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ROBUST FINITE-TIME STABILITY OF UNCERTAIN NEUTRAL NONHOMOGENEOUS FRACTIONAL-ORDER SYSTEMS WITH TIME-VARYING DELAYS

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Dedicated to the memory of Professor Veljko Vujičić

Abstract. This article addresses the problem of finite-time stability for uncertain neutral nonhomogeneous fractional-order systems with time-varying delays where a stability test procedure is suggested. Based on the extended form of the generalized Grönwall inequality, a new sufficient condition for robust finite-time stability of such systems is established. Finally, a numerical example is given to show the effectiveness of the obtained result.

1. Introduction

Time delay often appears in many real-world engineering systems either in the state, the control input, or the measurements [1]. Namely, time delay is usually involved in dynamical systems whose change of the present state depends on the past states. It can appear as the inherent state delay [2], or as the input delay usually introduced in control loops [3]. Delay systems are largely encountered in modeling propagation and transportation phenomena, describing dynamics, and representing interactions between interconnected dynamics through material, energy, and communication flow [1, 4].

Time-delay dynamical systems which can be modeled as first functional differential equations (FDEs) [1, 5] have been investigated over the years. The first FDEs were considered by, among others, such great mathematicians as Euler, Bernoulli, Lagrange, Laplace, and Poisson in the eighteenth century, and arose from various geometry problems. In the early twentieth century, a number of practical problems of time-delay systems were considered such as viscoelasticity problems, the

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predator-prey model used in population dynamics and ship stabilization problems. On the stability of time-delay systems, Pontryagin obtained some fundamental results concerning the zeros of quasipolynomials in 1942, and Chebotarev published a number of papers devoted to the Routh–Hurwitz criterion for quasipolynomials in the early 1940's [5].

Stability and control design of time-delay systems are widely studied due to the effect of delay phenomena on system dynamics [5, 6]. Analysis of stability of time-delay systems is more complex than that of delay-free systems because time-delay systems involve the derivative of the time-delayed state and the existence of time delay is frequently the source of instability.

Many scientific contributions have been published on this matter, with particular emphasis on the application of Lyapunov's second method and on using the idea of matrix measure [7, 8]. In this paper, we consider another approach – system stability in the non-Lyapunov sense (finite-time stability) – because the boundedness properties of system responses are very important from the engineering point of view. Therefore, increasing attention in the scientific community has been devoted to the problems of finite-time stability (FTS) and practical stability for both systems without delay [9,10,11] and systems with time-delay [12,13,14].

On the other hand, neutral delay is the leading example of many types of time delay, which not only exists in the system state but also has to do with the derivative of the system state. So, neutral systems represent a very general class which includes, as particular cases, ordinary time-delay systems [15]. The neutral time-delay system is common in practical engineering applications (population ecology, distribution network in a lossless transmission line, heat exchanger, robot in the rigid environment, etc.).

Also, over the last decade, dynamical systems with a fractional-order derivative presented by fractional (non-integer) differential equations (integer-order derivatives are replaced by fractional derivatives) have drawn considerable attention from researchers and engineers since fractional calculus provides an excellent tool for the description of memory and hereditary properties of various materials and processes [16, 17]. For example, viscoelastic materials are used widely as damping materials in vibrating systems or in bioengineering applications, i.e. the use of fractional order models was studied in [18, 19], where it was shown that they are advantageous for predicting the behavior of systems with memory.

Recently, some authors devoted attention to a class of fractional-delay systems (FDS) [20, 21, 22]. FDS stand for dynamical systems with both a fractional-order derivative and time delays. FDS can be classified into two categories: retarded type and neutral type. For retarded fractional-delay systems, the term with the highest order of derivative does not involve any delays, whereas for neutral fractional-delay systems, the term with the highest order of derivative involves at least one time delay. This kind of differential equations may come from delayed control of dynamical systems with a fractional derivative. For example, when a viscoelastic material is used as damping in vibration systems, the Scott–Blair model assumes that the damping is proportional to a fractional order derivative of the displacement variable with the order of the derivative ranging from 0 to 1 and is presented as neutral fractional-delay systems: ${}^{c}D^{\alpha}\,\mathbf{x}(t) + A^{c}\,D^{\alpha}\,\mathbf{x}(t-\tau) = Bx(t) + u(t)$, $\alpha = 1/2$ [23]. Besides, deriving the governing equation of many mechanical, electrical, and control systems with fractional terms results in a more general form as multi-order fractional delay differential equations (FDDEs) given as ${}^{c}D^{\alpha}\,\mathbf{x}(t) = f\left(t, x(t),\ {}^{c}D^{\beta}\,\mathbf{x}(t-\tau)\right)$, $0 < \beta \leqslant \alpha < 1$.

Also, as we previously stated, the stability problem is one of the most vital problems in the investigation dynamical system theory of time-delay fractional-order systems. In recent years, there have been some advances in control theory of fractional (non-integer) order dynamical systems, particularly for different kinds of stability such as internal stability, bounded input–bounded output (BIBO) stability, robust stability, finite-time stability, practical stability, etc. [24, 25, 26, 27, 28, 29]. FTS analysis of fractional delay systems is initially investigated and presented in [20, 28] using the generalized Grönwall inequality, as well as the FTS test procedure, whereas the stabilization of perturbed nonlinear (non)homogeneous fractional systems with time-varying delay is proposed in [29]. Some of the recently obtained results concerning this topic can be found in [30, 31, 32, 33, 34, 35].

To the best of the authors' knowledge, there are only a few results concerning FTS of neutral fractional order time-delay systems. In [36], authors performed FTS analysis using the generalized Grönwall inequality, and in [37] authors presented a sufficient condition using the generalized Grönwall inequality for the particular class of two-term neutral fractional systems with the time-varying delay and nonlinear perturbation. Also, in another recent work [38], authors obtained sufficient conditions for FTS of the neutral fractional two-term time-delay systems with Lipschitz nonlinearities by virtue of the method of steps and a more generalized Grönwall inequality.

Motivated by the aforementioned discussions, for the first time we investigate robust FTS problems for perturbed nonhomogeneous two-term neutral fractional order systems with time-varying delays in state and control.

This article is organized as follows. Some basic definitions and lemmas on fractional calculus as well as on neutral time-delay systems of integer and fractional order are given in Section 2. In Section 3, a new criterion of robust finite-time stability of uncertain neutral nonhomogeneous fractional-order systems with time-varying delays is established. A numerical example is presented to illustrate the application and verify the effectiveness of our results in Section 4. Finally, this paper ends with a conclusion in Section 5.

2. Preliminaries

2.1. Preliminary notes on fractional calculus. In this section, we consider the main definitions and properties of fractional derivative operators.

Definition 2.1. [39, 40]: Let $[a,b]$ be a finite interval, $-\infty < a < b < \infty$, $[a,b] \subset \mathbb{R}$, and $f(t)$ be a continuous function defined on $[a,b]$. The Riemann–Liouville (RL) fractional derivative of order α is given by

$$
{}^{RL}_{\ a}D^{\alpha}_{t}\, f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t (t-s)^{n-\alpha-1}\, f(s)\,\mathrm{d}s,
$$

$$\begin{cases} t \in [a,b], & \alpha \in \mathbb{C},\ Re(\alpha) \geqslant 0 \\ n = [Re(\alpha)] + 1, & n \in \mathbb{N}, \end{cases}$$

where $\mathbb{C}$ is the set of complex numbers and $\Gamma(\cdot)$ is Euler's gamma function, which generalizes factorial for non-integer arguments:

$$\Gamma(\alpha) = \int_0^\infty e^{-t}\, t^{\alpha-1}\, \mathrm{d}t, \quad \Gamma(\alpha+1) = \alpha\,\Gamma(\alpha), \quad \alpha \in \mathbb{C}.$$

Definition 2.2. Let $f(t)$ be a continuous function on $[a,b]$. The Riemann–Liouville fractional integral of order α is [39, 40]:

$${}^{RL}_{\ a}D_t^{-\alpha} f(t) \equiv {}^{RL}_{\ a}I_t^{\alpha} f(t) = \frac{1}{\gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s,\ t \in [a,b],\ \alpha \in \mathbb{C},\ Re(\alpha) > 0,$$

where $Re(\alpha)$ denotes the real part of α. The Caputo derivative is defined for a function $f(\cdot) : [a,b] \to \mathbb{C}$ which belongs to the space of absolutely continuous functions: $f(t) \in AC^n[a,b] = \{f(t) : \mathrm{d}^{n-1} f(t)/\mathrm{d}t^{n-1} \in AC[a,b]\}$, $n \in \mathbb{N}$.

Definition 2.3. The fractional derivative, in the Caputo sense, of order α, $\alpha \in \mathbb{C}$, $Re(\alpha) \geqslant 0$, for any function $f(t) \in AC^n[a,b]$ is given by [39, 40]:

$${}^{C}_{a}D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)\mathrm{d}s, & \alpha \notin \mathbb{N}_0,\ n = [Re(\alpha)] + 1,\ n \in \mathbb{N}, \\ f^{(n)}(t) = \dfrac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}, & \alpha = n \in \mathbb{N}_0. \end{cases}$$

Lemma 2.1 ([41] Generalized Grönwall Inequality). *Suppose $x(t)$, $a(t)$ are nonnegative and locally integrable on $0 \leqslant t < T$, $T \leqslant +\infty$, and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \leqslant t < T$, $g(t) \leqslant M = const$, $\alpha > 0$ with*

$$x(t) \leqslant a(t) + g(t) \int_0^t (t-s)^{\alpha-1}\, x(s)\,\mathrm{d}s$$

on this interval. Then

$$x(t) \leqslant a(t) + \int_0^t \Big[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1}\, a(s) \Big] \mathrm{d}s, \quad 0 \leqslant t < T.$$

Corollary 2.1. [42] *Under the hypothesis of Lemma 2.1, let $a(t)$ be a nondecreasing function on $[0,T)$. Then it holds:*

$$x(t) \leqslant a(t) E_\alpha(g(t)\Gamma(\alpha)t^\alpha),$$

where E_α is the Mittag-Leffler function defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}.$$

Lemma 2.2. [43] *Suppose fractional orders $\alpha > 0$, $\beta > 0$, $a(t)$ is a nonnegative function locally integrable on $[0,T)$, $g_1(t)$ and $g_2(t)$ are nonnegative, nondecreasing,* *continuous functions defined on* $[0,T)$*;* $g_1(t)\leqslant N_1$*,* $g_2(t)\leqslant N_2$*,* $(N_1,N_2=\text{const.})$*. Suppose* $x(t)$ *is nonnegative and locally integrable on* $[0,T)$ *with*

$$x(t)\leqslant a(t)+g_1(t)\int_0^t (t-s)^{\alpha-1}\,x(s)\,\mathrm{d}s+g_2(t)\int_0^t (t-s)^{\beta-1}\,x(s)\,\mathrm{d}s,\ \ t\in[0,T).$$

Then

$$x(t)\leqslant a(t)+\int_0^t [g(t)]^n\cdot\sum_{k=0}^{n}\frac{C_n^k[\Gamma(\alpha)]^{n-k}\,[\Gamma(\beta)]^k}{\Gamma((n-k)\alpha+k\beta)}(t-s)^{(n-k)\alpha+k\beta-1}\,a(s)\,\mathrm{d}s,\ \ t\in[0,T),$$

where $g(t)=g_1(t)+g_2(t)$ *and* $C_n^k=\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$*.*

Corollary 2.2. *Under the hypothesis of Lemma 2.2, let* $a(t)$ *be a nondecreasing function on* $[0,T)$*. Then*

$$x(t)\leqslant a(t)E_\kappa\left[g(t)(\Gamma(\alpha)t^\alpha+\Gamma(\beta)t^\beta)\right],$$

where $\kappa=\min(\alpha,\beta)$*.*

Lemma 2.3. [43] *Assume* $x(t)\in C^1([0,+\infty),\mathbb{R})$*,* $\dot{x}(t)\geqslant 0$*,* $\alpha>0$*. Then* $\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)\,\mathrm{d}s$ *is monotonically increasing with respect to* t*.*

Proof. Let

$$X(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)\,\mathrm{d}s.$$

Then

$$X(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)\,\mathrm{d}s=\int_0^t\frac{(s)^{\alpha-1}}{\Gamma(\alpha)}x(t-s)\,\mathrm{d}s.$$

Since

$$\dot{X}(t)=\int_0^t\frac{(s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\mathrm{d}x(t-s)}{\mathrm{d}t}\,\mathrm{d}s+\frac{t^{\alpha-1}}{\Gamma(\alpha)}x(0)\geqslant\int_0^t\frac{(s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\mathrm{d}x(t-s)}{\mathrm{d}t}\,\mathrm{d}s\geqslant 0,$$

one may conclude that $X(t)$ is monotonically increasing with respect to t. □

2.2. Motivation and preliminaries on neutral time-delay systems integer and fractional order. A homogeneous continuous linear time-invariant neutral system with multiple time-varying delays in state can be described by a linear neutral differential equation in state space:

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t}=A_0\mathbf{x}(t)+A_1\mathbf{x}(t-\tau_x(t))+A_{N_1}\frac{\mathrm{d}\mathbf{x}(t-\tau_{xN}(t))}{\mathrm{d}t},\tag{2.1}$$

with an associated continuous function of initial state:

$$\mathbf{x}(t)=\boldsymbol{\psi}_x(t),\quad t\in[-\tau_{xm},0],$$

where $x(t)\in\mathbb{R}^n$ is the state vector, A_0, A_1, and A_{N_1} are constant matrices with appropriate dimensions, $\tau_x(t)$, $\tau_{xN}(t)$ are the time-varying discrete delay and neutral delay in state, respectively, which satisfy:

$$0\leqslant\tau_x(t)\leqslant\tau_{xM},\quad 0\leqslant\tau_{xN}(t)\leqslant\tau_{xN},\quad\forall t\in J=[t_0,t_0+T],\ t_0\in\mathbb{R},\ T>0,\tag{2.2}$$

where t_0 is the initial time of observation of the system, τ_{xM} and τ_{xN} are constants, and τ_{xm} is defined to be $\max(\tau_{xM},\tau_{xN})$.

A nonhomogeneous system with time-varying delays in state and input (control) can be presented by the state-space equation:

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau_x(t)) + A_{N_1}\frac{\mathrm{d}\mathbf{x}(t-\tau_{xN}(t))}{\mathrm{d}t} + B_0\mathbf{x}(t) + B_1\mathbf{x}(t-\tau_u(t)), \tag{2.3}$$

with associated continuous functions of initial state and input (control):

$$\mathbf{x}(t) = \boldsymbol{\psi}_x(t), \quad t\in[-\tau_{xm},0], \quad \mathbf{u}(t) = \boldsymbol{\psi}_u(t), \quad t\in[-\tau_{uM},0], \tag{2.4}$$

where $u(t) \in \mathbb{R}^m$ is the control input B_0, and B_1 are constant matrices with appropriate dimensions, and $\tau_u(t)$ is time-varying delay in input (control) that satisfies $0 \leqslant \tau_u(t) \leqslant \tau_{uM}$. If the system (2.3) contains structured uncertainties, it can be written as:

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = (A_0+\Delta A_0)\mathbf{x}(t) + (A_1+\Delta A_1)\mathbf{x}(t-\tau_x(t)) + (A_{N_1}+\Delta A_{N_1})\frac{\mathrm{d}\mathbf{x}(t-\tau_{xN}(t))}{\mathrm{d}t} + B_0\mathbf{u}(t) + B_1\mathbf{u}(t-\tau_u(t)), \tag{2.5}$$

where the matrices ΔA_0, ΔA_1, ΔA_N present structural perturbations of the system. Recently, in [36] authors considered a neutral fractional time-delay system:

$$^{C}D^{\alpha}\mathbf{x}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau) + A_{N_1}{}^{C}D^{\alpha}\mathbf{x}(t-\tau) \tag{2.6}$$

$$\mathbf{x}(t) = \boldsymbol{\psi}_x(t), \quad t\in[-\tau,0], \tag{2.7}$$

where $^{C}D^{\alpha}(\cdots)$ denotes the Caputo fractional derivative of order $0 < \alpha \leqslant 1$. Besides, in [37] a particular class of two-term neutral fractional systems with time-varying delay and nonlinear perturbation is considered as follows:

$$^{C}D^{\alpha}\mathbf{x}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau_1(t)) + A_2{}^{C}D^{\beta}\mathbf{x}(t-\tau_2(t)) + G\mathbf{w} + f(t,\mathbf{x}(t),\mathbf{x}(t-\tau_1(t)),\mathbf{x}(t-\tau_2(t)),\mathbf{w}(t)) \tag{2.8}$$

$$\mathbf{x}(t) = \boldsymbol{\psi}_x(t), \quad t\in[-\tau,0], \quad \tau = \max(\tau_1,\tau_2), \tag{2.9}$$

where $^{C}D_t^{\alpha}$, $^{C}D_t^{\beta}$ denote Caputo fractional derivatives of order α, β, $0 < \beta < \alpha < 1$, and $0 \leqslant \tau_1(t) \leqslant \tau_1$, $0 \leqslant \tau_2(t) \leqslant \tau_2$.

Moreover, a nonhomogeneous neutral two-term system with time-varying delays in state and input (control) can be presented by the state-space equation:

$$^{C}D^{\alpha}\mathbf{x}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau_1(t)) + A_2{}^{C}D^{\beta}\mathbf{x}(t-\tau_2(t)) + B_0\mathbf{u}(t) + B_1\mathbf{u}(t-\tau_u(t)) \tag{2.10}$$

with associated continuous functions of initial state and input:

$$\begin{aligned} \mathbf{x}(t) &= \boldsymbol{\psi}_x(t), \quad t\in[-\tau,0], \quad \tau = \max(\tau_1,\tau_2), \\ \mathbf{u}(t) &= \boldsymbol{\psi}_u(t), \quad t\in[-\tau_{uM},0], \quad 0\leqslant\tau_u(t)\leqslant\tau_{uM}. \end{aligned} \tag{2.11}$$

The behavior of systems (2.1), (2.3) or (2.5), (2.6), (2.8) with given initial functions (2.4) is observed over a time interval $J = [t_0, t_0 + T] \subset \mathbb{R}$, where T may be either a real positive number or the ∞ symbol.

Therefore, both finite-time stability ($T \in \mathbb{R}^+$) and practical stability ($T = \infty$) may be analyzed at the same time. System trajectories and control actions are bounded by the time-invariant sets: S_δ – set of all initial states (state vectors) of the system, S_ε – set of all allowable states of the system, S_{γ_u} – set of all allowable control actions (control vectors), and S_{γ_0} – set of all initial control actions; $\delta, \varepsilon, \gamma_u, \gamma_0 \in \mathbb{R}^+$, $\delta < \varepsilon$. All these sets are bounded, connected and open.

Generally, the set S_ϱ may be defined as: $S_\varrho = \{\mathbf{x} : \|\mathbf{x}\|_Q^2 < \varrho\}$, $Q \in \mathbb{R}^{n\times n}$, $\varrho \in \mathbb{R}^+$, where $\|\mathbf{x}\|_Q^2 = \sqrt{\mathbf{x}^T Q \mathbf{x}}$ denotes the weighted norm and Q is a symmetric, positive definite matrix ($Q = Q^T > 0$). In this paper, the norm $\|(\cdot)\|$ will denote any vector norm, i.e. $\|(\cdot)\|_1$, $\|(\cdot)\|_2$, or $\|(\cdot)\|_\infty$, or the corresponding matrix norm induced by the equivalent vector norm, i.e. 1-, 2-, or ∞- norm, respectively. The initial functions (2.4) and their norm can be given in general form as:

$$\mathbf{x}(t_0 + t) = \boldsymbol{\psi}_x(t), \quad t \in [-\tau_{xm}, 0], \quad \begin{cases} \boldsymbol{\psi}_x(t) \in C([-\tau_{xm}, 0], \mathbb{R}^n) \\ \|\boldsymbol{\psi}_x\|_C = \sup\{\|\boldsymbol{\psi}_x(t)\|\} : t \in [-\tau_{xm}, 0], \end{cases}$$

$$\mathbf{x}(t_0 + t) = \boldsymbol{\psi}_u(t), \quad t \in [-\tau_{xm}, 0], \quad \begin{cases} \boldsymbol{\psi}_u(t) \in C([-\tau_{uM}, 0], \mathbb{R}^n) \\ \|\boldsymbol{\psi}_u\|_C = \sup\{\|\boldsymbol{\psi}_u(t)\|\} : t \in [-\tau_{uM}, 0], \end{cases}$$

where $C([-\tau_{xm}, 0], \mathbb{R}^n)$ and $C([-\tau_{uM}, 0], \mathbb{R}^n)$ denote Banach spaces of all continuous real vector functions defined on time intervals $[-\tau_{xm}, 0]$ and $[-\tau_{uM}, 0]$, respectively, which map these time intervals into $\mathbb{R}^n$, where norm is defined as: $\|\boldsymbol{\psi}\|_C = \sup_{-\tau \leqslant \boldsymbol{\psi}(\theta) \leqslant 0} \|\boldsymbol{\psi}(\theta)\|$. The usual smoothness condition is assumed, which means that there are no problems with existence, uniqueness and continuity of the solutions for the systems with respect to initial conditions [38].

Before proceeding further, the definitions of finite-time stability will be given for the homogeneous system (2.6), (2.8) and for the nonhomogeneous system (2.10) with associated initial functions (2.7), (2.9) and (2.11), respectively.

Definition 2.4. [12]: The time-delay system given by homogeneous state equation (2.6) satisfying initial condition (2.7) is finite-time stable with respect to $\{\delta, \varepsilon, t_0, J, \|(\cdot)\|\}$, $\delta < \varepsilon$, if and only if:

$$\|\boldsymbol{\psi}_x\|_C < \delta \Rightarrow \|\mathbf{x}(t)\| < \varepsilon, \quad \forall t \in J.$$

Definition 2.5. [13]The time-delay system given by nonhomogeneous state equation (2.10) satisfying initial conditions (2.11) is finite-time stable with respect to $\{\delta, \varepsilon, \gamma_u, \gamma_0, t_0, J, \|(\cdot)\|\}$, $\delta < \varepsilon$, if and only if:

$$\left.\begin{matrix} \|\boldsymbol{\psi}_x\|_C < \delta, \quad \|\boldsymbol{\psi}_u\|_C < \gamma_0 \\ \|\mathbf{u}(t)\| < \gamma_u, \quad \forall t \in J, \end{matrix}\right\} \Rightarrow \|\mathbf{x}(t)\| < \varepsilon, \quad \forall t \in J.$$

3. Main Results

3.1. Robust finite time stability of uncertain neutral fractional-order system with time-varying delays. Here, we examine the problem of sufficient conditions that enable system trajectories to stay within the a priori given sets for the class of uncertain neutral two-term fractional systems with time-varying delays in state and control, presented by the state equation:

$$\begin{aligned}(3.1)\quad {}^{C}D^{\alpha}\mathbf{x}(t) &= (A_0+\Delta A_0)\mathbf{x}(t)+(A_1+\Delta A_1)\mathbf{x}(t-\tau_x(t))+\\ &+(A_{N_1}+\Delta A_{N_1})\,{}^{C}D^{\beta}\mathbf{x}(t-\tau_{xN}(t))+B_0\mathbf{u}(t)+B_1\mathbf{u}(t-\tau_u(t)),\end{aligned}$$

with associated continuous functions of initial state and input (control):

$$(3.2)\qquad \mathbf{x}(t)=\boldsymbol{\psi}_x(t),\quad t\in[-\tau_{xm},0],\quad \mathbf{u}(t)=\boldsymbol{\psi}_u(t),\quad t\in[-\tau_{uM},0],$$

where time varying delays satisfy (2.2), as well as $\tau_{xm}=\max(\tau_{xM},\tau_{xN})$, $0\leqslant\tau_u(t)\leqslant\tau_{uM}$ and ${}^{C}D_t^{\alpha}$, ${}^{C}D_t^{\beta}$ denote Caputo fractional derivatives of order α,β, with $0<\beta<\alpha<1$.

THEOREM 3.1. *The nonhomogeneous neutral two-term fractional order time varying delay system (3.1) satisfying initial conditions (3.2) is finite-time stable with respect to $\{\delta,\varepsilon,\gamma_u,\gamma_0,J_0,\|(\cdot)\|\}$, $\delta<\varepsilon$, if the following condition is satisfied:*

$$\begin{aligned}\Big[1+\frac{\mu_{N_1}|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{\mu_\Sigma|t|^{\alpha}}{\Gamma(\alpha+1)}\Big]E_{\alpha-\beta}\Big[\Big(\frac{\mu_{N_1}}{\Gamma(\alpha-\beta)}+\frac{\mu_\Sigma}{\Gamma(\alpha)}\Big)(\Gamma(\alpha-\beta)t^{\alpha-\beta}+\Gamma(\alpha)t^{\alpha})\Big]\\ +\frac{\gamma_{0u}|t|^{\alpha}}{\Gamma(\alpha+1)}+\frac{\gamma_{01}\,\tau_{uM}^{\alpha}}{\Gamma(\alpha+1)}+\frac{\gamma_{1u}|t-\tau_u|^{\alpha}}{\Gamma(\alpha+1)}\leqslant\frac{\varepsilon}{\delta},\quad \forall t\in J_0,\end{aligned}$$

where:

$$\begin{gathered}\mu_\Sigma=\sum_{i=0}^{1}\mu_{A_i},\quad \mu_{A_i}=\sigma_{\max}(A_i)+\sigma_{\max}(\Delta A_i),\quad i=0,1,\\ \mu_{N_1}=\sigma_{\max}(A_{N_1})+\sigma_{\max}(\Delta A_{N_1}),\quad b_j=\|B_j\|,\quad j=0,1,\quad J_0=[0,T],\\ \gamma_{0u}=b_0\gamma_u/\delta,\quad \gamma_{1u}=b_1\gamma_u/\delta,\quad \gamma_{01}=b_1\gamma_0/\delta,\end{gathered}$$

with $\sigma_{\max}$ being the largest singular value of the matrix $(\cdot)$.

PROOF. In accordance with the property of the fractional order $0<\beta<\alpha<1$, a solution can be obtained in the form of the equivalent Volterra integral equation, where $t_0=0$:

$$\begin{aligned}(3.3)\quad \mathbf{x}(t)&=\boldsymbol{\psi}_x(0)-(A_{N_1}+\Delta A_{N_1})\boldsymbol{\psi}_x(-\tau_{xm})\frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\\ &+\frac{1}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta+1}(A_{N_1}+\Delta A_{N_1})\mathbf{x}(s-\tau_{xN}(s))\mathrm{d}s\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[(A_0+\Delta A_0)\mathbf{x}(s)+(A_1+\Delta A_1)\mathbf{x}(s-\tau_x(s))\\ &+B_0\mathbf{u}(s)+B_1\mathbf{u}(s-\tau_u(s))]\mathrm{d}s.\end{aligned}$$

Now, by using the norm $\|(\cdot)\|$ in equation (3.3), an estimate of solution $\mathbf{x}(t)$ can be obtained:

$$(3.4)\quad \|\mathbf{x}(t)\|\leqslant\|\boldsymbol{\psi}_x(0)\|+\|(A_{N_1}+\Delta A_{N_1})\|\,\|\boldsymbol{\psi}_x(-\tau_{xm})\|\,\frac{|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}$$

$$+\frac{1}{\Gamma(\alpha-\beta)}\int_0^t |(t-s)|^{\alpha-\beta-1}\,\|(A_{N_1}+\Delta A_{N_1})\|\,\|\mathbf{x}(s-\tau_{xN}(s))\|\,\mathrm{d}s$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t |(t-s)|^{\alpha-1}[\|(A_0+\Delta A_0)\mathbf{x}(s)+(A_1+\Delta A_1)\mathbf{x}(t-\tau_x(s))$$
$$+B_0\mathbf{u}(s)+B_1\mathbf{u}(s-\tau_u(s))\|]\mathrm{d}s.$$

Also, one can obtain:

$$\begin{aligned}\|(A_0+\Delta A_0)\mathbf{x}(t)+(A_1+\Delta A_1)\mathbf{x}(t-\tau_x(t))B_0\mathbf{u}(t)+B_1\mathbf{u}(t-\tau_u(t))\|\\ \leqslant(\sigma_{\max}(A_0)+\sigma_{\max}(\Delta A_0))\,\|\mathbf{x}(t)\|\\ +(\sigma_{\max}(A_1)+\sigma_{\max}(\Delta A_1))\,\|\mathbf{x}(t-\tau_x(t))\|\\ +\|B_0\|\,\|\mathbf{u}(t)\|+\|B_1\|\,\|\mathbf{u}(t-\tau_u(t))\|,\end{aligned}\tag{3.5}$$

where:

$$\|\mathbf{x}(t-\tau_x(t))\|\leqslant\sup\{\|\mathbf{x}(t^{\cdot})\|:t^{\cdot}\in[t-\tau_{xm},t]\},$$

Applying this inequality, (3.5) can be presented in the following manner:

$$\begin{aligned}&\|(A_0+\Delta A_0)\mathbf{x}(t)+(A_1+\Delta A_1)\mathbf{x}(t-\tau_x(t))+B_0\mathbf{u}(t)+B_1\mathbf{u}(t-\tau_u(t))\|\\ &\leqslant\mu_{A_0}\|\mathbf{x}(t)\|+\mu_{A_1}\|\mathbf{x}(t-\tau_x(t))\|+b_0\|\mathbf{u}(t)\|+b_1\|\mathbf{u}(t-\tau_u(t))\|\\ &\leqslant\mu_\Sigma\sup_{t^{\cdot}\in[t-\tau_{xm},t]}\|\mathbf{x}(t^{\cdot})\|+b_0\|\mathbf{u}(t)\|+b_1\|\mathbf{u}(t-\tau_u(t))\|,\quad t>\tau_{xm},\end{aligned}\tag{3.6}$$

$$\begin{aligned}&\|(A_0+\Delta A_0)\mathbf{x}(t)+(A_1+\Delta A_1)\mathbf{x}(t-\tau_x(t))+B_0\mathbf{u}(t)+B_1\mathbf{u}(t-\tau_u(t))\|\\ &\leqslant\mu_\Sigma\left(\sup_{t^{\cdot}\in[t-\tau_{xm},t]}\|\mathbf{x}(t^{\cdot})\|+\|\boldsymbol{\psi}_x\|_C\right)+b_0\|\mathbf{u}(t)\|+b_1\|\mathbf{u}(t-\tau_u(t))\|,\quad t>0^+.\end{aligned}$$

After combining (3.4) and (3.6), it holds:

$$\begin{aligned}\|\mathbf{x}(t)\|&\leqslant\|\boldsymbol{\psi}_x\|_C\left[1+\frac{(\sigma_{\max(A_{N_1})}+\sigma_{\max(\Delta A_{N_1})})|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right]\\ &+\frac{1}{\Gamma(\alpha-\beta)}\int_0^t|(t-s)^{\alpha-\beta-1}|(\sigma_{\max(A_{N_1})}+\sigma_{\max(\Delta A_{N_1})})\,\|\mathbf{x}(s-\tau_{xN}(s))\|\,\mathrm{d}s\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t|(t-s)^{\alpha-1}|\Bigg[\mu_\Sigma\left(\sup_{t^{\cdot}\in[s-\tau_{xm},s]}\|\mathbf{x}(t^{\cdot})\|+\|\boldsymbol{\psi}_x\|_C\right)\\ &\qquad+b_0\|\mathbf{u}(s)\|+b_1\|\mathbf{u}(s-\tau_u(s))\|\Bigg]\mathrm{d}s.\end{aligned}$$

Therefore, we can obtain

$$\begin{aligned}\|\mathbf{x}(t)\|&\leqslant\|\boldsymbol{\psi}_x\|_C\left[1+\frac{\mu_{N_1}|t|^{\alpha-b}}{\Gamma(\alpha-\beta+1)}+\frac{\mu_\Sigma|t|^\alpha}{\Gamma(\alpha+1)}\right]\\ &\quad+\frac{\mu_{N_1}}{\Gamma(\alpha-\beta)}\int_0^t|t-s|^{\alpha-\beta-1}\sup_{t^{\cdot}\in[s-\tau_{xm},s]}\|\mathbf{x}(t^{\cdot})\|\,\mathrm{d}s\end{aligned}$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t |(t-s)^{\alpha-1}|\Big[\mu_\Sigma\Big(\sup_{t^\cdot\in[s-\tau_{xm},s]}\|\mathbf{x}(t^\cdot)\|\Big)\Big]\mathrm{d}s$$
$$+\frac{b_0\gamma_u|t|^\alpha}{\Gamma(\alpha+1)}+\frac{b_1\gamma_0\tau_{uM}^\alpha}{\Gamma(\alpha+1)}+\frac{b_1}{\Gamma(\alpha)}\int_{\tau_{uM}}^t |(t-s)^{\alpha-1}|\,\|\mathbf{u}(s-\tau_{uM})\|\,\mathrm{d}s$$

$$\|\mathbf{x}(t)\|\leqslant\|\boldsymbol{\psi}_x\|_C\Big[1+\frac{\mu_{N_1}|t|^{\alpha-b}}{\Gamma(\alpha-\beta+1)}+\frac{\mu_\Sigma|t|^\alpha}{\Gamma(\alpha+1)}\Big]$$
$$+\frac{\mu_{N_1}}{\Gamma(\alpha-\beta)}\int_0^t |t-s|^{\alpha-\beta-1}\sup_{t^\cdot\in[s-\tau_{xm},s]}\|\mathbf{x}(t^\cdot)\|\,\mathrm{d}s$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t |(t-s)^{\alpha-1}|\Big[\mu_\Sigma\Big(\sup_{t^\cdot\in[s-\tau_{xm},s]}\|\mathbf{x}(t^\cdot)\|\Big)\Big]\mathrm{d}s$$
$$+\frac{b_0\gamma_u|t|^\alpha}{\Gamma(\alpha+1)}+\frac{b_1\gamma_0\tau_{uM}^\alpha}{\Gamma(\alpha+1)}+\frac{b_1}{\Gamma(\alpha)}\int_0^{\tau_{uM}} |(t-\tau_{uM}-s)^{\alpha-1}|\,\|\mathbf{u}(s)\|\,\mathrm{d}s$$

$$\|\mathbf{x}(t)\|\leqslant\|\boldsymbol{\psi}_x\|_C\Big[1+\frac{\mu_{N_1}|t|^{\alpha-b}}{\Gamma(\alpha-\beta+1)}+\frac{\mu_\Sigma|t|^\alpha}{\Gamma(\alpha+1)}\Big]$$
$$+\frac{\mu_{N_1}}{\Gamma(\alpha-\beta)}\int_0^t |t-s|^{\alpha-\beta-1}\sup_{t^\cdot\in[s-\tau_{xm},s]}\|\mathbf{x}(t^\cdot)\|\,\mathrm{d}s$$
$$+\frac{\mu_\Sigma}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\Big(\sup_{t^\cdot\in[s-\tau_{xm},s]}\|\mathbf{x}(t^\cdot)\|\Big)\Big]\mathrm{d}s$$
$$+\frac{b_0\gamma_u|t|^\alpha}{\Gamma(\alpha+1)}+\frac{b_1\gamma_0\tau_{uM}^\alpha}{\Gamma(\alpha+1)}+\frac{b_1\gamma_u|t-\tau_{uM}|^\alpha}{\Gamma(\alpha+1)}.$$

The nondecreasing function $a(t)$ can be introduced as:

$$a(t)=\|\boldsymbol{\psi}_x\|_C\Big[1+\frac{\mu_{N_1}|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{\mu_\Sigma|t|^\alpha}{\Gamma(\alpha+1)}\Big].$$

Since $\frac{\mu_{N_1}}{\Gamma(\alpha-\beta)}$, $\frac{\mu_\Sigma}{\Gamma(\alpha)}$ are monotonically increasing, nonnegative continuous functions on $J_0=[0,T]$, by applying Lemma 2.3 [42], we have

$$\sup_{t^\cdot\in[s-\tau_{xM},s]}\|\mathbf{x}(t^\cdot)\|\leqslant a(t)+\frac{\mu_{N_1}}{\Gamma(\alpha-\beta)}\int_0^t |(t-s)|^{\alpha-\beta-1}\sup_{t^\cdot\in[s-\tau_{xm},s]}\|\mathbf{x}(t^\cdot)\|\,\mathrm{d}s$$
$$+\frac{\mu_\Sigma}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\Big(\sup_{t^\cdot\in[s-\tau_{xm},s]}\|\mathbf{x}(t^\cdot)\|\Big)\Big]\mathrm{d}s$$
$$+\frac{b_0\gamma_u|t|^\alpha}{\Gamma(\alpha+1)}+\frac{b_1\gamma_0\tau_{uM}^\alpha}{\Gamma(\alpha+1)}+\frac{b_1\gamma_u|t-\tau_{uM}|^\alpha}{\Gamma(\alpha+1)}.$$

From Lemma 2.2 [40], we obtain:

$$\|\mathbf{x}(t)\|\leqslant\sup_{t^\cdot\in[s-\tau_{xM},s]}\|\mathbf{x}(t^\cdot)\|\leqslant a(t)E_\kappa[g(t)(\Gamma(\alpha-\beta)t^{\alpha-\beta}+\Gamma(\alpha)t^\alpha)],$$

where $g=g_1+g_2$, $g_1=\frac{\mu_{N_1}}{\Gamma(\alpha-\beta)}$, $g_2=\frac{\mu_\Sigma}{\Gamma(\alpha)}$ and $\kappa=\min(\alpha,\alpha-\beta)$ and

$$\|\mathbf{x}(t)\| \leqslant \delta\Big[1 + \frac{\mu_{N_1}|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\mu_{\Sigma}|t|^{\alpha}}{\Gamma(\alpha+1)}\Big]E_{\kappa}[g(t)(\Gamma(\alpha-\beta)t^{\alpha-\beta} + \Gamma(\alpha)t^{\alpha})] + \frac{b_0\gamma_u|t|^{\alpha}}{\Gamma(\alpha+1)} + \frac{b_1\gamma_0\tau_{uM}^{\alpha}}{\Gamma(\alpha+1)} + \frac{b_1\gamma_u|t-\tau_{uM}|^{\alpha}}{\Gamma(\alpha+1)}.$$

Finally, using the basic condition of Theorem 3.1, we can obtain the required finite time stability condition:

$$\|\mathbf{x}(t)\| < \varepsilon, \quad \forall t \in J_0.$$

This completes the proof. □

From Theorem 3.1, the result below follows.

THEOREM 3.2. *The homogeneous system given by* (2.6)*, when* $\mathbf{u}(t) \equiv 0$*,* $\mathbf{u}(t-\tau_u(t)) \equiv 0$*,* $\forall t \in J_0$*,* $\forall j \in \{1, 2, \dots, m\}$*, satisfying function of initial state* (2.7) *is finite-time stable with respect to* $\{\delta, \varepsilon, J_0, \|(\cdot)\|\}$*,* $\delta < \varepsilon$*, if the following condition is satisfied:*

$$\Big[1 + \frac{\mu_{N_1}|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\mu_{\Sigma}|t|^{\alpha}}{\Gamma(\alpha+1)}\Big]E_{\kappa}[g(t)(\Gamma(\alpha-\beta)t^{\alpha-\beta} + \Gamma(\alpha)t^{\alpha})] \leqslant \frac{\varepsilon}{\delta}, \quad \forall t \in J_0.$$

PROOF. The proof immediately follows from the proof of Theorem 3.1. □

REMARK 3.1. In this contribution, the neutral part of the system is different from other neutral systems, namely $D^{\beta}\mathbf{x}(t-\tau_{xN}(t))$, where the fractional order $\alpha \neq \beta$, so the given system becomes more general.

4. Numerical example

A nonhomogeneous perturbed neutral fractional time-varying delay system is considered:

$$^{C}D^{0.5}\mathbf{x}(t) = (A_0 + \Delta A_0)\mathbf{x}(t) + (A_1 + \Delta A_1)\mathbf{x}(t-\tau_x(t)) + (A_{N_1} + \Delta A_{N_1})^{C}D^{0.1}\mathbf{x}(t-\tau_{xN}(t)) + B_0\mathbf{u}(t) + B_1\mathbf{u}(t-\tau_u(t)),$$

where:

$$A_0 = \begin{bmatrix} -0.2 & 0 \\ -0.1 & 0.3 \end{bmatrix}, \quad \Delta A_0 = \begin{bmatrix} -0.02 & 0.01 \\ -0.01 & 0.03 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} -0.2 & 0.1 \\ 0 & -0.1 \end{bmatrix}, \quad \Delta A_1 = \begin{bmatrix} -0.05 & 0.01 \\ 0.02 & -0.03 \end{bmatrix},$$
$$A_{N_1} = \begin{bmatrix} 0.3 & 0 \\ -0.05 & 0.2 \end{bmatrix}, \quad \Delta A_{N_1} = \begin{bmatrix} 0.04 & 0 \\ 0 & 0.02 \end{bmatrix},$$

where $t_0 = 0$, $\tau_x = \tau_{xN} = 0.1$, $\tau_{xm} = 0.1$, $\tau_u = \tau_{uM} = 0.04$, with associated functions:

$$\mathbf{x}(t) = \boldsymbol{\psi}_x(t) = [0.05\ 0.05]^T, \quad t \in [t_0 - \tau_{xm}, t_0] = [-0.1\ 0],$$
$$\mathbf{u}(t) = \boldsymbol{\psi}_u(t) = [0.1\ 0], \quad t \in [t_0 - \tau_{um}, t_0] = [-0.04\ 0].$$

The task is to analyze the finite-time stability with respect to $\{\delta = 0.1,\ \varepsilon = 50,\ \gamma_0 = 0.2,\ \gamma_u = 2,\ J_0 = [0,3]\,s\}$. From the initial functions and the given state equation, it follows:

$$\|\boldsymbol{\psi}_x\|_C = \max_{t\in[-0.1,0]} \|\boldsymbol{\psi}_x(t)\| = \|\boldsymbol{\psi}_x\| = (0.05^2 + 0,05^2)^{1/2} = 0,071 < \delta = 0.1,$$

$$\|\boldsymbol{\psi}_u\|_C = \max_{t\in[-0.04,0]} \|\boldsymbol{\psi}_u(t)\| = \|\boldsymbol{\psi}_u\| = (0.1^2 + 0^2)^{1/2} = 0.1 < \gamma_0 = 0.2.$$

Moreover, other values are calculated as follows:

$$\sigma_{\max}(A_0) = 0.3257,\ \sigma_{\max}(\Delta A_0) = 0.0362,\ \sigma_{\max}(A_1) = 0.2288,\ \sigma_{\max}(\Delta A_1) = 0.04,$$
$$\sigma_{\max}(A_{N_1}) = 0.3071,\ \sigma_{\max}(\Delta A_{N_1}) = 0.2146,$$
$$\mu_{A_0} = \sigma_{\max}(A_0) + \sigma_{\max}(\Delta A_0) = 0.3619,\ \mu_{A_1} = \sigma_{\max}(A_1) + \sigma_{\max}(\Delta A_1) = 0.2688,$$
$$\mu_{N_1} = \sigma_{\max}(A_{N_1}) + \sigma_{\max}(\Delta A_{N_1}) = 0.5217,\ \mu_\Sigma = \mu_{A_0} + \mu_{A_1} = 0.6307,$$
$$b_0 = \|B_0\| = \sigma_{\max}(B_0) = 3,\ b_1 = \|B_1\| = \sigma_{\max}(B_1) = 2,$$
$$\gamma_{0u} = b_0\gamma_u/\delta = 60,\ \gamma_{1u} = b_1\gamma_u/\delta = 40,\ \gamma_{01} = b_1\gamma_0/\delta = 4.$$

Applying the condition of Theorem 3.1, it follows

$$\Big[1 + \frac{0.5217|T_e|^{0.4}}{\Gamma(1.4)} + \frac{0.6307|T_e|^{0.5}}{\Gamma(1.5)}\Big] E_{0,4}\Big[\frac{0.5217}{\Gamma(0.4)} + \frac{0.6037}{\Gamma(0.5)}(\Gamma(0.4)T_e^{0.4}) + (\Gamma(0.5)T_e^{0.5})\Big]$$
$$+\frac{60|T_e|^{0.5}}{\Gamma(1.5)} + \frac{4\cdot 0.04^{0.5}}{\Gamma(1.5)} + \frac{40|T_e - 0.04|^{0.5}}{\Gamma(1.5)} \leqslant \frac{50}{0.1}$$

and the estimated time of finite-time stability is $T_e \approx 1.273\,$s.

5. Conclusion

In this paper, finite-time stability analysis for a class of nonhomogeneous perturbed neutral fractional system with multiple time-varying delays was considered. By use of the generalized Gronwall inequality, a sufficient condition for robust finite-time stability for this class of neutral fractional time-varying delay systems has been proposed. Finally, a numerical example for this class of system was presented.

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РОБУСНА СТАБИЛНОСТ НА КОНАЧНОМ ВРЕМЕНСКОМ ИНТЕРВАЛУ НЕИЗВЕСНИХ НЕХОМОГЕНИХ НЕУТРАЛНИХ СИСТЕМА ФРАКЦИОНОГ РЕДА СА ВРЕМЕНСКИ ПРОМЕНЉИВИМ КАШЊЕЊИМА

Резиме. Разматра се проблем стабилности на коначном временском интервалу неизвесних нехомогених неутралних система фракционог реда са временски променљивим кашњењима где се предлаже поступак тестирања робусне стабилности. На основу проширеног облика уопштене Гронвалове неједнакости, добијен је нови довољан услов за робусност стабилности на коначном временском интервалу. На крају, ефикасност добијеног поступка илустрована је једним нумерички примером.

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