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A BRIEF SURVEY OF THE SPECTRAL NUMBERS IN FLOER HOMOLOGY

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ABSTRACT. We give a brief introduction and a partial survey of some of the results about spectral numbers in symplectic topology that we are aware of. Without attempting to be comprehensive, we will select just some of the constructions and ideas that, according to our personal taste and our point of view, give a flavor of this fast developing theory.

1. Introduction

Although symplectic geometry is defined in a smooth setting, the following classical theorem by Eliashberg [26] and Gromov [40] was the first indication that some symplectic notions and properties can be carried out in a C^0 world:

THEOREM 1.1 (Gromov–Eliashberg). *If a sequence of symplectic diffeomorphisms φ_k of a symplectic manifold converges in a C^0 sense to a diffeomorphism φ, then φ is also symplectic.*

In other words, the group of symplectic diffeomorphisms is closed in C^0 topology in the group of all diffeomorphisms of any symplectic manifold. This phenomenon is also called *the symplectic rigidity*. Since then, this kind of phenomena has been intensively studied and has grown into a huge research area, C^0 *symplectic topology* (see for example [19, 20, 41, 73] and the references cited therein).

Spectral numbers belong to a class of symplectic invariants of C^0-type.

Let $f\colon X \to \mathbb{R}$ be a continuous function on a topological space X. For $t \in \mathbb{R}$ denote by

$$f^t := \{x \in X \mid f(x) \leqslant t\}$$

the *sublevel sets* of f. Let $H_*(\cdot)$ be a homology theory.

DEFINITION 1.1. For $\alpha \in H_*(X) \setminus \{0\}$ the *spectral number* of f is defined as

$$c(\alpha, f) := \inf\{t \in \mathbb{R} \mid \alpha \in \operatorname{im}(\iota^t_*)\},$$

where $\iota^t\colon f^t \to X$ is the inclusion.

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If $H_*(\cdot)$ is singular homology, the definition above can be rephrased as

$$c(\alpha, f) := \inf_{[a]=\alpha} \max f(|a|),$$

where $|a|$ is the support of the singular cycle a and the infimum is taken over all singular cycles representing α.

If X is a compact smooth manifold and f is a smooth function, then $c(\alpha, f)$ is a critical value of f (and, since its definition does not involve derivatives, can be considered as a C^0-definition of a critical value). For that reason, the term *critical value selector* is sometimes used as a synonym for spectral numbers (another term in use is *spectral invariants*). For example, if X is path connected, then $c([\mathrm{pt}], f) = \min f$, and, if X is oriented and $[X]$ its fundamental class, then $c([X], f) = \max f$.

For fixed α, we have the function $c(\alpha, \cdot)\colon C(X) \to \mathbb{R}$, where $C(X)$ is the space of continuous functions on X with the uniform norm. An important property of this function is the following

Proposition 1.1. *The function $c(\alpha, \cdot)\colon C(X) \to \mathbb{R}$ is Lipschitz continuous.*

As a consequence, $c(\alpha, \cdot)$ is uniquely determined by its values on some dense subset of $C(X)$.

For example, if X is a compact smooth manifold, it is enough to know the values of $c(\alpha, \cdot)$ on the set of Morse functions. As the singular homology of a compact smooth manifold can be computed with the help of Morse homology (see [**10**, **82**, **83**]), we can rephrase Definition 1.1 in the following way. For a Morse function $f\colon X \to \mathbb{R}$, denote by $C_k(f)$ the free abelian group generated by its critical points with Morse index k, and by $C_k^t(f)$ the subgroup generated by critical points in f^t. It is well known that $C_k(f)$ is a chain complex with respect to the boundary operator which is well defined in terms of negative gradient lines of f, and that (since f decreases along the negative gradient lines) $C_k^t(f)$ is its subcomplex. The corresponding homology groups are denoted by $H_k(f)$ and $H_k^t(f)$. The obvious map $\jmath^t\colon C_k^t(f) \to C_k(f)$ gives rise to the map $\jmath_*^t\colon H_k^t(f) \to H_k(f)$. Now, for $\alpha \in H_k(f)$ we can set

$$c(\alpha, f) := \inf\{t \in \mathbb{R} \mid \alpha \in \mathrm{im}(\jmath_*^t)\}.$$

The function $c(\alpha, \cdot)$ is Lipshitz continuous on the dense subset of $C(X)$ consisting of Morse functions, and thus it extends to a Lipschitz continuous function on $C(X)$.

The advantage of this approach is that instead of exploiting the topology of the ambient X, it is based solely on the algebra of $C_*(f)$. This gives rise to the following possibility to modify the definition of spectral numbers in a symplectic environment, following the first work in this direction by Viterbo [**102**].

Let (M, ω) be a closed (or convex at infinity in the sense of [**27**]) symplectic manifold. The idea of construction of spectral numbers in symplectic topology is as follows. Instead of a topological space X in Definition 1.1, one takes the space of paths $\Gamma\colon [0,1] \to M$ (with prescribed boundary conditions), and instead of the function f one takes the Hamiltonian action functional

$$\mathcal{A}_H(\Gamma) := \int_\Gamma p\, dq - H\, dt.$$

Spectral numbers of the action functional are sometimes called *action selectors* (see [41]). Here, instead of the homology theory $H_*(\cdot)$, one takes the Floer homology theory defined by the chosen boundary condition. For periodic boundary conditions on closed symplectically aspherical symplectic manifolds, this construction is due to Schwarz [84], and for Lagrangian boundary conditions on a pair (zero-section, conormal bundle of a closed submanifold) in cotangent bundles, the construction is due to Oh [63, 64]. There are several generalizations of these constructions, such as [25, 45, 46, 54, 55, 57, 61, 63, 64].

The previous constructions can be considered as an infinite dimensional version of the construction given earlier by Viterbo [102]. Let E be a finite dimensional smooth vector bundle over the smooth closed manifold B, and let $S\colon E \to \mathbb{R}$ be a smooth function which is quadratic in fibers outside a compact subset of E. If the derivative d_vS in vertical directions is transversal to the zero section of the dual bundle E^*, so that $\Sigma_S := \{e \in E \mid d_vS(e) = 0\}$, then

$$i_S\colon \Sigma_S \to T^*B, \quad i_S(e) := dS(e)$$

is an exact Lagrangian immersion, i.e. $L_S := i_S(\Sigma_S)$ is an immersed exact Lagrangian submanifold. Viterbo's spectral numbers are the spectral numbers of S. If instead of the finite dimensional bundle S one takes the infinite dimensional fibration Ω over M consisting of smooth paths $\Gamma\colon [0,1] \to T^*B$ starting at the zero section, with the fibration map $\Gamma \mapsto \pi \circ \Gamma(1)$ (here $\pi\colon T^*B \to B$ is the canonical projection), and the Hamiltonian action functional $\mathcal{A}_H$ instead of S, and repeats the previous construction, a simple calculation shows that $\Sigma_{\mathcal{A}_H}$ is the set of Hamiltonian paths in T^*B starting at the zero section, and $L_{\mathcal{A}_H}$ is the Hamiltonian deformation of the zero section.

2. Construction of spectral numbers

2.1. Periodic boundary conditions. Here we will present the construction of spectral numbers in periodic orbits Floer homology for closed symplectically aspherical symplectic manifolds, following Schwarz [84]. For general closed symplectic manifolds we refer the reader to [39, 65, 66, 94, 95].

If (M,ω) is symplectically aspherical (i.e. $\omega|_{\pi_2(M)} = 0 = c_1|_{\pi_2(M)}$), then the Hamiltonian action functional is well defined on the space $\Omega(M)$ of contractible smooth loops. More precisely, if $\Gamma\colon [0,1] \to M$ is such a loop, $u\colon D^2 \to M$ a capping disc for Γ, and $H\colon [0,1] \times M \to \mathbb{R}$ a smooth Hamiltonian, then

$$\mathcal{A}_H(\Gamma) := \int_{D^2} u^*\omega - \int_0^1 H(t, \Gamma(t))dt$$

is well defined, i.e. does not depend on the choice of a capping disc. Its set of critical points $\mathrm{Crit}(\mathcal{A}_H)$ consists of time-one periodic orbits of the Hamiltonian flow ϕ_t^H defined by H. The free abelian group generated by $\mathrm{Crit}(\mathcal{A}_H)$, denoted by $CF_*(H)$, is a chain complex with the boundary map defined by counting pseudo-holomorphic cylinders connecting two periodic orbits. For $t \in \mathbb{R}$ the free abelian group $CF_*^t(H)$ generated by $\Gamma \in \mathrm{Crit}(\mathcal{A}_H)$ with $\mathcal{A}_H(\Gamma) \leqslant t$ is a chain subcomplex. The corresponding homology groups, called Floer homology groups, are denoted by $HF_*(H)$ and $HF^t_*(H)$. There exists a canonical isomorphism PSS: $H_*(M) \to HF_*(H)$, where $H_*(\cdot)$ is the Morse homology [76]. For $\alpha \in H_*(M) \setminus \{0\}$ the spectral number $\rho(\alpha, H)$ is defined by

$$\rho(\alpha, H) := \inf\{t \in \mathbb{R} \mid \mathrm{PSS}(\alpha) \in \mathrm{im}(\imath^t_*)\},$$

where $\imath^t_*\colon HF^t_*(H) \to HF_*(H)$ is the map induced by the inclusion $\imath^t\colon CF^t_*(H) \to CF_*(H)$.

Proposition 2.1. [84] *If $F, G\colon [0,1] \times M \to \mathbb{R}$ are smooth Hamiltonians, then*

$$|\rho(\alpha, F) - \rho(\alpha, G)| \leqslant \|F - G\|,$$

where

$$\|H\| := \int_0^1 \Big(\max_{x\in M} H(t,x) - \min_{x\in M} H(t,x)\Big)dt \tag{2.1}$$

is Hofer's norm.

As a corollary, $\rho(\alpha, H)$ can be defined, by continuity, for every continuous H.

For constant $C \in \mathbb{R}$, the Hamiltonians H and $H + C$ generate the same Hamiltonian paths. Thus it is natural to normalize the Hamiltonians so that

$$\int_M H\omega^{\wedge n} = 0.$$

The spectral numbers $\rho(\alpha, H)$ belong to the action spectrum $\mathrm{Spec}(H)$, which is the set of critical values of $\mathcal{A}_H\colon \Omega(M) \to \mathbb{R}$. If F and G are two normalized Hamiltonians generating the same time-one map $\phi^F_1 = \phi^G_1 = \phi$, such that the paths ϕ^F_t and ϕ^G_t in the group $\mathrm{Ham}(M)$ of Hamiltonian diffeomorphisms are homotopic with fixed endpoints, then $\mathrm{Spec}(F) = \mathrm{Spec}(G)$. However, if normalized Hamiltonians F and G generate the same time-one map ϕ, but ϕ^F_t and ϕ^G_t belong to different homotopy classes of paths, then their action spectra differ by a quantity associated to a certain group homomorphism

$$I\colon \pi_1(\mathrm{Ham}(M)) \to \mathbb{R}.$$

On symplectically aspherical manifolds this homomorphism vanishes [83, 87], and thus one can prove that spectral numbers $\rho(\alpha, H)$ depend only on the time-one map generated by H, i.e. that $\rho(\alpha, \cdot)$ is a function defined on $\mathrm{Ham}(M)$. In a general case $\rho(\alpha, \cdot)$ can be considered as a function defined on the universal cover $\widetilde{\mathrm{Ham}}(M)$ of $\mathrm{Ham}(M)$.

If F and G are two Hamiltonians which generate the same time-one map ϕ, then for any non-zero homology class α the difference $\rho(\alpha, F) - \rho(\alpha, G)$ is constant on $H_*(M) \setminus \{0\}$. Therefore, for $\alpha, \beta \in H_*(M) \setminus \{0\}$ the number

$$\gamma_\rho(\alpha, \beta, H) := \rho(\alpha, H) - \rho(\beta, H) \tag{2.2}$$

does not depend on the choice of H generating the time-one map $\phi := \phi^H_1$ and it is justified to denote it by $\gamma_\rho(\alpha, \beta, \phi)$.

A different approach to the construction of spectral numbers on aspherical symplectic manifolds, which uses only Gromov compactness, is given by Abbondandolo, Haug and Schlenk in [1].

In addition to C^0 continuity, spectral numbers have several important properties that were used for axiomatic definitions of *a weak action selector* and *an action selector* by Frauenfelder, Ginzburg and Schlenk in [38], where they proved that the existence of a (weak) action selector implies several energy-capacity inequalities.

Albers and Frauenfelder [5] defined spectral numbers for a perturbed action functional. Here the ambient symplectic manifold is exact and has contact type boundary, i.e. it is a compact smooth manifold W with boundary, with a symplectic form $\omega = -d\lambda$, such that there exists a vector field (called *the Liouville vector field*) Y pointing outside the boundary ∂W and defined by $L_Y\omega = \omega$. If (H, F) is a pair of Hamiltonians satisfying certain assumptions (the so called *Moser pair*, see [5]), the action functional under consideration is *the Rabinowitz action functional*

$$\mathcal{A}^{(H,F)}(\Gamma) := \int_0^1 \Gamma^*\lambda - \int_0^1 H(t, \Gamma(t))dt - \eta \int_0^1 F(t, \Gamma(t))dt$$

defined on the free loop space. Here η can be thought of as a Lagrange multiplier. The corresponding Floer homology theory is *the Rabinowitz Floer homology* (see [4–6], for the precise definitions).

2.2. Lagrangian boundary conditions. In [63, 64] Oh considered the case of cotangent bundle T^*B over the closed smooth manifold B, with canonical Liouville 1-form λ. Let $N \subset B$ be a closed submanifold and let $H\colon [0,1] \times T^*B \to \mathbb{R}$ be a smooth Hamiltonian function with compact support. As the action functional in this case consider

$$\mathcal{A}_H(\Gamma) := \int_0^1 \Gamma^*\lambda - \int_0^1 H(t, \Gamma(t))dt,$$

defined on the space $\Omega(B, N)$ of smooth paths $\Gamma\colon [0,1] \to T^*B$ with boundary condition $\Gamma(0) \in 0_B$, $\Gamma(1) \in \nu^*N$, where 0_B is the zero section in T^*B and ν^*N is the conormal bundle of N. The set $\mathrm{Crit}(\mathcal{A}_H)$ of critical points of $\mathcal{A}_H$ consists of Hamiltonian paths connecting 0_B and ν^*N. The free abelian group generated by $\mathcal{A}_H$ is the Floer chain group $CF_*(H, N)$, and the free abelian group generated by orbits $\Gamma \in \mathrm{Crit}(\mathcal{A}_H)$ with $\mathcal{A}_H(\Gamma) \leqslant t$ is the chain subgroup $CF_*^t(H, N)$. Here the boundary map is defined by counting perturbed pseudo-holomorphic strips, i.e. $u\colon [0,1] \times \mathbb{R} \to T^*B$ such that $\bar{\partial}_J u = -\nabla H(u)$, with boundary conditions $u(0, \cdot) \in 0_B$, $u(1, \cdot) \in \nu^*N$ and asymptotic conditions such that $u(\cdot, \pm\infty)$ are Hamiltonian orbits, i.e. generators of $CF_*(H, N)$. The corresponding Floer homology groups are denoted by $HF_*(H, N)$ and $HF_*^t(H, N)$. There exist isomorphisms $\mathrm{PSS} : H_*(N) \to HF_*(H, N)$, where $H_*(\cdot)$ is the Morse homology [80]. They can be made canonical, i.e. independent of the choice of Morse function [24, 43, 44]. For $\alpha \in H_*(N) \setminus \{0\}$ spectral numbers are defined by

$$\ell(\alpha, H; N) := \inf\left\{t \in \mathbb{R} \mid \mathrm{PSS}(\alpha) \in \mathrm{im}(\imath_*^t)\right\}, \tag{2.3}$$

where $\imath_*^t\colon HF_*^t(H, N) \to HF_*(H, N)$ is the map induced by the inclusion $\imath^t\colon CF_*^t(H, N) \to CF_*(H, N)$. We abbreviate

$$\ell(\alpha, H) := \ell(\alpha, H; B) \tag{2.4}$$

in case $N = B$. One can show (see for example [61, Lemma 2.7]) that if H_1, H_2 are properly normalized Hamiltonians such that their time-one maps coincide, then their spectral invariants coincide:

$$\phi_1^{H_1} = \phi_1^{H_2} \implies \ell(\alpha, H_1) = \ell(\alpha, H_2).$$

This justifies the notation $\ell(\alpha, \phi) := \ell(\alpha, H)$, where $\phi := \phi_1^H$.

The spectral numbers are contained in the action spectrum $\mathrm{Spec}(H; 0_B, \nu^*N)$, i.e. in the set of critical values of $\mathcal{A}_H : \Omega(0_B, \nu^*N) \to \mathbb{R}$. If F and G are Hamiltonians such that $\phi_1^F(0_B) = \phi_1^G(0_B)$, then $\mathrm{Spec}(F; 0_B, \nu^*N) = \mathrm{Spec}(G; 0_B, \nu^*N) + C$ for some constant $C \in \mathbb{R}$. Using this fact, and combining the geometric and dynamic definitions of Lagrangian Floer homology, one can show (see [63] for more details) that

$$\gamma_\ell(\alpha, \beta, L) := \ell(\alpha, H) - \ell(\beta, H)$$

does not depend on H as long as $\phi_1^H(0_B) =: L$ is a fixed Lagrangian submanifold.

It is known [21, 52] that every Hamiltonian deformation of zero section can be generated by a generating function $S\colon E \to \mathbb{R}$ fiberwise quadratic outside a compact set on a finite dimensional vector bundle E in a way that we mentioned in Introduction. Hence for $L = \phi_1^H(0_B)$ the spectral numbers $c(\alpha, S)$ introduced earlier by Viterbo are defined. In that case, again,

$$\gamma_c(\alpha, \beta, L) := c(\alpha, S) - c(\beta, S) \tag{2.5}$$

depends only on L. It turns out that these two constructions give the same result (see [56,57] for more details; see also [42] for similar relations in a different context and for the surfaces).

In [45, 46] the previous construction is generalized to the case where N is a submanifold with boundary in B. The assumption that B is closed can also be somewhat weakened [57].

Leclercq [54] constructed spectral numbers in the following situation. Let a symplectic manifold M be closed or convex at infinity. Let $L_0 \subset M$ be a closed Lagrangian submanifold, such that $\omega|_{\pi_2(M,L_0)} = 0$ and $\mu|_{\pi_2(M,L_0)} = 0$, where μ is the Maslov index. If $L_1 := \phi_1^H(L_0)$ is a Hamiltonian deformation of L_0, then the Floer homology $HF_*(H; L_0)$ and its filtrations by action $HF_*^t(H; L_0)$ are well defined. In this generality, there exists a morphism $\mathrm{PSS}\colon H_*(L_0) \to HF_*(H; L_0)$ [3,11,104]. This gives rise to the definition of spectral numbers in this case. Using spectral sequences machinery developed by Barraud and Cornea [12], Leclercq also introduced the higher order invariants, and showed that spectral invariants are actually their special case.

Leclercq and Zapolsky generalized the construction of spectral numbers to monotone Lagrangian submanifolds in [55]. Another generalization is given in [39].

Lagrangian spectral numbers are continuous with respect to Hofer's norm. Therefore, their definition can be extended by continuity to the cases that avoid several generic choices needed in definitions of Floer homology, and even to continuous Hamiltonians. In case of Lagrangian submanifolds, the Hofer distance has a remarkable generalization to *the cobordism distance* discovered by Cornea and Shelukhin [23]; its relation to spectral invariants is studied by Bisgaard [16].

For some symplectic manifolds and some Lagrangian submanifolds the Lagrangian Floer homology has the structure of a module over the Floer homology for periodic orbits, i.e. there exists a bilinear product

$$\circ\colon HF_*(H_1)\otimes HF_*(H_2;L)\to HF_*(H_3;L).$$

This product is due to the holomorphic "chimneys" (see [2, 3]) and it leads to a comparison of the Lagrangian spectral invariants with the spectral invariants for periodic orbits:

$$\ell(\mathrm{PSS}^{-1}(a\circ b),H_1\sharp H_2)\leqslant \rho(\mathrm{PSS}^{-1}(a),H_1)+\ell(\mathrm{PSS}^{-1}(b),H_2),$$

where $\sharp$ denotes the concatenation of Hamiltonians with respect to the time variable. Similar comparison formulae for spectral numbers with periodic and Lagrangian boundary conditions were proved in [61] in cotangent bundles, in [25] for weakly exact Lagrangian submanifolds and in [55] in a more general case.

3. Some objects constructed by spectral numbers

3.1. Graph selectors. Let B be a closed smooth manifold. Recall that the Lagrangian submanifold $L\subset T^*B$ is called *exact* if the restriction $\lambda|_L$ of the Liouville form is exact. In other words, if $\imath : L\to T^*B$ is the inclusion map, then L is exact if $\imath^*\lambda = dh_L$, for some function $h_L\colon L\to\mathbb{R}$. The simplest example of an exact Lagrangian submanifold is a graph $\mathrm{graph}(df) := \{(q, df(q)) \mid q\in B\}$ of the differential of a smooth function $f\colon B\to\mathbb{R}$. In this case $h_L = f\circ\pi|_L$. Every exact Lagrangian submanifold L in T^*B such that the restriction $\pi|_L\colon L\to B$ is a diffeomorphism is the graph of df for $f = h_L\circ(\pi|_L)^{-1}$.

More general examples are Lagrangian submanifolds generated by a generating function $S\colon E\to\mathbb{R}$ that we mentioned in Introduction. If L is a Hamiltonian deformation of a zero section, i.e. $L := \phi_1^H(0_B)$, then it is not difficult to see that L is exact with

$$h_L(x) = \mathcal{A}_H(\phi_t^H(\phi^1)^{-1}(x)).$$

Unlike the case of $L := \mathrm{graph}(df)$, where the projection $\pi|_L$ is a diffeomorphism and $f = h_L\circ(\pi|_L)^{-1}$, for the general exact Lagrangian submanifold L the primitive $h_L\colon L\to\mathbb{R}$ of the Liouville form $\lambda|_L$ is not related to a function on B.

Suppose that L is closed and that $\pi|_L$ is surjective. The composition $f = h_L\circ(\pi|_L)^{-1}$ can be considered as a multi-valued function on B. The Lagrangian submanifold L can be considered as an image of the multi-valued section $(\pi|_L)^{-1}$. The set of singular values of the projection $\pi|_L\colon L\to B$ is a set of measure zero. On the complement of the set of singular points $\pi|_L$ is a local diffeomorphism.

A genuine section $\sigma\colon B\to T^*B$, such that $\sigma(B)\subset L$ can be understood as a section that selects a branch of L understood as an image of a multi-valued section $(\pi|_L)^{-1}$. For that reason, it is natural to call such a section *a Lagrangian selector* of L (see [72, 75]).

A related object is introduced in the following

Definition 3.1. A *graph selector* of a closed exact Lagrangian submanifold $L\subset T^*B$ is a Lipschitz function $f\colon B\to\mathbb{R}$ which is differentiable on a dense open subset $U \subset B$ of full measure, such that for all $q \in U$ we have $(q, df(q)) \in L$ and $f \circ \pi|_L = h_L$.

Clearly, if f is a graph selector, then df is a Lagrangian selector.

It turns out that every closed exact Lagrangian submanifold admits a graph selector f. Moreover, if $df \equiv 0$ on U, then L coincides with the zero section. The construction is given by Chaperon [22], Sikorav [91] and Paternain, Polterovich, Siburg [75] via generating functions and by Oh [63] and Amorim, Oh and Dos Santos [7] via Floer homology. See also [8] for some discussion and the application.

The construction in [63] assumes that L is a Hamiltonian deformation of the zero section. In that case f is defined as $f(q) := \ell([q], H; \{q\})$, where $\ell([q], H; \{q\})$ is defined in (2.3), with $N = \{q\}$ and H such that $\phi_1^H(0_B) = L$. For a closed exact L, in [7] Amorim, Oh and Dos Santos used the wrapped Floer homology without making a reference to a Hamiltonian deformation.

3.2. Spectral norm. The spectral norm for compactly supported Hamiltonian diffeomorphisms in $\mathbb{R}^{2n}$ is first constructed by Viterbo [102]. By identifying the graph of Hamiltonian diffeomorphism ϕ with the Hamiltonian deformation L of the zero section in $T^*\Delta$ (where Δ is the diagonal in $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$), and compactifying Δ to $\mathbb{S}^{2n}$, Viterbo defined the norm $\gamma_c(\phi) := \gamma_c([\mathbb{S}^{2n}], [\mathrm{pt}], L)$, where γ_c is defined by (2.5).

In a similar way, for a closed symplectically aspherical manifold M, the function

$$\gamma\colon \operatorname{Ham}(M) \to \mathbb{R}, \quad \gamma(\phi) := \gamma_\rho([M], [\mathrm{pt}], \phi),$$

where $\gamma_\rho(\alpha, \beta, \phi)$ is defined by (2.2), is a norm on the group of Hamiltonian diffeomorphisms of M, i.e. it holds:

(1) $\gamma(\phi) \geqslant 0$
(2) $\gamma(\phi) = 0$ if and only if $\phi = \mathrm{id}$
(3) $\gamma(\phi^{-1}) = \gamma(\phi)$
(4) $\gamma(\phi\psi) \leqslant \gamma(\phi) + \gamma(\psi)$.

This norm is called the *spectral norm*. It is invariant under the conjugation by symplectomorphisms, i.e. $\gamma(\phi) = \gamma(\theta\phi\theta^{-1})$ for any $\phi \in \operatorname{Ham}(M)$ and any symplectomorphism θ. It is bounded from above by Hofer's distance

$$\gamma(\phi) \leqslant d_{\mathrm{Hofer}}(\mathrm{id}, \phi).$$

Here $d_{\mathrm{Hofer}}(\mathrm{id}, \phi) := \inf \|H\|$, where $\|\cdot\|$ is Hofer's norm (2.1) and the infimum is taken over all Hamiltonians H having ϕ as the time-one map. The norm γ satisfies the energy–capacity inequality $\gamma(\phi) \leqslant 2e(\operatorname{supp}(\phi))$, where $e(\cdot)$ denotes the displacement energy of the set $e(A) := \inf\{d_{\mathrm{Hofer}}(\mathrm{id}, \psi) \mid \psi(A) \cap A = \emptyset\}$ and $\operatorname{supp}(\phi) := \overline{\{x \in M \mid \phi(x) \neq x\}}$ is the support of a diffeomorphism ϕ. The proofs can be found in [66,69,84,95,96]. Buhovsky, Humilière and Seyfaddini proved that the spectral norm is continuous in C^0 topology [19], answering the question posed by Oh [70]. Some estimates relating the difference of spectral invariants with the C^0 distance of the corresponding Hamiltonian flows are obtained by Seyfaddini [85,86].

Note that, unlike Hofer's distance, the spectral norm is not intrinsic, i.e. it is not obtained as an infimum of the length of paths. However, it has many applications to Hofer's geometry. For example, it is exploited to describe the geodesics and length minimizing paths in Hofer's geometry, following the first work by Bialy and Polterovich [13] (see e.g. [58–60, 67, 68, 71]).

Using the Lagrangian spectral numbers, one can define the spectral distance between Lagrangian submanifolds in a similar way. For example, if B is a closed smooth manifold and L_0, L_1 are two Hamiltonian deformations of the zero section in T^*B, their spectral distance $\gamma(L_0, L_1)$ is a generalization of the spectral distance for Hamiltonian diffeomorphisms. It satisfies the inequality $\gamma(L_0, L_1) \leqslant d_{\text{Hofer}}(L_0, L_1)$. It is conjectured by Viterbo [103] that if the $U \subset T^*B$ is a bounded domain, the set of Hamiltonian deformations of the zero section contained in U is bounded in the metric γ. It is proved for some manifolds by Shelukhin [88, 89]. Some bounds on spectral metric that depend on the boundary depth of Floer complexes for the pairs (L_0, F) and (L_1, F) for fiber $F \subset T^*B$ are given by Biran and Cornea in [14]. Note that, in contrast to these results, it is proved by Khanevsky [48] that Hofer's metric on the same space is unbounded in some cases, and conjectured that it also holds in general.

3.3. Quasi-morphisms. By the theorem proved by Banyaga [9] in 1978, the group $\mathrm{Ham}(M)$ of Hamiltonian diffeomorphisms of a closed symplectic manifold is simple. Therefore, there are no nontrivial homomorphisms from this group to the additive group of real numbers. Similarly, the universal cover $\widetilde{\mathrm{Ham}}(M)$ for closed M is a perfect group; thus this group also does not admit non-trivial homomorphisms to $(\mathbb{R}, +)$.

A weaker notion, which has been in use in algebra, topology, geometry and dynamics for a while, is that of a *quasi-morphism* (see [50]). Its application to the group of Hamiltonian diffeomorphisms or its universal cover was extensively studied by Entov, Polterovich [29–31] and several other authors [15, 17, 18, 39, 51, 61, 74, 95, 97] (see also [28, 77] and the references therein).

If $\mathcal{G}$ is a group, a map $\mu\colon \mathcal{G} \to \mathbb{R}$ is a *quasi-morphism* if there exists a constant $C \geqslant 0$ such that $|\mu(gh) - \mu(g) - \mu(h)| \leqslant C$ for all $g, h \in \mathcal{G}$. This property is called *quasi-additivity*. A quasi-morphism is *homogeneous* if $\mu(g^k) = k \cdot \mu(g)$ for all $g \in \mathcal{G}$, $k \in \mathbb{Z}$. Every quasi-morphism can be homogenized by defining

$$\bar{\mu}(g) := \lim_{k \to \infty} \frac{\mu(g^k)}{k}.$$

It can be proved that the above limit exists and that $\bar{\mu}$ is a homogeneous quasi-morphism (see [77]). The quasi-morphism $\bar{\mu}$ is called *the homogenization* of μ.

If $\mathcal{G}$ is a Lie group and $\mathfrak{g}$ its Lie algebra, the derivative of a quasi-morphism is a mapping $\zeta\colon \mathfrak{g} \to \mathbb{R}$ called *a quasi-state* (see [28, 30, 77] for a definition in full generality).

Sometimes a quasi-morphism on $\widetilde{\mathrm{Ham}}(M)$ can be defined via spectral numbers (see [77] and the references therein). In [77] the *subadditive spectral invariant* is defined axiomatically as a function $c\colon \widetilde{\mathrm{Ham}}(M) \to \mathbb{R}$ satisfying the following:

(1) $c(\psi\phi\psi^{-1}) = c(\phi)$
(2) $c(\phi\psi) \leqslant c(\phi) + c(\psi)$

(3) $\int_0^1 \min(F_t - G_t)dt \leqslant c(\phi) - c(\psi) \leqslant \int_0^1 \max(F_t - G_t)dt$, where ϕ and ψ are generated by normalized Hamiltonians F and G

(4) $c(\phi) \in \mathrm{Spec}(\phi)$ for nondegenerate ϕ (see [77] for details).

The term *subadditive* is related to the second axiom above. In the presence of subadditive spectral invariant c the function defined by $\nu_c(\phi) := c(\phi) + c(\phi^{-1})$ for $\phi \neq \mathrm{id}$ and $\nu_c(\mathrm{id}) = 0$ is a pseudo-norm on $\widetilde{\mathrm{Ham}}(M)$, called *the spectral pseudo-norm.*

It is proved in [77, Proposition 4.8.1] that if a closed symplectic manifold M admits a subadditive spectral invariant c with bounded spectral pseudo-norm, then $\sigma(\phi) := \lim_{k\to\infty} c(\phi^k)/k$ is a homogeneous quasi-morphism. Moreover, the normalized quasi-morphism $\mu := -V \cdot \sigma$ (where $V := \int_M \omega^{\wedge n}$ is the volume of M) also satisfies *the Calabi property*: for every open displaceable subset $U \subset M$ the restriction $\mu|_{\widetilde{\mathrm{Ham}}(U)}$ is equal to *the Calabi homomorphism*

$$\mathrm{Cal}_U \colon \widetilde{\mathrm{Ham}}(U) \to \mathbb{R}, \quad \mathrm{Cal}_U(\phi) := \int_0^1 \int_M H_t \omega^{\wedge n} dt.$$

Note that Cal_U is a *homomorphism* – in contrast to the closed case, the group of Hamiltonian diffeomorphisms (or its universal cover) of an open symplectic manifold admits a non-trivial homomorphism Cal to $\mathbb{R}$. A theorem by Banyaga [9] states that the kernel of Calabi homomorphism is a simple group.

Spectral numbers $\rho(\alpha, \phi)$ are defined in maximal generality for every $\phi \in \widetilde{\mathrm{Ham}}(M)$ and for every non-zero *quantum* homology class (see [65, 66, 94, 95]). They satisfy the triangle inequality $\rho(\alpha \star \beta, \phi\psi) \leqslant \rho(\alpha, \phi) + \rho(\beta, \psi)$, where $\star$ is the quantum product. If there is an idempotent non-zero element in the quantum homology ring, i.e. an element e such that $e^2 = e$, then $c(\cdot) := \rho(e, \cdot)$ satisfies the subadditivity axiom. In [77, Theorem 12.6.1] gives sufficient conditions for a corresponding spectral pseudo-norm to be bounded. As a consequence, if the quantum homology of a closed monotone symplectic manifold M has a field as a factor, then there exists a homogeneous quasi-morphism μ on $\widetilde{\mathrm{Ham}}(M)$, which satisfies the Calabi property (see [28, Theorem 3.1] and [77, Corollary 12.6.2]).

In general, if e is an idempotent non-zero element in a quantum homology the ring, μ constructed as before has weaker properties (see [28, Theorem 3.2]). Instead of being quasi-additive, it is *partially quasi-additive*, i.e. for a displaceable open set U there exists $C > 0$ such that

$$|\mu(gh) - \mu(g) - \mu(h)| \leqslant C \min\{\|\phi\|_U, \|\psi\|_U\},$$

where $\|\cdot\|_U$ is *Banyaga's fragmentation norm* (see [9]). Instead of being homogeneous, it is *partially homogeneous*, i.e. $\mu(\phi^k) = k\mu(\phi)$ for any *non-negative* integer k. Such μ is called *a partial quasi-morphism.*

For a closed manifold B, Monzner, Vichery and Zapolsky [61] defined the mapping $\mu_0 \colon \mathrm{Ham}(T^*B) \to \mathbb{R}$ on a group of compactly supported Hamiltonian diffeomorphisms in T^*B by

$$\mu_0(\phi) := \lim_{k\to\infty} \frac{\ell([B], \phi^k)}{k},$$

where ℓ is defined by (2.3) and (2.4). More generally, for $a \in H^1(B)$ and for a one-form $\alpha\colon B \to T^*B$ representing cohomology class a they defined

$$\mu_a(\phi) := \mu_0(T_{-\alpha}\phi T_\alpha),$$

where T_α is a symplectomorphism of T^*B defined by $T_\alpha(q,p) := (q, p + \alpha(q))$, and proved that it has the properties analogous to those of a partial quasi-morphism. Due to the results of Shelukhin [88] and Kislev–Shelukhin [49], μ_a are genuine quasi-morphisms for suitable bases B (Zoll spaces etc). In a special case of $B = \mathbb{T}^n$, Monzner, Vichery and Zapolsky in [61] proved that $\mu_p(\phi_1^H) = \bar{H}(p)$, where $\bar{H}$ is the Viterbo's homogenization [103] of H and $p \in \mathbb{R}^n \cong H^1(\mathbb{T}^n)$ (see also [62] for the relation between symplectic homogenization and Calabi quasi-states).

4. Concluding remarks

Floer homology, the infinite dimensional version of Morse theory for the Hamiltonian action functional, is one of the main tools in symplectic topology. It was created by Andreas Floer in a series of papers [32–36] as a part of the proof of the Arnold conjecture about the minimal number of periodic Hamiltonian orbits (or the Lagrangian intersections in another version). Under certain transversality assumptions, it is defined on a large class of symplectic manifolds, and, as an algebraic object, independent of the choice of Hamiltonian. This feature of Floer homology makes it suitable for the study of the properties shared by all Hamiltonians, such as the aforementioned Arnold conjecture.

In order to obtain an algebraic object adapted to a particular Hamiltonian, one needs a construction that would depend on its choice. The use of filtration by the levels of the action functional meets that requirement. Chronologically, the first construction of this type were the spectral numbers. Our intention in this paper was to illustrate the basic ideas of their construction and some of its applications in the simplest form. We have not touched on the constructions in which these ideas were further developed. Using the chain level Floer theory, Oh [65] developed a more delicate version of similar constructions in order to give some descriptions of minimizing paths in Hofer's geometry. For Usher's notion of boundary depth and application we refer the reader to [95, 98, 99, 101]. The recent generalization of the ideas of filtered Floer's theory within the theory of persistent modules and barcodes has been the subject of a large number of papers [19, 37, 47, 49, 53, 78, 79, 81, 88–90, 92, 93, 100, 101, 105], to mention just a few.

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КРАТКИ ПРЕГЛЕД ИСТРАЖИВАЊА О СПЕКТРАЛНИМ БРОЈЕВИМА У ФЛОРОВОЈ ХОМОЛОГИЈИ

Резиме. Дајемо кратак и делимичан преглед резултата о спектралним бројевима у симплектичкој топологији који су нам познати. Без намере да будемо свеобухватни, издвајамо само неке конструкције и идеје, које, у складу са нашим субјективним избором и тачком гледишта, приказују начин на који се ова теорија брзо развија.

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