the curves on the movable and fixed sphere have the form D: D_1, see Figure 6, D_2 see Figure 7, and D_3, see Figure 8. If, however, $x^I = 1$ or $x^{IV} = -1$, the curves representing the motion of the point M on the

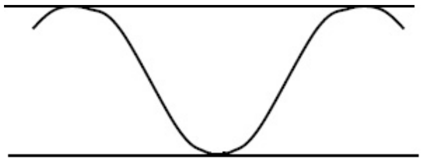

FIGURE 2. Demchenko: Figure 4, p. 53: curve A.

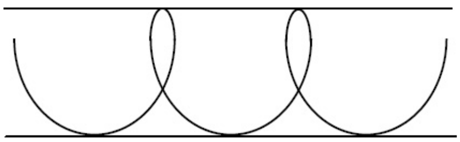

FIGURE 3. Figure 5, p. 53: curve B.

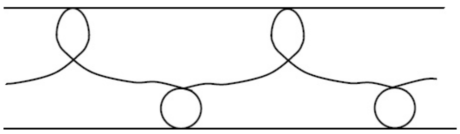

FIGURE 4. Demchenko: Figure 6, p. 53: curve C_1.

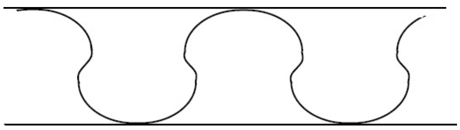

FIGURE 5. Demchenko: Figure 7, p. 53: curve C_2.

movable sphere are presented as E_1, see Figure 9, and E_2, see Figure 10. In these cases the curves on the fixed sphere do not possess singularities.

2.9. Special Solutions. The last chapter is devoted to particular cases and particular solutions. These considerations reduce to more elementary situations than the general ones or to approximate formulae. Demchenko heavily used the capital four-volume treatise of dynamics of the top by Felix Klein and Arnold Sommerfeld [**70**]. Demchenko establishes four classes of particular motion. In each class, he also resolves the issue of stability.

The classes are:

1) Regular precessions: they are possible. The curves which M describes on both fixed and movable spheres coincide with parallels. The precession is stable if

$$\frac{\partial^2 X}{\partial x^{I2}} < 0,$$

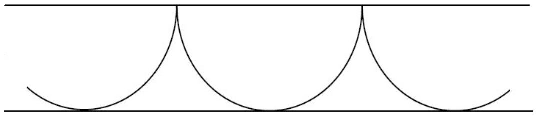

FIGURE 6. Demchenko: Figure 8, p. 56: curve D_1.

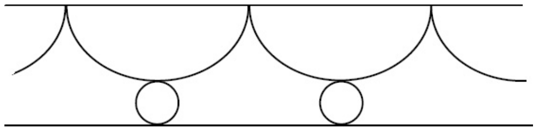

FIGURE 7. Demchenko: Figure 9, p. 56: curve D_2.

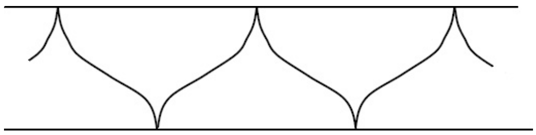

FIGURE 8. Demchenko: Figure 10, p. 57: curve D_3.

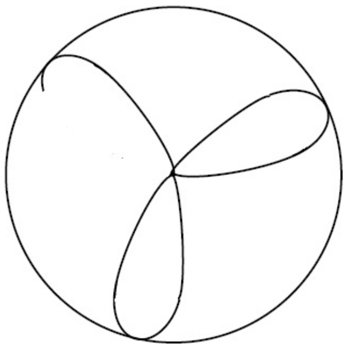

FIGURE 9. Demchenko: Figure 11, p. 57: curve E_1.

and unstable if

$$\frac{\partial^2 X}{\partial x^{I2}} > 0.$$

In the cases

$$\frac{\partial^2 X}{\partial x^{I2}} = 0,$$

the stable situation corresponds to

$$\frac{\partial^3 X}{\partial x^{I3}} = 0,$$

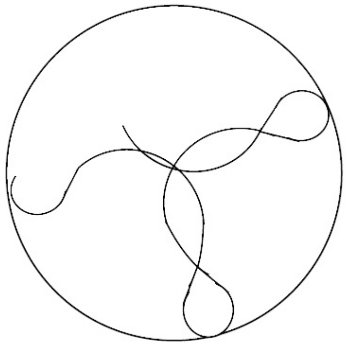

FIGURE 10. Demchenko: Figure 12, p. 58: curve E_2.

and the unstable situation corresponds to

$$\frac{\partial^3 X}{\partial x^{I3}} \neq 0.$$

2) Pseudo-regular precessions are possible if the rotations of the gyroscope are much bigger than the initial rotations s_0, τ_0 of the gyroscopic ball. These precessions are always stable.
3) Stationary motions are possible. The trajectories of the point M consist of a single point both on movable and fixed spheres. The angular velocity of the ball is constant and the orientation of the axis of the gyroscope is constant. Such motion is always stable.
4) Rolling of an ordinary ball, when the gyroscope stays at rest.

The last Subchapter 6.7 is devoted to the remarkable trajectories, the notion introduced by Painlevé [**84**] and Bilimović [**13**]. These are trajectories independent of the initial energy. A detailed analysis shows that in the dynamics of a gyroscopic ball rolling without sliding over a sphere such remarkable trajectories exist. These are regular precessions when the axis of the gyroscope rests parallel to the fixed vector of the moment of the gyroscopic ball with respect to the point M.

3. V. S. Zhardecki, K. P. Voronec, the Paris period and fluid mechanics

A few months before Demchenko, another immigrant from Russia, Viacheslav Sigmundovich Zhardecki (1896–1962) defended his thesis [**103**] also on rigid body dynamics, having Anton Bilimović as the advisor and Milutin Milanković as a co-signatory of the report. The Zhardecki family belongs to the Polish nobility. Viacheslav knew Bilimović from their time in Odessa and at the Novorossisk University. In Belgrade he was also influenced by Milanković, a notable geoscientist and mathematician. Thus, later on, Zhardecki shifted his interests more toward geoscience and obtained remarkable results, see, for example, [**63–65**]. In 1943, during the German occupation of Belgrade, Zhardecki refused to serve at the reformed University. As a consequence, he was retired at the age of 48. He managed to move to Austria in 1944 and in 1946–47 he served as Acting Director of the

Institute of Physics and Astronomy in Graz. During that period he got experimental confirmation of his theory of formation of continents and moved further to the US, see [**16**]. His son, Oleg, who became a scientist himself left interesting notes about his family and the dramatic time of their emigration from Russia, see [**104**].

After defending his thesis, V. Demchenko taught mathematics in Subotica, a city 200 km north of Belgrade. His father, Grigorij Vasilievich Demchenko was a Professor of Law School there, and served as Dean of the School 1929–30. Both the father and the son were delegates of the Congress of Russians from Abroad in Paris in 1926, as representatives of the Yugoslav Committee, [**48, 86**]. The same year, Bilimović and Demchenko reacted together to the paper [**89**] of the notable Bulgarian scientist Ivan Cenov and indicated three papers of P. Voronec [**93–95**], three papers of Bilimović [**9, 11, 13**] and the doctoral thesis of Demchenko [**35**] as relevant and a source of the results close to those presented in [**89**]. Their remark was published in a Liouvile's journal editorial comment [**85**].

Around that time, Vasilije moved to Paris. Vasilije Demchenko, now as Basile Demtchenko, defended his second doctoral dissertation in mathematical sciences in Paris in 1928. He switched his field from nonholonimic to fluid mechanics. The thesis was entitled "I. Sur les cavitations solitaires dans un liquid infini. II. Sur l'influence des bords sur mouvement d'un corps solide dans une liquide" [**36**] It was defended on June 2 with the committee consisting of three major French mathematicians, Paul Painlevé (1863–1933), Henri Villat (1879–1972), and Paul Montel (1876–1975). The thesis was completed under the direction of Painlevé and was dedicated to Peter Voronec, the teacher: "À mon cher et regretté Maitre, Pierre Voronetz." Demchenko's work in Paris was also associated with the group of the renowned expert in hydro and aerodynamics, D. P. Ryabushinsky (1882–1962), see [**2**]. Some of the notable works of Demchenko include [**37–39**].

Vasilije Demchenko was an invited speaker at the International Congresses of Mathematicians [**62**] in Bologna 1928 and Zurich 1932. Two members of his Belgrade thesis committee, Bilimović and Petrović, were also invited speakers at the same Congresses. In addition, Petrović was also an invited speaker in Rome 1908, Cambridge 1912, and Toronto 1924. (Demchenko's advisor from Paris, P. Painlevé was a plenary speaker in Heidelberg 1904.)

There is an interesting parallel between Demchenko and Konstantin Voronec, the above mentioned son of P. Voronec. Konstantin defended his doctoral dissertation in Belgrade in 1930 [**97**], having the same committee as Demchenko's thesis, Bilimović, Milanković, Petrović. The Voronec thesis was very much influenced by Demchenko's thesis. After the defense, Konstantin also moved to Paris and switched to fluid mechanics too. He also defended his second doctoral thesis in Paris [**98, 99**] see [**48, 87**]. In his second thesis, Voronec was again influenced by Demchenko, this time by [**37**]. Demchenko and Voronec worked together in an institute for fluid mechanics within the French Ministry of Air (Ministeȓe de l'Air) in Paris. They also published a joint monograph in 1939 [**40**].

4. Demchenko's PhD thesis and contemporary nonholonomic mechanics

The doctoral dissertation of Tatomir Andjelić can be seen as one of the important links between the works of Voronec, Bilimović, and Demchenko and contemporary science, [**2**]. The thesis was completed just before the Second Word War, but was defended after the war, in 1946. That was one more example of the principle adopted by many notable Serbian scientists who did not participate in the university matters during the German occupation. Although formally Bilimović did not serve as a committee member, Andjelić made it clear that the problem was posed by Bilimović and written under his guidance. He studied application of the Voronec principle to the problem of motion of a nonholonomic system placed in an incompressible fluid.

An important reference in nonholonomic mechanics after the Second World War is the monograph by Neimark and Fufaev [**82**]. There is a whole chapter devoted to the Voronec and Chaplygin equations. Among others, the monograph referred to several contributions of members of the Bilimović school [**3**, **8**–**12**, **90**, **91**].

Let us note that the Chaplygin systems have a natural geometrical framework – the nonholonomic constraints define connections on principal bundles (see Koiler [**71**]). On the other hand, Bloch, Krishnaprasad, Marsden, and Murray [**15**] incorporated nonholonomic systems into the geometrical framework of the Ehresmann connections. It was pointed out in Bakša [**6**] that the equations used in [**15**] are literally the same as the original Voronec equations [**93**]. The same year de Leon also referred to the Voronec equations in [**76**]. Now we can say that the Voronec equations, together with the Chaplygin equations and the equations of nonholonomic systems written in terms of quasi-velocities, known as *the Euler–Poincaré–Chetayev–Hamel* equations, form the central tools in the study of nonholonomic systems (e.g., see [**15**, **45**, **46**, **82**, **102**]).

Consider a Lagrangian nonholonomic system $(Q, L, \mathcal{D})$ where the constraints define a nonintegrable distribution $\mathcal{D}$ of the tangent bundle TQ, i.e., the constraints are homogeneous and do not depend on time, and the Lagrangian is the difference of the kinetic and the potential energy $L(q, \dot q) = T(q, \dot q) - V(q)$.

Further, we assume that Q has a structure of the fiber bundle $\pi\colon Q \to S$ over the base space S and that $\mathcal{D}$ is transverse to the fibers of π:

$$T_qQ = \mathcal{D}_q \oplus \mathcal{V}_q, \qquad \mathcal{V}_q = \ker d\pi(q).$$

The space $\mathcal{V}_q$ is called the *vertical space* at q. The distribution $\mathcal{D}$ can be seen as the kernel of a vector-valued one form A on Q, which defines the *Ehresmann connection*, which satisfies

(i) $A_q\colon T_qQ \to \mathcal{V}_q$ is a linear mapping, $q \in Q$;
(ii) A is a projection: $A(X_q) = X_q$, for all $X_q \in \mathcal{V}_q$.

The distribution $\mathcal{D}$ is called *the horizontal space* of the Ehresmann connection A. By X^h and X^v we denote the horizontal and the vertical component of the vector field $X \in \mathfrak{X}(Q)$. The *curvature* B of the connection A is a vertical vector-valued two-form defined by

$$B(X, Y) = -A([X^h, Y^h]).$$

In the local coordinates, we have

$$
\begin{aligned}
&\pi\colon (q_1,\dots,q_n,q_{n+1},\dots,q_{n+k}) \longmapsto (q_1,\dots,q_n),\\
&A=\sum_{\nu=1}^{k}\omega^{\nu}\frac{\partial}{\partial q_{n+\nu}},\quad \omega^{\nu}=dq_{n+\nu}-\sum_{i=1}^{n}a_{\nu i}dq_i,\\
&X^h=\Big(\sum_{l=1}^{n+\nu}X_l\frac{\partial}{\partial q_l}\Big)^h=\sum_{i=1}^{n}X_i\frac{\partial}{\partial q_i}+\sum_{\nu=1}^{k}\sum_{i=1}^{n}a_{\nu i}X_i\frac{\partial}{\partial q_{n+\nu}},\\
&X^v=\Big(\sum_{l=1}^{n+\nu}X_l\frac{\partial}{\partial q_l}\Big)^v=\sum_{\nu=1}^{k}\Big(X_{n+\nu}-\sum_{i=1}^{n}a_{\nu i}X_i\Big)\frac{\partial}{\partial q_{n+\nu}},\\
&B\Big(\frac{\partial}{\partial q_i},\frac{\partial}{\partial q_j}\Big)=\sum_{\nu=1}^{k}B^{\nu}_{ij}\frac{\partial}{\partial q_{n+\nu}},\qquad i,j=1,\dots,n,
\end{aligned}
$$

where $B^{\nu}_{ij}=A^{(\nu)}_{ij}$ in the Voronec equations (2.11). Also, in the case when the generalized forces Q_s, $s=1,\dots,n+k$ are potential: $Q_s=-\partial V/\partial q_s$, the Voronec equations (2.11) take the form:

$$
\frac{d}{dt}\frac{\partial L_c}{\partial \dot q_i}=\frac{\partial L_c}{\partial q_i}+\sum_{\nu=1}^{k}a_{\nu i}\frac{\partial L_c}{\partial q_{n+\nu}}+\sum_{\nu=1}^{k}\sum_{j=1}^{n}\frac{\partial L}{\partial \dot q_{n+\nu}}B^{\nu}_{ij}\dot q_j\quad (i=1,\dots,n), \tag{4.1}
$$

where $L_c=L(q,\dot q^h)$ is the constrained Lagrangian, i.e., $L_c=\Theta-V$ in Demchenko's notation. The Voronec principle (2.12) for the equations (4.1) in an invariant form can be expressed as (see [**15**]):

$$
\delta L_c=\mathbb{F}L(B(\dot q,\delta q)) \tag{4.2}
$$

for all virtual displacements

$$
\delta q=\sum_{s=1}^{n+k}\delta q_s\frac{\partial}{\partial q_s}\in\mathcal{D}_q.
$$

Here δL_c is the variational derivative of the constrained Lagrangian along the variation δq and $\mathbb{F}L$ is the fiber derivative of L:

$$
\delta L_c=\sum_{s=1}^{n+k}\Big(\frac{\partial L_c}{\partial q_s}-\frac{d}{dt}\frac{\partial L_c}{\partial \dot q_s}\Big)\delta q_s,\quad \mathbb{F}L(B(\dot q,\delta q))=\sum_{\nu=1}^{k}\frac{\partial L}{\partial \dot q_{n+\nu}}B^{\nu}(\dot q,\delta q),
$$

In the case when the constraints are nonhomogeneous and time dependent (2.10), the coefficients $A^{(\nu)}_{ij}$, $A^{(\nu)}_{i}$ can also be interpreted as the component of the curvature of the Ehresmann connection of the fiber bundle $\pi\colon Q\times\mathbb{R}\to S\times\mathbb{R}$ (see Bakša [**6**]).

Assume that the fibration $\pi\colon Q\to S$ is determined by a free action of a Lie group G on Q ($S=Q/G$) and that the constraint distribution $\mathcal{D}$ and the Lagrangian $L=T-V$ are G–invariant. Then A is a principal connection and the nonholonomic system (4.2) is G–invariant and reduces to the tangent bundle of the base manifold S. The equations take the form

$$(4.3)\qquad \delta L_{\rm red}=\sum_{i=1}^{n}\Big(\frac{\partial L_{\rm red}}{\partial x_i}-\frac{d}{dt}\frac{\partial L_{\rm red}}{\partial \dot x_i}\Big)\delta x_i = JK(\dot x,\delta x)\quad \text{for all}\quad \delta x\in T_xS,$$

where the reduced Lagrangian $L_{\rm red}$ is obtained from the constrained Lagrangian L_c by the identification $TS=\mathcal{D}/G$, and $JK(X,Y)$ is induced by the right hand side of (4.2).

The system $(Q,L,\mathcal{D},G)$ is referred to as *a G–Chaplygin system*, as a generalization of the classical Chaplygin systems with Abelian symmetries [**5**, **32**, **34**, **59**, **71**, **82**, **88**].

Demchenko noticed that Voronec had derived his principle in order to relate the nonholonomic systems to the Hamiltonian variational principle of least action (see [**35**, pages 16–19]). Obviously, Voronec and his followers were aware of the fact that the equations were not variational, or, in modern terminology, that they were not Hamiltonian. However, as it was pointed out by Chaplygin [**34**], some systems have an invariant measure, which puts them rather close to Hamiltonian systems. The existence of an invariant measure for various nonholomic problems has been well studied (e.g., see [**50**–**52**, **54**, **67**, **72**, **92**, **101**]). A closely related problem is the Hamiltonization of nonholonomic systems (e.g., see [**7**, **18**, **20**, **22**, **26**, **28**, **32**, **34**, **46**, **46**, **53**, **88**]). Chaplygin was also one of the first who considered a time reparameterization in order to transform nonholonomic systems to the Hamiltonian form [**34**]. In the case of integrability, the dynamics over regular invariant m–dimensional tori, in the original time, has the form

$$(4.4)\qquad \dot\varphi_1=\omega_1/\Phi(\varphi_1,\dots,\varphi_m),\dots,\dot\varphi_m=\omega_m/\Phi(\varphi_1,\dots,\varphi_m),\qquad \Phi>0.$$

Also, after [**33**], one of the most famous solvable problems in nonholonomic mechanics, describing the rolling without slipping of a balanced ball over a horizontal surface, is referred to as the *Chaplygin ball*, see [**4**, **23**, **26**, **73**]. On the other hand, the rolling without slipping of the Chaplygin ball over a sphere generically is not integrable. The only known integrable case is given by Borisov and Fedorov [**21**]. Let R_2, m, and $\mathbb{I}=\operatorname{diag}(I_1,I_2,I_3)$ be the radius, mass and the inertia operator of the ball B, and let R_1 be the radius of the fixed sphere S. There are three possible configurations:

(i) rolling of B over the outer surface of S;
(ii) rolling of B over the inner surface of S ($R_1>R_2$);
(iii) rolling of B over the outer surface of S, but S is within B ($R_1<R_2$, in this case, the rolling ball B is actually a spherical shell).

Let

$$\epsilon=\frac{R_1}{R_1\pm R_2},$$

where we take "+" for the case (i) and "−" in the cases (ii) and (iii) and let $D=mR_2^2$. The equations of motion in the frame attached to the ball can be written in the form

$$\dot{\mathbf{G}} = \mathbf{G} \times \omega, \qquad \dot{\gamma} = \epsilon\gamma \times \omega, \tag{4.5}$$

where $\mathbf{G} = \mathbb{I}\omega + D\omega - D(\omega,\gamma)\gamma = \mathbf{I}\omega - D(\omega,\gamma)\gamma$ is the angular momentum of the ball with respect to the point of contact, and γ is the unit normal to the sphere S at the contact point. Here $\mathbf{I} = \mathbb{I} + D\mathbb{E}$, $\mathbb{E} = \mathrm{diag}(1,1,1)$. When R_1 tends to infinity, ϵ tends to 1, γ tends to the unit vector that is constant in the fixed reference frame. This way we obtain the equations of motion of the Chaplygin ball rolling over the plane orthogonal to γ.

In the space $\mathbb{R}^6(\omega,\gamma)$ the system has an invariant measure with the density

$$\mu(\gamma) = \sqrt{(\gamma - D\mathbf{I}^{-1}\gamma,\gamma)}, \tag{4.6}$$

the expression given by Chaplygin for $\epsilon = 1$ [**33**], and by Yaroshchuk for $\epsilon \neq 1$ [**100**]. Also, the system (4.5) always has three integrals

$$F_1 = (\gamma,\gamma) = 1, \quad F_2 = \tfrac{1}{2}(\mathbf{G},\omega), \quad F_3 = (\mathbf{G},\mathbf{G}).$$

For $\epsilon = 1$, there is a fourth first integral $F_4 = (\mathbf{G},\gamma)$. The problem is integrable by the Euler–Jacobi theorem (see [**4, 73**]): the phase space is almost everywhere foliated by two-dimensional invariant tori with quasi-periodic, non-uniform motion (4.4) (see Chaplygin [**33**]). Moreover, Borisov and Mamaev proved that the system (4.5) is Hamiltonizable with respect to a certain nonlinear Poisson bracket on $\mathbb{R}^6$ ([**22**], see also [**7, 20, 26, 28**]).

Remarkably, for $\epsilon = -1$ (the case (iii) with $R_2 = 2R_1$) Borisov and Fedorov (see [**21**]) found an integrable case with the following fourth first integral

$$\tilde{F}_4 = (I_2 + I_3 - I_1 + D)\mathbf{G}_1\gamma_1 + (I_3 + I_1 - I_2 + D)\mathbf{G}_2\gamma_2 + (I_1 + I_2 - I_3 + D)\mathbf{G}_3\gamma_3.$$

The system is integrated on an invariant hypersurface $\tilde{F}_4^{-1}(0)$ [**25**]. Its topological analysis is given in [**27**].

One can consider the additional nonholonomic constraint $(\omega,\gamma) = 0$ describing the *no-twisting* condition: the ball B does not rotate around the normal at the contact point (the so called *rubber Chaplygin ball*). Then the momentum with respect to the contact point can be expressed as $\mathbf{G} = \mathbb{I}\omega + D\omega = \mathbf{I}\omega$, and the equations take the form

$$\dot{\mathbf{G}} = \mathbf{G} \times \omega + \lambda\gamma, \qquad \dot{\gamma} = \epsilon\gamma \times \omega, \qquad (\omega,\gamma) = 0,$$

where the Lagrange multiplier $\lambda = -(\mathbf{G}, \mathbf{I}^{-1}(\mathbf{G} \times \omega))/(\gamma, \mathbf{I}^{-1}\gamma)$ is determined by differentiation of the constraint $(\omega,\gamma) = 0$. The system has an invariant measure with the density $\mu_\epsilon(\gamma) = (\mathbf{I}^{-1}\gamma,\gamma)^{\frac{1}{2\epsilon}}$ (see [**46**] for $\epsilon = 1$ and [**47**] for $\epsilon \neq 1$). Apart from the integrability of the rolling over a horizontal plane ($\epsilon = 1$) [**46**], as in the case of non-rubber rolling, Borisov and Mamaev proved the integrability for $\epsilon = -1$ [**24**]. Note that for $\epsilon = 1$, the above equations coincide with the equations of nonholonomic rigid body motion studied by Veselov and Veselova [**92**].

The problem is Hamiltonizable for all ϵ [**46, 47**]. On the other hand, the rubber rolling of the ball where the mass center does not coincide with the geometrical center over a horizontal plane provides an example of the system having the following interesting property (see [**19**]). The appropriate phase space is foliated on invariant tori, such that the foliation is isomorphic to the foliation of the integrable Euler case of the rigid body motion about a fixed point, but the system itself does not have an analytic invariant measure and is not Hamiltonizable.

The gyroscopic generalizations of the mentioned Chaplygin ball problems are also well studied. Markeev proved that the addition of a gyroscope to the Chaplygin ball problem of rolling of a dynamically non-symmetric ball without slipping over a plane remains integrable [**77**]. As in Demchenko's thesis described in Sections 2.5 and 2.6, the addition of a gyroscope is equivalent to the addition of a constant angular momentum κ, directed as the axis of the gyroscope, to $\mathbf{G}$ (with the new inertia operator described in Sections 2.5 and 2.6). In the above notations, we can write the equations of the Chaplygin ball with the gyroscope rolling without slipping over the

$$\dot{\mathbf{G}} = (\mathbf{G} + \kappa) \times \omega, \qquad \dot{\gamma} = \epsilon\gamma \times \omega.$$

When $\epsilon = 1$, we have the Markeev integrable case [**77**]. The system has an invariant measure with the same density (4.6) and four first integrals

$$F_1 = (\gamma, \gamma) = 1, \quad F_2 = \tfrac{1}{2}(\mathbf{G}, \omega), \quad F_3 = (\mathbf{G} + \kappa, \mathbf{G} + \kappa), \quad F_4 = (\mathbf{G} + \kappa, \gamma).$$

The analysis of the bifurcation diagram and the topology of the phase space of the Chaplygin ball with the gyroscope case is studied in [**78**] and [**106**], respectively.

The functions F_1, F_2, and F_3 are integrals for all ϵ. When the ball is dynamically symmetric with the gyroscope directed along the axis of the symmetry, it is the Bobylev–Zhukovsky case for $\epsilon = 1$ [**17, 105**], while when $\epsilon \neq 1$ and the Zhukovsky condition (2.16) on the moments of inertia of the ball and the gyroscope are satisfied, we obtain the Demchenko integrable case. The integrability without the Zhukovsky condition for the dynamically symmetric ball can be found in Borisov and Mamaev [**23**]. Existence of an integrable case for a dynamically nonsymmetric ball with a gyroscope rolling over a sphere is still an open problem.

The Voronec approach to the problem of rolling bodies given in [**95, 96**] can be found also in the recent papers [**31, 74, 75**]. In [**74, 75**], the problem of rolling without sliding of a rotationally symmetric body on a fixed sphere is studied. It is assumed that the resultant of active forces is directed from the center of masses of the body to the center of a sphere. The problem reduces to a linear differential equation of second order. In a special case of motion of the nonhomogeneous dynamically symmetric ball, they proved the existence of Liouvillian solutions. In [**31**] the problem of rolling without slipping of a body with a gyroscope on a moving sphere is considered. It is assumed that the central ellipsoid of the system body + gyroscope is an ellipsoid of revolution. In a special case when the body is a sphere, the motion of the contact point is determined by quadratures. Analysis of trajectories of the contact point is given. This analysis, including pictures, given in [**31**] is very similar to the analysis presented by Demchenko in [**35**], see Figures 2–10.

Another line of current research is the application of the Voronec equations (4.1), (4.2) and their reductions in the case of symmetries (4.3) to the study of their multi-dimensional versions, describing motions of the n–dimensional ball rolling without slipping (and twisting) over a hyperplane or a sphere in $\mathbb{R}^n$ (see

[**49**, **50**, **52**, **55**–**58**, **66**, **68**, **69**]). These examples, together with the classical one, form a rich pool of nonholonomic systems. They motivate further study of the geometry and dynamics of nonholonomic systems, including their integrabilty and Hamiltonization.

There are very recent papers which are built on the results of Bilimović, e.g. [**29**, **30**]. We hope that the current paper will further draw attention to the heritage of the Bilimović scientific school and their contribution to nonholonomic mechanical problems. Demchenko's integrable case and his comprehensive analysis provided in his doctoral thesis seem to be completely forgotten nowadays although still very modern and deserve to be known by a wider community.

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ДЕМЧЕНКОВ НЕХОЛОНОМНИ СЛУЧАЈ КОТРЉАЊА БЕЗ КЛИЗАЊА ГИРОСКОПСКЕ ЛОПТЕ ПО СФЕРИ НА ОСНОВУ ЊЕГОВЕ БЕОГРАДСКЕ ДОКТОРСКЕ ДИСЕРТАЦИЈЕ ИЗ 1923.

Резиме. Представљамо интеграбилни нехолономски случај котрљања без клизања гироскопске лопте по сфери. Овај случај је добио и детаљно проучио Василије Демченко у својој докторској дисертацији под руководством Антона Билимовића, одбрањеној на Универзитету у Београду 1923. Ови резултати су непознати савременим истраживачима. Рад се заснива на Нојмановим координатама и Вороњецовом принципу. Коришћењем напредне технике елиптичних функција дата је детаљна анализа кретања. Издвојено је неколико посебних класа трајекторија, укључујући правилну и псеудо-регуларну прецесију. Описане су и такозване изванредне трајекторије, које су увели Пол Пенлеве и Антон Билимовић. Историјски контекст и значај резултата у савременој механици су детаљно представљени.

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Department of Mathematical Sciences
University of Texas at Dallas
Dallas
USA

Mathematical Institute
Serbian Academy of Sciences and Arts
Belgrade
Serbia
Vladimir.Dragovic@utdallas.edu (corresponding author)

Mathematical Institute
Serbian Academy of Sciences and Arts
Belgrade
Serbia
gajab@mi.sanu.ac.rs

Mathematical Institute
Serbian Academy of Sciences and Arts
Belgrade
Serbia
bozaj@mi.sanu.ac.rs