THEORETICAL AND APPLIED MECHANICS
Volume 47 (2020) Issue 2, 181–204 DOI: https://doi.org/10.2298/TAM200826014S

CLASSIFICATION OF LEFT INVARIANT METRICS ON 4-DIMENSIONAL SOLVABLE LIE GROUPS

Tijana Šukilović

Abstract. In this paper the complete classification of left invariant metrics of arbitrary signature on solvable Lie groups is given. By identifying the Lie algebra with the algebra of left invariant vector fields on the corresponding Lie group G, the inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g} = \operatorname{Lie} G$ extends uniquely to a left invariant metric g on the Lie group. Therefore, the classification problem is reduced to the problem of classification of pairs $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ known as the metric Lie algebras. Although two metric algebras may be isometric even if the corresponding Lie algebras are non-isomorphic, this paper will show that in the 4-dimensional solvable case isometric means isomorphic.

Finally, the curvature properties of the obtained metric algebras are considered and, as a corollary, the classification of flat, locally symmetric, Ricci-flat, Ricci-parallel and Einstein metrics is also given.

1. Preliminaries

The pair $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$, where $\mathfrak{g}$ is a Lie algebra and $\langle\cdot,\cdot\rangle$ an inner product of arbitrary signature, is called a *metric Lie algebra*. We denote by $\mathfrak{f}$ a metric Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ and by $M(\mathfrak{f})$ the (pseudo-)Riemannian space of the group G with the metric g of arbitrary signature. Note that the inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ extends uniquely to a left invariant metric g on the corresponding simply connected Lie group G. Therefore, the problem of classification of metrics on solvable Lie groups is reduced to the problem of classification of metric algebras.

Milnor's classification of 3-dimensional Lie groups with a left invariant positive definite metric [17] has become a classic reference, while the corresponding Lorentz classification [11] followed twenty years later. In dimension four, only partial results are known. The classification of 4-dimensional Riemannian Lie groups is due to Bérard-Bérgery [7]. Jensen [14] studied Einstein homogeneous spaces with a Riemannian (positive definite) metric, while Karki and Thompson [15] studied Einstein manifolds that arise from right invariant Riemannian metrics on a 4-dimensional Lie group. Calvaruso and Zaeim [9] have classified Lorentz left invariant metrics on the Lie groups that are Einstein or Ricci-parallel. Also, the classification in thecase

2010 *Mathematics Subject Classification:* 22E25; 53B30.
Key words and phrases: solvable Lie groups, left invariant metrics, metric algebra, Ricci-parallel metrics, Einstein spaces.

of nilpotent Lie groups was extensively studied in both the Riemannian [16] and the pseudo-Riemannian setting [8, 20].

One of the open questions and a possible direction for future research would be the classification of sub-Riemannian and nonholonomic Riemannian structures on 4-dimensional Lie groups. In the case of 3-dimensional Lie groups, a complete classification of left-invariant sub-Riemannian structures in terms of the basic differential invariants has been given by Agrachev and Barilari in [1], while the nonholonomic Riemannian structures have been investigated by Barrett et al. [5, 6].

In this paper the case of 4-dimensional solvable Lie groups is considered. The Lie group G is *solvable* if its corresponding Lie algebra is *solvable*, meaning that its derived series terminates in the zero subalgebra. The solvable Lie algebra $\mathfrak{g}$ is said to be *completely solvable* if ad_x has real eigenvalues for all $x \in \mathfrak{g}$.

From the following proposition it is clear that all simply connected 4-dimensional solvable Lie groups can be divided into three types.

Proposition 1.1. [4, 7]. *The simply connected 4-dimensional solvable Lie groups are:*

- I *the non-trivial semi-direct products* $\mathbb{R} \ltimes E(2)$ *and* $\mathbb{R} \ltimes E(1,1)$*;*
- II *the non-nilpotent semi-direct products* $\mathbb{R} \ltimes H_3$*, where* H_3 *denotes the Heisenberg group;*
- III *the semi-direct products* $\mathbb{R} \ltimes \mathbb{R}^3$*.*

On the algebra level, it can be said that all 4-dimensional solvable Lie algebras are given by $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{g}_3$, where the generator of $\mathfrak{r}$ acts as a derivation over $\mathfrak{g}_3$. A more refined classification of the corresponding Lie algebras is given by the following proposition.

Proposition 1.2. [18]. *Let* $\mathfrak{g}$ *be a* 4*-dimensional solvable Lie algebra. Then if* $\mathfrak{g}$ *is not abelian, it is equivalent to one of the algebras listed below:*

$\mathfrak{g}_2 + 2\mathfrak{g}_1$: $[e_4, e_1] = e_1$;
$2\mathfrak{g}_2$: $[e_1, e_2] = e_2$, $[e_3, e_4] = e_4$;
$\mathfrak{g}_{3,1} + \mathfrak{g}_1$: $[e_1, e_2] = e_3$;
$\mathfrak{g}_{3,2} + \mathfrak{g}_1$: $[e_4, e_2] = e_2$, $[e_4, e_3] = e_2 + e_3$;
$\mathfrak{g}_{3,3} + \mathfrak{g}_1$: $[e_4, e_2] = e_2$, $[e_4, e_3] = e_3$;
$\mathfrak{g}_{3,4}^{\alpha} + \mathfrak{g}_1$: $[e_4, e_2] = e_2$, $[e_4, e_3] = \alpha e_3$, $-1 \leqslant \alpha < 1$, $\alpha \neq 0$;
$\mathfrak{g}_{3,5}^{\alpha} + \mathfrak{g}_1$: $[e_4, e_2] = \alpha e_2 - e_3$, $[e_4, e_3] = e_2 + \alpha e_3$, $\alpha \geqslant 0$;
$\mathfrak{g}_{4,1}$: $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$;
$\mathfrak{g}_{4,2}^{\alpha}$: $[e_4, e_1] = \alpha e_1$, $[e_4, e_2] = e_2$, $[e_4, e_3] = e_2 + e_3$, $\alpha \neq 0$;
$\mathfrak{g}_{4,3}$: $[e_4, e_1] = e_1$, $[e_4, e_3] = e_2$;
$\mathfrak{g}_{4,4}$: $[e_4, e_1] = e_1$, $[e_4, e_2] = e_1 + e_2$, $[e_4, e_3] = e_2 + e_3$;
$\mathfrak{g}_{4,5}^{\alpha,\beta}$: $[e_4, e_1] = e_1$, $[e_4, e_2] = \alpha e_2$, $[e_4, e_3] = \beta e_3$, $-1 < \alpha \leqslant \beta \leqslant 1$, $\alpha\beta \neq 0$;
$\mathfrak{g}_{4,6}^{\alpha,\beta}$: $[e_4, e_1] = \beta e_1$, $[e_4, e_2] = \alpha e_2 - e_3$, $[e_4, e_3] = e_2 + \alpha e_3$, $\alpha \geqslant 0$, $\beta \neq 0$;
$\mathfrak{g}_{4,7}$: $[e_1, e_2] = e_4$, $[e_1, e_3] = e_1$, $[e_2, e_3] = e_1 + e_2$, $[e_4, e_3] = 2e_4$;
$\mathfrak{g}_{4,8}^{\alpha}$: $[e_1, e_2] = e_4$, $[e_1, e_3] = e_1$, $[e_2, e_3] = \alpha e_2$, $[e_4, e_3] = (1 + \alpha)e_4$, $|\alpha| \leqslant 1$;
$\mathfrak{g}_{4,9}^{\alpha}$: $[e_1, e_2] = e_4$, $[e_1, e_3] = \alpha e_1 - e_2$, $[e_2, e_3] = e_1 + \alpha e_2$, $[e_4, e_3] = 2\alpha e_4$, $\alpha \geqslant 0$;

$\mathfrak{g}_{4,10}$: $[e_1, e_3] = e_1,\ [e_1, e_4] = -e_2,\ [e_2, e_3] = e_2,\ [e_2, e_4] = e_1.$

REMARK 1.1. All the possible complex Lie algebras of dimension at most four were classified by Lie himself. The classification by Mubarakzyanov is listed here since it was the first complete and correct classification of real Lie algebras of dimension $\leqslant 4$. Also, the original notation is used with one addition: if the algebra definition uses parameters, these parameters will be exponents. For the relations between the lists of algebras in different notations, refer to Table A.1 in the Appendix.

The metric algebras $\mathfrak{f}$ and $\bar{\mathfrak{f}}$ are said to be *isometric* if there exists an isomorphism $\phi\colon \mathfrak{f} \to \bar{\mathfrak{f}}$ of Euclidean spaces preserving the curvature tensor and its covariant derivatives. This translates to the condition that metric algebras are isometric if and only if they are isometric as Riemannian spaces (see Proposition 1.3 below). Although two isomorphic metric algebras are also isometric, the converse is not true. In general, two metric algebras may be isometric even if the corresponding Lie algebras are non-isomorphic. The test to determine whether any two given solvable metric algebras (i.e. solvemanifolds) are isometric was developed by Gordon and Wilson in [**12**] and it will be discussed in detail later. However, in the completely solvable case, isometric means isomorphic, as stated in the following proposition.

PROPOSITION 1.3. [**2**]. *The metric Lie algebras $\mathfrak{f}$ and $\bar{\mathfrak{f}}$ are isometric if and only if the corresponding Riemannian spaces $M(\mathfrak{f})$ and $M(\bar{\mathfrak{f}})$ are isometric. Isometric completely solvable metric Lie algebras are isomorphic.*

The paper is organized as follows. In Section 2, the idea of classification is explained. Each type of Lie algebra is separately considered and the main result is stated case by case in Theorem 2.5 (type I), Theorem 2.4 (type II) and Theorems 2.1, 2.2 and 2.3 (type III). Although many non-isomorphic Lie algebras share the same set of inner products, they have distinct curvature properties and therefore completely different geometry. This is best illustrated in Section 3, where the curvature properties of the metrics obtained in the classification are further investigated. As a corollary, the metrics that are flat, locally symmetric, Ricci-flat, Ricci-parallel and Einstein are classified (Table 1 and Corollary 3.2).

2. Classification of left invariant metrics

There are two, in some way dual, approaches to the classification problem. The first one is to fix a Lie algebra basis in a way to make the commutator relations as simple as possible and then to adapt the inner product to it. The second one is to start from the basis that makes the inner product take the most basic form, while allowing the Lie brackets to be arbitrary, but satisfying the Jacobi identity. Although the second approach is useful when a curvature condition is introduced (see [**9**]), the main drawback is that the complete classification must be obtained in order for the type of Lie algebra to be recognized. In general, if no additional restriction is imposed, the problem becomes too complicated. Therefore, the first approach is used here.

Let us fix the basis $\{e_1, e_2, e_3, e_4\}$ of the Lie algebra $\mathfrak{g}$ in such a way that the structural equations are as in Proposition 1.2. Denote by $\mathrm{Aut}(\mathfrak{g})$ the group of automorphisms of the Lie algebra $\mathfrak{g}$ defined by

$$\mathrm{Aut}(\mathfrak{g}) := \{F\colon \mathfrak{g} \to \mathfrak{g} \ \mid F - \text{linear, bijective}, \ \ [Fx, Fy] = F[x, y], \ \ x, y \in \mathfrak{g}\}.$$

The isomorphic classes of different left invariant metrics can be seen as orbits of the automorphism group $\mathrm{Aut}(\mathfrak{g})$ naturally acting on a space of left invariant metrics. This allows us to use the algebraic approach, although often more geometrical tools are required. In the fixed basis, the set $\mathcal{S}(\mathfrak{g})$ of non-equivalent inner products of the algebra $\mathfrak{g}$ is identified with the symmetric matrices S of arbitrary signature modulo action $S \mapsto F^{\mathsf{T}} S F$, $F \in \mathrm{Aut}(\mathfrak{g})$. From now on the Lie algebra $\mathfrak{g}$ is identified with the algebra of left invariant vector fields on G. Thus, the matrix S representing the inner product on $\mathfrak{g}$ is at the same time representing the metric g in the left invariant basis.

Remark 2.1. The automorphisms of solvable Lie algebras given by Proposition 1.2 are calculated in Table A.2. For the list of automorphisms corresponding to the classification [19], one can refer to [10].

Two non-equivalent inner products on $\mathfrak{g}$ define two metric algebras on the same space. Since those inner products correspond to the orbits of $\mathrm{Aut}(\mathfrak{g})$ acting on the space of symmetric matrices, the metric algebras are isomorphic, and hence isometric. For the completely solvable algebras, the converse follows directly from Proposition 1.3.

If the algebras are not completely solvable, then the more general result by Gordon and Wilson [12] must be used. Although proven in a rather general setting, only the version of the algorithm corresponding to the simply connected Lie groups is presented here.

The two Riemannian solvemanifolds $M(\mathfrak{f})$ and $M(\bar{\mathfrak{f}})$ are isometric if and only if the corresponding standard positions of $\mathfrak{f} = (\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ and $\bar{\mathfrak{f}} = (\bar{\mathfrak{g}}, \langle\cdot\,,\cdot\rangle_1)$ are isomorphic. Let us first define the *standard modification* $(\bar{\mathfrak{g}}, \langle\cdot\,,\cdot\rangle_1)$ of $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ as follows: $\bar{\mathfrak{g}}$ is defined to be the orthogonal complement of the algebra $N_{\mathfrak{l}}(\mathfrak{g})$ of all skew-symmetric derivations of $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ in $\mathfrak{h} = N_{\mathfrak{l}}(\mathfrak{g}) \oplus \mathfrak{g}$ relative to the Killing form of $\mathfrak{h}$ ($\bar{\mathfrak{g}}$ is a solvable ideal of $\mathfrak{h}$). Then define $\langle\cdot\,,\cdot\rangle_1$ by $\langle (Id+\phi)x, (Id+\phi)y\rangle_1 = \langle x, y\rangle$, where $\phi\colon \mathfrak{g} \to \bar{\mathfrak{g}}$ is the modification map. By replacing $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ with $(\bar{\mathfrak{g}}, \langle\cdot\,,\cdot\rangle_1)$, the computation of the standard modification should be repeated until the pair $(\bar{\bar{\mathfrak{g}}}, \langle\cdot\,,\cdot\rangle_2)$ in the standard position is obtained: $\bar{\bar{\mathfrak{g}}}$ is in the *standard position* if and only if $\bar{\bar{\mathfrak{g}}}$ equals its standard modification.

Therefore, in order to find all non-equivalent inner products on the Lie algebra, in addition to the action of the automorphism group, one must also consider the action of the modification map.

2.1. Groups of the type $\mathbb{R} \ltimes \mathbb{R}^3$. Let us start with a group to which the majority of the solvable Lie groups belong: the abelian group, the nilpotent groups and the algebras with a codimension one nilradical. Note that all algebras of this type, except $\mathfrak{g}_{3,5}^{\alpha} + \mathfrak{g}_1$ and $\mathfrak{g}_{4,6}^{\alpha,\beta}$, are completely solvable. Interestingly, the nilpotent Lie groups stand out and should therefore be considered separately. However, the nilpotent case was the main subject of the previous, much more detailed research, hence for the sake of completeness, only the main conclusions are stated.

Let us first consider the trivial abelian case $4\mathfrak{g}_1$. Since the automorphism group is isomorphic to $GL(4,\mathbb{R})$, every symmetric matrix S can be reduced by the action of the automorphism group to the diagonal form:

$$S = \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & \epsilon_3 & 0 \\ 0 & 0 & 0 & \epsilon_4 \end{pmatrix}, \quad \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \{-1, 1\}. \tag{2.1}$$

In the 4-dimensional case, there exist two non-trivial nilpotent Lie algebras: $\mathfrak{g}_{3,1} + \mathfrak{g}_1$ (which is the trivial extension of the 3-dimensional Heisenberg algebra $\mathfrak{h}_3$) and $\mathfrak{g}_{4,1}$ (the filiform Lie algebra, i.e. the algebra whose nil index is maximal). In other notations, these two algebras are often referred to as $\mathfrak{h}_3 + \mathbb{R}$ and $\mathfrak{n}_4$. The classification of inner products on these two algebras has been made in [16] for the Riemannian case and later completed for the pseudo-Riemannian case in [8] (Lorentz case) and [20] (neutral signature case). The results are summarized in the following theorem, whose proof is omitted.

Theorem 2.1. [8, 20]. *Let $\mu \neq 0$, $\epsilon_1, \epsilon_2, \epsilon_3 \in \{-1, 1\}$.*

(i) *The set of non-isometric inner products of $\mathfrak{g}_{3,1} + \mathfrak{g}_1$ is represented by:*

$$\begin{gathered} S^{\epsilon}_{\mu} = \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \epsilon_3 \end{pmatrix}, \quad S^{\epsilon} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ S^{\epsilon}_{01} = \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_2 \end{pmatrix}, \quad S^{\epsilon}_{02} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \epsilon_1 & 0 & 0 \\ 0 & 0 & \epsilon_2 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad S^{0}_{0} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \end{gathered} \tag{2.2}$$

(ii) *The non-isometric inner products of $\mathfrak{g}_{4,1}$ are represented by the matrix*

$$S^{\epsilon}_{A} = \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & b & c \end{pmatrix}, \tag{2.3}$$

where the signature of the matrix $A = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ is determined by the signature of the inner product, or by one of the following matrices:

$$\begin{gathered} S^{\epsilon}_{1,\mu} = \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \mu & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad S^{\epsilon}_{2,\mu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \epsilon_1 & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad S^{\epsilon}_{3,\mu} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \epsilon_1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}, \\ S^{\epsilon}_{4,\mu} = \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}, \quad S^{0}_{1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad S^{0}_{2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \epsilon_1 \\ 1 & 0 & 0 & 0 \\ 0 & \epsilon_1 & 0 & 0 \end{pmatrix}. \end{gathered} \tag{2.4}$$

Now, let us focus on the non-nilpotent completely solvable algebras. Observe that Table A.2 from the Appendix implies that all automorphism groups are subgroups of $\mathrm{Aff}_3(\mathbb{R})$. Thus, every $F \in \mathrm{Aut}(\mathfrak{g})$ can be obtained as a product of the translation matrix with scaling, rotation or shear transformation matrix. Rather than observing this as a simple matrix calculus, one can approach it from a purely geometrical point of view: the quadratic surfaces are classified with respect to the action of some subgroup of affine transformations. Note that the nature of the vector e_4 cannot be changed, so essentially two separate cases have to be considered. First, let us presume that e_4 is not null, i.e. $\langle e_4, e_4\rangle \neq 0$. In this case, the basis can always be changed so that e_4 is orthogonal to the subalgebra $\mathfrak{h} = \mathcal{L}(e_1, e_2, e_3)$. The corresponding inner products have the form:

$$S^A = \begin{pmatrix} A & 0 \\ 0 & \mu \end{pmatrix}, \quad A = A^{\intercal} \in GL(3, \mathbb{R}),\ \mu \neq 0. \tag{2.5}$$

In the second case, when e_4 is a null vector, the restriction of the inner product to the subalgebra $\mathfrak{h}$ is degenerate, and more careful analysis is required.

Let us consider in detail the case of algebra $\mathfrak{g}_{3,2} + \mathfrak{g}_1$. It is easy to see that the matrix $F \in \mathrm{Aut}(\mathfrak{g}_{3,2} + \mathfrak{g}_1)$ can be written as a product of matrices:

$$F = F_1F_2F_3 = \begin{pmatrix} 1 & 0 & 0 & a_{14} \\ 0 & 1 & 0 & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{22} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{a_{23}}{a_{22}} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The matrix F_1 is a translation matrix, while F_2 is a simple scaling and F_3 is a shear matrix influencing only the vector e_3. In addition to the vector e_4, the nature of the vectors e_1 and e_2 cannot be changed by any of these transformations. Furthermore, since the vector e_1 can only be scaled, an appropriate factor can be chosen such that, if it is not null, e_1 becomes a unit vector, i.e. $\langle e_1, e_1\rangle = \kappa \in \{-1, 0, 1\}$.

First, presume that the vector e_4 is not null. As previously discussed, the inner product takes a form (2.5) and the subalgebra $\mathfrak{h} = \mathcal{L}(e_1, e_2, e_3)$ is non-degenerated, meaning that the restriction A of the inner product S on $\mathfrak{h}$ is a regular matrix. The matrix F_1 is acting identically on $\mathfrak{h}$, thus only an action of the matrices F_2 and F_3 should be observed. If e_2 is space-like (or time-like), a parameter of shear transformation can always be set such that e_2 becomes orthogonal to e_3, and by setting the appropriate scaling factors, e_2 can become a unit vector. Specifically, by changing the signs of scaling factors, $\langle e_1, e_2\rangle \geqslant 0$ can be obtained. Now, let e_2 be a null vector. Note that F_3 is preserving the signature of the lower-right 2×2 sub-matrix of A. Therefore, one must consider the cases when that matrix is regular and when it is singular. The corresponding forms of the matrix A are given by (2.6), respectively.

Now, let us consider the case when e_4 is null. Similarly to the previous, by applying scaling and shear transformation, the restriction matrix A takes the forms from (2.6), but with one major addition – the matrix A is now singular. By translating the vector e_4, using the conditions of regularity of the matrix S and singularity of the matrix A, the inner products (2.7) are obtained. In order to get a more compact form, in some cases, additional scaling must be applied.

The other algebras can be considered in a similar manner and the results are summarized in the following theorem.

THEOREM 2.2. *Let $\mu \neq 0$ and $\epsilon_i \in \{-1, 1\}$, $\kappa_i \in \{-1, 0, 1\}$, $i = 1, 2, 3$.*

(i) *The non-isometric inner products on $\mathfrak{g}_{3,2} + \mathfrak{g}_1$, $\mathfrak{g}_{4,2}^{\alpha}$, $\alpha \neq 1$, and $\mathfrak{g}_{4,3}$ are represented by (2.5), where A is given by one of the matrices:*

$$A_1 = \begin{pmatrix} \kappa_1 & a & b \\ a & \epsilon_2 & 0 \\ b & 0 & c \end{pmatrix}, \quad A_2 = \begin{pmatrix} \kappa_1 & d & b \\ d & 0 & \epsilon_2 \\ b & \epsilon_2 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} \kappa_1 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & \epsilon_2 \end{pmatrix}, \tag{2.6}$$

or by one of the matrices:

$$\begin{gathered} S_1^0 = \begin{pmatrix} \kappa_1 & a & b & 0 \\ a & \epsilon_2 & 0 & 0 \\ b & 0 & \frac{\epsilon_1 b}{\kappa_1 \epsilon_2 - a^2} & d \\ 0 & 0 & d & 0 \end{pmatrix}, \quad S_2^0 = \begin{pmatrix} \epsilon_1 & 1 & 0 & 0 \\ 1 & \epsilon_1 & 0 & e \\ 0 & 0 & d & 0 \\ 0 & e & 0 & 0 \end{pmatrix}, \quad S_3^0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \epsilon_1 & 0 & 0 \\ 0 & 0 & d & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\ S_4^0 = \begin{pmatrix} 2\epsilon_1 ad & d & a & 1 \\ d & 0 & \epsilon_1 & 0 \\ a & \epsilon_1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad S_5^0 = \begin{pmatrix} 0 & 0 & b & d \\ 0 & 0 & \epsilon_1 & 0 \\ b & \epsilon_1 & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix}, \quad S_6^0 = \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & d & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \end{gathered} \tag{2.7}$$

with $a \geqslant 0$, $b, c \in \mathbb{R}$, $d, e \neq 0$.

(ii) *The non-isometric inner products on $\mathfrak{g}_{4,2}^{1}$ are represented by (2.5), where A is one of the matrices:*

$$\begin{gathered} A_1 = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & a \end{pmatrix}, \quad A_2 = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & 0 & \epsilon_2 \\ 0 & \epsilon_2 & 0 \end{pmatrix}, \\ A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \epsilon_1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \epsilon_1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \end{gathered} \tag{2.8}$$

or given by one of the matrices:

$$\begin{gathered} S_1^0 = \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad S_2^0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & a & 0 & 0 \\ 0 & 0 & \epsilon_1 & b \\ 1 & 0 & b & 0 \end{pmatrix}, \quad S_3^0 = \begin{pmatrix} 0 & 1 & 0 & b \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ b & 0 & 1 & 0 \end{pmatrix}, \\ S_4^0 = \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad S_5^0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \epsilon_1 & 0 \\ 0 & \epsilon_1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad S_6^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \end{gathered} \tag{2.9}$$

where $a \neq 0$ and $b \geqslant 0$.

(iii) *The non-isometric inner products on $\mathfrak{g}_{3,4}^{\alpha} + \mathfrak{g}_1$ (for $|\alpha| < 1$, $\alpha \neq 0$) and $\mathfrak{g}_{4,5}^{\alpha,\beta}$, $-1 < \alpha < \beta < 1$, $\alpha\beta \neq 0$, are represented by (2.5), where A is* *given by:*

$$A=\begin{pmatrix}\kappa_1 & a & b\\ a & \kappa_2 & c\\ b & c & \kappa_3\end{pmatrix},\qquad a,b\geqslant 0,\ c\in\mathbb{R}, \tag{2.10}$$

or by one of the matrices:

$$\begin{aligned} S_1^0&=\begin{pmatrix}\kappa_1 & a & b & 0\\ a & \kappa_2 & c & 0\\ b & c & d & 1\\ 0 & 0 & 1 & 0\end{pmatrix}, & S_2^0&=\begin{pmatrix}\epsilon_1 & 1 & b & a\\ 1 & \epsilon_1 & \epsilon_1 b & 0\\ b & \epsilon_1 b & \kappa_2 & 0\\ a & 0 & 0 & 0\end{pmatrix},\\ S_3^0&=\begin{pmatrix}\kappa_1 & 0 & a & 0\\ 0 & 0 & 0 & 1\\ a & 0 & \kappa_2 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}, & S_4^0&=\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & \kappa_1 & a & 0\\ 0 & a & \kappa_2 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}, \end{aligned} \tag{2.11}$$

with $a,b\geqslant 0$, $\kappa_1\kappa_2-a^2\neq 0$, $c\in\mathbb{R}$, $d=\frac{2abc-\kappa_2 b^2-\kappa_1 c^2}{a^2-\kappa_1\kappa_2}$.

(iv) *The non-isometric inner products on* $\mathfrak{g}_{3,4}^{-1}+\mathfrak{g}_1$ *are represented by* (2.5), (2.10) *or by one of the matrices* S_1^0 *and* S_4^0 *from* (2.11).

(v) *The non-isometric inner products on* $\mathfrak{g}_2+2\mathfrak{g}_1$, $\mathfrak{g}_{3,3}+\mathfrak{g}_1$ *and* $\mathfrak{g}_{4,5}^{\alpha,\alpha}$, $|\alpha|<1$, $\alpha\neq 0$, *are represented by* (2.5), (2.10) (*with* $\kappa_2=\epsilon_2$, $c=0$) *or by the matrices* S_1^0 (*with* $\kappa_2=\epsilon_2$, $c=0$, $d=\frac{-\epsilon_2 b^2}{a^2-\kappa_1\epsilon_2}$ *or* $\kappa_2=0$, $a=1$, $b=0$, $c\geqslant 0$, $d=-\kappa_1 c^2$) *and* S_4^0 (*with* $\kappa_i=\epsilon_i$, $a=0$) *from* (2.11).

(vi) *The non-isometric inner products on* $\mathfrak{g}_{4,5}^{\alpha,1}$, $|\alpha|<1$, $\alpha\neq 0$, *are represented by* (2.5), (2.10) (*with* $\kappa_1=\epsilon_1$, $b=0$, $c\geqslant 0$) *or by the matrices* S_1^0 (*with* $\kappa_1=\epsilon_1$, $b=0$, $c\geqslant 0$, $d=\frac{-\epsilon_1 c^2}{a^2-\kappa_1\epsilon_2}$ *or* $\kappa_1=0$, $a=1$, $c=0$, $d=-\kappa_2 b^2$) *and* S_3^0 (*with* $\kappa_i=\epsilon_i$, $a=0$) *from* (2.11).

(vii) *The non-isometric inner products on* $\mathfrak{g}_{4,5}^{1,1}$ *are given by* (2.5), (2.10) (*with* $\kappa_i=\epsilon_i$, $a=b=c=0$) *or by* S_4^0 (*with* $\kappa_i=\epsilon_i$, $a=0$) *from* (2.11).

(viii) *The non-isometric inner products on* $\mathfrak{g}_{4,4}$ *are represented by* (2.5), *where* A *is one of the matrices:*

$$A_1=\begin{pmatrix}\epsilon_1 & 0 & 0\\ 0 & a & b\\ 0 & b & c\end{pmatrix},\ A_2=\begin{pmatrix}0 & \epsilon_1 & 0\\ \epsilon_1 & a & 0\\ 0 & 0 & d\end{pmatrix},\ A_3=\begin{pmatrix}0 & 0 & \epsilon_1\\ 0 & d & a\\ \epsilon_1 & a & 0\end{pmatrix}, \tag{2.12}$$

where $a,b,c\in\mathbb{R}$, $ac-b^2\neq 0$ *and* $d\neq 0$, *or by one of the matrices:*

$$\begin{aligned} S_1^0&=\begin{pmatrix}c & 0 & 0 & 0\\ 0 & a & b & 0\\ 0 & b & \frac{b^2}{a} & 1\\ 0 & 0 & 1 & 0\end{pmatrix}, & S_2^0&=\begin{pmatrix}c & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & a & 0\\ 0 & 1 & 0 & 0\end{pmatrix}, & S_3^0&=\begin{pmatrix}0 & a & 0 & 0\\ a & b & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix},\\ S_4^0&=\begin{pmatrix}0 & 0 & a & 0\\ 0 & 0 & 0 & 1\\ a & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}, & S_5^0&=\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & a & 0 & 0\\ 0 & 0 & c & 0\\ 1 & 0 & 0 & 0\end{pmatrix}, & S_6^0&=\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & a & 0\\ 0 & a & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}, \end{aligned} \tag{2.13}$$

where $a,\ c\neq 0$, $b\in\mathbb{R}$.

Finally, let us consider the algebras $\mathfrak{g}_{3,5}^{\alpha}+\mathfrak{g}_1$ and $\mathfrak{g}_{4,6}^{\alpha,\beta}$ that are not completely solvable. Note that the algebra $\mathfrak{g}_{3,5}^{\alpha}+\mathfrak{g}_1$ can be considered as a special case of $\mathfrak{g}_{4,6}^{\alpha,\beta}$, for $\beta=0$. Applying a procedure similar to the one in the completely solvable case, we get that the inner products take one of the forms:

$$
\begin{aligned}
&S^A=\begin{pmatrix}\kappa_1 & a & b & 0\\ a & \kappa_2 & 0 & 0\\ b & 0 & c & 0\\ 0 & 0 & 0 & d\end{pmatrix}, \quad S_1^0=\begin{pmatrix}\epsilon_1 & 0 & a & 0\\ 0 & \epsilon_2 & b & 0\\ a & b & \epsilon_1 a^2+\epsilon_2 b^2 & d\\ 0 & 0 & d & 0\end{pmatrix},\\
&S_2^0=\begin{pmatrix}0&0&0&1\\0&\epsilon_1&0&0\\0&0&d&0\\1&0&0&0\end{pmatrix}, \quad S_3^0=\begin{pmatrix}0&0&a&1\\0&0&\epsilon_1&0\\a&\epsilon_1&c&0\\1&0&0&0\end{pmatrix}, \quad S_4^0=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&c&0\\0&1&0&0\end{pmatrix},
\end{aligned}
\tag{2.14}
$$

where $\kappa_1,\kappa_2\in\{-1,0,1\}$, $\epsilon_1,\epsilon_2\in\{-1,1\}$, $a\geqslant 0$, $b,c\in\mathbb{R}$, $d\neq 0$. Specially, for the unimodular algebra $\mathfrak{g}_{3,5}^{0}+\mathfrak{g}_1$, the additional constraint $b\geqslant 0$ can be obtained.

Now, the standard modification algorithm has to be performed. In both cases the algebra of all skew-symmetric derivations is spanned by $E_{23}-E_{32}$, where E_{ij} denotes the matrix with the unit on the cross of the i-th row and the j-th column and the zero otherwise. Calculating the standard modification algebra, it is shown that it equals the original one. Hence, the standard position is obtained in the very first step. Furthermore, the action of the corresponding modification map is a special case of the action of the automorphism group and the inner products cannot be more simplified.

The previous consideration yields the following theorem.

THEOREM 2.3. *The non-isometric inner products on* $\mathfrak{g}_{3,5}^{\alpha}+\mathfrak{g}_1$ *and* $\mathfrak{g}_{4,6}^{\alpha,\beta}$ *are represented by* (2.14).

2.2. Groups of the type $\mathbb{R}\ltimes H_3$. For a Lie algebra $\mathfrak{g}$ a *nil ideal* is an ideal $\mathfrak{n}$ of $\mathfrak{g}$ such that $\operatorname{ad} x$ is nilpotent for every $x\in\mathfrak{n}$. An ideal $\mathfrak{n}$ is a nil ideal if and only if, it is, as an algebra, nilpotent. Every Lie algebra $\mathfrak{g}$ has a unique maximal nil ideal containing all other nil ideals and it is called the *nilradical* of $\mathfrak{g}$. The algebra $\mathfrak{g}$ is nilpotent if and only if it is equal to its nilradical.

The 3-dimensional Heisenberg algebra $\mathfrak{h}_3$ is a nilpotent Lie algebra, while the algebra of 3-dimensional real hyperbolic space $\mathfrak{g}_{3,3}$ is an example of a Lie algebra with a codimension one abelian nilradical. The algebras of type II are the Lie algebras for which a codimension one nilradical is isomorphic to the Heisenberg algebra $\mathfrak{h}_3$. They can also be obtained from the 3-dimensional Lie algebras having a codimension one abelian nilradical by applying the following extension procedure introduced in [18].

Let $\mathfrak{g}$ be the 3-dimensional Lie algebra that has a codimension one abelian nilradical. Then $\mathfrak{g}$ is one of the following algebras:

$\mathfrak{g}_{3,2}$ $[e_1,e_3]=e_1,\ [e_2,e_3]=e_1+e_2;$

$\mathfrak{g}_{3,3}$ $[e_1,e_3]=e_1,\ [e_2,e_3]=e_2;$

$\mathfrak{g}_{3,4}^{\alpha}$: $[e_1,e_3]=e_1,\ [e_2,e_3]=\alpha e_2,\quad -1\leqslant\alpha<1,\ \alpha\neq 0;$

$\mathfrak{g}_{3,5}^{\alpha}$: $[e_1,e_3]=\alpha e_1-e_2,\ [e_2,e_3]=e_1+\alpha e_2,\ \alpha\geqslant 0.$

If an additional vector e_4 that gives an extended algebra $\mathfrak{g}'$ is introduced, then the only possibility for the extra brackets is $[e_1, e_2] = e_4$. Note that e_4 is the center of the nilradical, i.e. center of the Heisenberg algebra $\mathfrak{h}_3$, and the remaining bracket is given by: $[e_4, e_3] = (c_{13}^1 + c_{23}^2)e_4$, where c_{ij}^k are the structural constants of $\mathfrak{g}$. Now, it is easy to compute that $\mathfrak{g}_{3,2}$ extends to $\mathfrak{g}_{4,7}$, $\mathfrak{g}_{3,5}^\alpha$ extends to $\mathfrak{g}_{4,9}^\alpha$, while $\mathfrak{g}_{3,3}$ and $\mathfrak{g}_{3,4}^\alpha$ give the algebra $\mathfrak{g}_{4,8}^\alpha$ for $\alpha = 1$ and $-1 \leqslant \alpha < 1, \alpha \neq 0$, respectively.

REMARK 2.2. Note that the algebra $\mathfrak{g}_{4,8}^0$ was excluded from the previous consideration. This is only a technicality, since it is obtained by the Heisenberg extension of the reducible algebra $\mathfrak{g}_2 + \mathfrak{g}_1$ that can be considered as a special case of the algebra $\mathfrak{g}_{3,4}^\alpha$, for $\alpha = 0$.

REMARK 2.3. It is interesting that the algebra of complex hyperbolic plane $\mathfrak{g}_{4,8}^1$ is obtained here as the Heisenberg extension of the algebra $\mathfrak{g}_{3,3}$ of 3-dimensional real hyperbolic space. The algebra of 4-dimensional real hyperbolic space $\mathfrak{g}_{4,5}^{1,1}$ could be obtained from $\mathfrak{g}_{3,3}$, but by taking the abelian extension, i.e. by adding the vector e_4 such that $[e_4, e_3] = e_4$, $[e_4, e_2] = [e_4, e_1] = 0$.

The described construction can be generalized to higher dimensions. However, more possibilities for the extra bracket will occur and more detailed analysis is required.

Bearing in mind the previous construction, the following technical lemma holds.

LEMMA 2.1. *Let $\mathfrak{g}$ be the 3-dimensional irreducible algebra with the codimension one abelian nilradical given by the structural constants c_{ij}^k. Then the group of automorphisms of its 4-dimensional Heisenberg extension $\mathfrak{g}'$ consists of the matrices of the form:*

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & 0 & \pm 1 & 0 \\ \frac{c_{13}^1\Delta_{13}+c_{13}^2\Delta_{23}}{C} & \frac{c_{23}^1\Delta_{13}+c_{23}^2\Delta_{23}}{C} & a_{43} & \Delta_{12} \end{pmatrix}, \quad \Delta_{12} \neq 0, \tag{2.15}$$

where the upper-left 3×3 matrix $\bar{F}$ is the matrix of the automorphisms of $\mathfrak{g}$, $\Delta_{ij} = \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix}$, $1 \leqslant i < j \leqslant 3$, and $C = c_{13}^2 c_{23}^1 - c_{13}^1 c_{23}^2$.

PROOF. The proof follows directly from the following facts: the restriction of the automorphisms to the algebra $\mathfrak{g}$ must preserve its commutators, the algebra $\mathfrak{g}$ has a codimension one abelian radical $\mathbb{R}^2 = \mathcal{L}(e_1, e_2)$ and e_4 is the center of the Heisenberg subalgebra $\mathfrak{h}_3 = \mathcal{L}(e_1, e_2, e_4)$. □

Observe that, in this way, the problem was reduced to the problem of classification in dimension three. By selecting the matrix $\bar{F} \in \mathrm{Aut}(\mathfrak{g})$, the inner product on the extension algebra $\mathfrak{g}'$ is almost completely determined and since all algebras of this type are completely solvable, the obtained products are final. However, one must keep in mind that the degenerate products on $\mathfrak{g}$ must also be considered since they can rise to the non-degenerate product on $\mathfrak{g}'$. Also, the center e_4 of the nilradical cannot change its character; the cases when it is space-like ($\langle e_4, e_4\rangle > 0$), time-like ($\langle e_4, e_4\rangle < 0$) and null ($\langle e_4, e_4\rangle = 0$) must be separately examined.

Now, the main theorem of this subsection can be stated.

THEOREM 2.4. *Let* $\epsilon_1, \epsilon_2 \in \{-1, 1\}$ *and* $\kappa_1, \kappa_2 \in \{-1, 0, 1\}$.

(i) *The non-isometric inner products on* $\mathfrak{g}_{4,7}$ *are represented by:*

$$S_1^\epsilon = \begin{pmatrix} a & 0 & f & 0 \\ 0 & b & c & 0 \\ f & c & d & 0 \\ 0 & 0 & 0 & \epsilon_1 \end{pmatrix}, \quad S_2^\epsilon = \begin{pmatrix} 0 & b & f & 0 \\ b & c & 0 & 0 \\ f & 0 & d & 0 \\ 0 & 0 & 0 & \epsilon_1 \end{pmatrix}$$
$$S_0^0 = \begin{pmatrix} b & 0 & 0 & 1 \\ 0 & c & e & 0 \\ 0 & e & d & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad S_1^0 = \begin{pmatrix} b & 0 & c & 0 \\ 0 & 0 & 0 & 1 \\ c & 0 & d & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad S_2^0 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & \kappa_1^2 & 0 \\ 0 & \kappa_1^2 & 0 & c \\ 0 & 0 & c & 0 \end{pmatrix}, \tag{2.16}$$

with $a \neq 0$, $b, c, d, e \in \mathbb{R}$, $f \geqslant 0$.

(ii) *The non-isometric inner products on* $\mathfrak{g}_{4,8}^\alpha$ $(|\alpha| < 1, \alpha \neq 0)$ *are represented by:*

$$S^\epsilon = \begin{pmatrix} \kappa_1 & a & d & 0 \\ a & b & c & 0 \\ d & c & e & 0 \\ 0 & 0 & 0 & \epsilon_1 \end{pmatrix}, \quad S_0^0 = \begin{pmatrix} 0 & 0 & 0 & \epsilon_1 \\ 0 & a & c & \epsilon_2 \\ 0 & c & b & d \\ \epsilon_1 & \epsilon_2 & d & 0 \end{pmatrix},$$
$$S_1^0 = \begin{pmatrix} 0 & c & 0 & 1 \\ c & a & \kappa_1 & 0 \\ 0 & \kappa_1 & b & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad S_2^0 = \begin{pmatrix} a & c & \kappa_1 & 0 \\ c & 0 & 0 & 1 \\ \kappa_1 & 0 & b & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad S_3^0 = \begin{pmatrix} b & a & 0 & 0 \\ a & \epsilon_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{2.17}$$
$$S_4^0 = \begin{pmatrix} \kappa_1 & -1 & 0 & 0 \\ -1 & 0 & c & 0 \\ 0 & c & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad S_5^0 = \begin{pmatrix} \kappa_1 & 1 & c & 0 \\ 1 & 0 & 0 & 0 \\ c & 0 & 0 & \alpha \\ 0 & 0 & \alpha & 0 \end{pmatrix}, \quad S_6^0 = \begin{pmatrix} f & a & c & 0 \\ a & \epsilon_1 & 0 & 0 \\ c & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

with $a, b, e \in \mathbb{R}$, $c, d \geqslant 0$ *and* $f = \frac{\epsilon_1}{\alpha}(a+1)(\alpha a - 1)$.

(iii) *The non-isometric inner products on* $\mathfrak{g}_{4,8}^{-1}$ *are represented by the matrices* S^ϵ, S_0^0, S_1^0, S_3^0, S_5^0 *(with* $\alpha = -1$ *and* $\kappa_1 = 0$*) and* S_6^0 *(with* $f = \epsilon_1(a+1)^2$*) from* (2.17).

(iv) *The non-isometric inner products on* $\mathfrak{g}_{4,8}^0$ *are represented by the matrices:*

$$S^\epsilon = \begin{pmatrix} \kappa_1 & a & d & 0 \\ a & b & c & 0 \\ d & c & e & 0 \\ 0 & 0 & 0 & \epsilon_1 \end{pmatrix}, \quad S_0^0 = \begin{pmatrix} 0 & a & 0 & 1 \\ a & \kappa_1 & b & d \\ 0 & b & e & 0 \\ 1 & d & 0 & 0 \end{pmatrix}, \quad S_1^0 = \begin{pmatrix} \kappa_1 & 0 & b & 0 \\ 0 & 0 & 0 & 1 \\ b & 0 & e & d \\ 0 & 1 & d & 0 \end{pmatrix},$$
$$S_2^0 = \begin{pmatrix} \kappa_1 & a & 0 & 0 \\ a & b & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad S_3^0 = \begin{pmatrix} \kappa_1 & -1 & 0 & 0 \\ -1 & b & d & 0 \\ 0 & d & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{2.18}$$

with $a, b, e \in \mathbb{R}$, $c, d \geqslant 0$.

(v) *The non-isometric inner products on* $\mathfrak{g}_{4,8}^1$ *are represented by:*

$$S_1^\epsilon = \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & a & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & \epsilon_2 \end{pmatrix}, \quad S_2^\epsilon = \begin{pmatrix} 0 & c & 0 & 0 \\ c & 0 & \kappa_1 & 0 \\ 0 & \kappa_1 & b & 0 \\ 0 & 0 & 0 & \epsilon_2 \end{pmatrix}, \quad S_0^0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \kappa_1 & a & 0 \\ 0 & a & b & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$S_1^0 = \begin{pmatrix} 0 & c & 0 & 1 \\ c & 0 & \epsilon_1 & 0 \\ 0 & \epsilon_1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad S_2^0 = \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad S_3^0 = \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & -\epsilon_1 & c & 0 \\ 0 & c & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{2.19}$$

with $a, b \in \mathbb{R}$, $c \geqslant 0$.

(vi) *The non-isometric inner products on* $\mathfrak{g}_{4,9}^\alpha$, $\alpha \geqslant 0$, *are represented by:*

$$S^\epsilon = \begin{pmatrix} a & d & 0 & 0 \\ d & b & e & 0 \\ 0 & e & c & 0 \\ 0 & 0 & 0 & \epsilon_1 \end{pmatrix}, \quad S_0^0 = \begin{pmatrix} a & 0 & 0 & 1 \\ 0 & b & d & 0 \\ 0 & d & c & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$S_1^0 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & \epsilon_1 \\ 0 & 0 & \epsilon_1 & 0 \end{pmatrix}, \quad S_2^0 = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & a & e & 0 \\ 0 & e & 0 & \epsilon_1 \\ 0 & 0 & \epsilon_1 & 0 \end{pmatrix}, \tag{2.20}$$

with $a, b, c, d \in \mathbb{R}$, $e \geqslant 0$ *and* $f = \frac{a\epsilon_1 - 1}{a(\alpha^2+1)-\epsilon_1}$. *If* $\alpha = 0$, *then* $d \geqslant 0$ *and* $f = \epsilon_1$.

2.3. Groups of the type $\mathbb{R} \ltimes \boldsymbol{E(1,1)}$ **and** $\mathbb{R} \ltimes \boldsymbol{E(2)}$**.** Finally, let us consider the type I groups, i.e. the semi-direct products of $\mathbb{R}$ with the group of rigid motions of the Minkowski and Euclidean space, respectively, with the corresponding Lie algebras $2\mathfrak{g}_2 \cong \mathrm{aff}(\mathbb{R}) \oplus \mathrm{aff}(\mathbb{R})$ and $\mathfrak{g}_{4,10} \cong \mathrm{aff}(\mathbb{C})$.

THEOREM 2.5. *Let* $\epsilon_1, \epsilon_2 \in \{-1, 1\}$, $\lambda_1, \lambda_2 \in \mathbb{R}$.

(i) *The non-isometric inner products on* $\mathrm{aff}(\mathbb{R}) \oplus \mathrm{aff}(\mathbb{R})$ *are represented by the matrices:*

$$S_1 = \begin{pmatrix} \lambda_1 & 0 & a & e \\ 0 & \epsilon_1 & f & d \\ a & f & \lambda_2 & 0 \\ e & d & 0 & \epsilon_2 \end{pmatrix}, \quad S_2 = \begin{pmatrix} \lambda_1 & 0 & 0 & 1 \\ 0 & \epsilon_1 & f & 0 \\ 0 & f & \lambda_2 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} \lambda_1 & 0 & a & b \\ 0 & \epsilon_1 & 0 & 1 \\ a & 0 & \lambda_2 & 0 \\ b & 1 & 0 & 0 \end{pmatrix},$$

$$S_4 = \begin{pmatrix} \lambda_1 & 0 & a & 0 \\ 0 & 0 & 0 & 1 \\ a & 0 & \lambda_2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad S_5 = \begin{pmatrix} \lambda_1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & \lambda_2 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad S_6 = \begin{pmatrix} 0 & 1 & a & b \\ 1 & 0 & c & d \\ a & c & 0 & 1 \\ b & d & 1 & 0 \end{pmatrix}, \tag{2.21}$$

$$S_7 = \begin{pmatrix} 0 & 1 & a & e \\ 1 & 0 & c & d \\ a & c & \lambda_2 & 0 \\ e & d & 0 & \epsilon_2 \end{pmatrix}, \quad S_8 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & c & 0 \\ 0 & c & \lambda_2 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad S_9 = \begin{pmatrix} 0 & 1 & a & b \\ 1 & 0 & 0 & 1 \\ a & 0 & \lambda_2 & 0 \\ b & 1 & 0 & 0 \end{pmatrix},$$

with $a, b, c, d \in \mathbb{R}$, $e, f \geqslant 0$.

(ii) *The non-isometric inner products on* $\mathrm{aff}(\mathbb{C})$ *are represented by the matrices:*

$$S_1 = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & a & b & c \\ 0 & b & \lambda_1 & d \\ 0 & c & d & \lambda_2 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & a & c \\ 0 & a & 0 & d \\ 1 & c & d & \lambda_2 \end{pmatrix}, \tag{2.22}$$

with $a, d \in \mathbb{R}$, $b, c \geqslant 0$. *If* $a \neq 1$, *the inner product* S_2 *can be further simplified by making* $d = 0$.

PROOF. (i) Note that the automorphisms of $\mathrm{aff}(\mathbb{R}) \oplus \mathrm{aff}(\mathbb{R})$ are represented by the block matrices (see Table A.2):

$$\mathrm{Aut}(\mathrm{aff}(\mathbb{R}) \oplus \mathrm{aff}(\mathbb{R})) = \left\{ F_1 = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}, \ F_2 = \begin{pmatrix} 0 & F \\ F & 0 \end{pmatrix} \,\middle|\, F = \begin{pmatrix} 1 & 0 \\ f_1 & f_2 \end{pmatrix}, \ f_1, f_2 \in \mathbb{R} \right\}.$$

If the matrix S is also presented in the block form $S = \left(\begin{smallmatrix} A & B \\ B^{\intercal} & C \end{smallmatrix}\right)$, with $A = A^{\intercal}$, $C = C^{\intercal}$, by the action of the matrix F, the matrices A and C can be reduced to one of the following forms: $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \kappa \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\lambda \in \mathbb{R}, \kappa \in \{-1, 0, 1\}$. The basic matrix manipulation and case-by-case analysis yields the inner products given by (2.21).

(ii) While the proof of the previous part was purely algebraic, this part requires more geometrical tools. By taking a closer look at the form of the automorphisms (see Table A.2), one can observe that the upper-left 3×3 matrix is actually a composition of homothety, rotation/reflection and translation. Therefore, the problem is reduced to a problem of classification of second-order plane curves by the action of these affine transformations.

However, $\mathrm{aff}(\mathbb{C})$ is not a completely solvable algebra and the standard modification algorithm must be applied.

The algebra $N_{\mathfrak{l}}(\mathrm{aff}(\mathbb{C}))$ of all skew-symmetric derivations of $(\mathrm{aff}(\mathbb{C}), \langle \cdot, \cdot \rangle)$ is spanned by $E_{12} - E_{21}$. Then the standard modification of $\mathrm{aff}(\mathbb{C})$ is given by $\overline{\mathrm{aff}(\mathbb{C})} = \mathcal{L}(e_1 + e_2, -e_1 + e_2, e_3, e_4)$ (with e_i being the standard basis of the algebra $\mathrm{aff}(\mathbb{C})$). It is easy to verify that $\overline{\mathrm{aff}(\mathbb{C})}$ equals $\mathrm{aff}(\mathbb{C})$. Therefore, we already have the algebra in the standard position. The only thing left is to check the action of the modification map ϕ on the inner product, i.e. to see if the inner product $\langle (Id + \phi)x, (Id + \phi)y \rangle_1 = \langle x, y \rangle$, $x, y \in \mathrm{aff}(\mathbb{C})$ can take a simpler form. However, since $Id + \phi \in \mathrm{Aut}(\mathrm{aff}(\mathbb{C}))$, the obtained inner products (2.22) are the final ones. □

3. Geometrical properties

Once the complete classification of left invariant metrics, both Riemannian and pseudo-Riemannian, is obtained, the metrics with interesting geometrical properties can be further examined. First, let us recall a basic notion.

Let X, Y, Z be the left invariant vector fields. Then the Levi-Civita connection ∇ of the metric g can be calculated from Koszul's formula

$$2g(\nabla_X Y, Z) = g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y),$$

and the curvature R and Ricci tensor ρ are given by:

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z, \quad \rho(X, Y) = Tr(Z \mapsto R(Z, X)Y).$$

The metric algebra $\mathfrak{f}$ is said to be *flat* if the corresponding curvature tensor is zero everywhere, i.e. $R = 0$, and it is *locally symmetric* if $\nabla R = 0$, which is equivalent to

$$\nabla_W(R(X,Y)Z) = R(\nabla_W X, Y)Z + R(X, \nabla_W Y)Z + R(X,Y)\nabla_W Z,$$

for all tangent vector fields X, Y, Z, W. Similarly, the metric algebra is *Ricci-flat* if $\rho = 0$ and *Ricci-parallel* if $\nabla\rho = 0$, i.e. if for all tangent vector fields X, Y, Z:

$$\nabla_Z \rho(X,Y) = -\rho(\nabla_Z X, Y) - \rho(X, \nabla_Z Y) = 0.$$

Obviously, the flat metric algebra is also locally symmetric and Ricci-flat and Ricci-parallel. Also, Ricci-flat is simultaneously Ricci-parallel. However, in the classification below, these trivial cases are omitted, i.e. the classification of Ricci-parallel and locally symmetric metric algebras includes only the non-flat cases.

COROLLARY 3.1. *The classification of flat, locally symmetric, Ricci-flat and Ricci-parallel metric algebras is given in Table 1.*

We say that the metric is Einstein if the curvature tensor is proportional to a metric tensor. According to the results presented by Jensen [14], the 4-dimensional Lie group endowed with the left invariant Riemannian metric is Einstein space if and only if its Lie algebra is one of the following $2\mathfrak{g}_2$, $\mathfrak{g}_{3,5}^0 + \mathfrak{g}_1$, $\mathfrak{g}_{4,5}^{1,1}$ and $\mathfrak{g}_{4,6}^{\alpha,\alpha}$ or $\mathfrak{g}_{4,8}^1$ and $\mathfrak{g}_{4,9}^\alpha$, for $\alpha > 0$. As Riemannian spaces, these cases correspond to the direct product of a 2-dimensional solvable group manifold (of constant curvature $K = -1$) with itself, flat space, a real hyperbolic space with constant curvature $K = -1$ and a Hermitian hyperbolic space with sectional curvature $-4 \leqslant K \leqslant -1$, respectively. However, a more general result is obtained here.

COROLLARY 3.2. *The non-flat Einstein metric algebras are:*

$$\left(2\mathfrak{g}_2, S_1 = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \epsilon_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \epsilon_2 \end{pmatrix}\right); \quad \left(2\mathfrak{g}_2, S_1 = \begin{pmatrix} -\epsilon_1 f^2 & 0 & \epsilon_1 f^2 & f \\ 0 & \epsilon_1 & f & \epsilon_1 \\ \epsilon_1 f^2 & f & -\epsilon_1 f^2 & 0 \\ f & \epsilon_1 & 0 & \epsilon_1 \end{pmatrix}\right);$$

$$\left(2\mathfrak{g}_2, S_7 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & -2 & \lambda_2 & 0 \\ 0 & 0 & 0 & \epsilon_2 \end{pmatrix}\right); \quad \left(\mathfrak{g}_{3,4}^{\frac{1}{2}} + \mathfrak{g}_1, S^A = \begin{pmatrix} 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & \epsilon_1 & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}\right);$$

$$\left(\mathfrak{g}_{4,4}, S^{A_3} = \begin{pmatrix} 0 & 0 & \epsilon_1 & 0 \\ 0 & -\epsilon_1 & 0 & 0 \\ \epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}\right); \quad \left(\mathfrak{g}_{4,5}^{-\frac{1}{5},\frac{2}{5}}, S^A = \begin{pmatrix} 0 & a & 0 & 0 \\ a & \epsilon_2 & 0 & 0 \\ 0 & 0 & \epsilon_1 & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}\right);$$

$$\left(\mathfrak{g}_{4,5}^{-\frac{1}{2},\frac{1}{4}}, S^A = \begin{pmatrix} 0 & a & 0 & 0 \\ a & \epsilon_2 & \epsilon_1\sqrt{2} & 0 \\ 0 & \epsilon_1\sqrt{2} & \epsilon_2 & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}\right); \quad \left(\mathfrak{g}_{4,5}^{1,1}, S^A = \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & \epsilon_3 & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}\right);$$

$$\left(\mathfrak{g}_{4,5}^{\alpha,\beta}, S^A = \begin{pmatrix} 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & \epsilon_1 & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}\right) \quad \textit{for } \alpha = 2\beta - 1, 0 < \beta < 1;$$

$$\left(\mathfrak{g}^{\alpha,\alpha}_{4,6}, S^A = \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & \epsilon_2 & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}\right); \quad \left(\mathfrak{g}^{0}_{4,8}, S^\epsilon = \begin{pmatrix} 0 & a & 0 & 0 \\ a & 0 & \epsilon_1 a^2 & 0 \\ 0 & \epsilon_1 a^2 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon_1 \end{pmatrix}\right);$$

$$\left(\mathfrak{g}^{\alpha}_{4,8}, S^\epsilon = \begin{pmatrix} 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & -\epsilon_1(1+\alpha)^2 a^2 & 0 \\ 0 & 0 & 0 & \epsilon_1 \end{pmatrix}\right) \quad \textit{for } |\alpha| < 1, \alpha \neq 0;$$

$$\left(\mathfrak{g}^{\frac{1}{2}}_{4,8}, S^0_2 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & b & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right); \quad \left(\mathfrak{g}^{1}_{4,8}, S^\epsilon_1 = \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 4\epsilon_1\epsilon_2 a & 0 \\ 0 & 0 & 0 & \epsilon_2 \end{pmatrix}\right);$$

$$\left(\mathfrak{g}^{1}_{4,8}, S^\epsilon_2 = \begin{pmatrix} 0 & c & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & 0 & -4\epsilon_2 c^2 & 0 \\ 0 & 0 & 0 & \epsilon_2 \end{pmatrix}\right); \quad \left(\mathfrak{g}^{\alpha}_{4,9}, S^\epsilon = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 4\epsilon_1\alpha^2 a^2 & 0 \\ 0 & 0 & 0 & \epsilon_1 \end{pmatrix}\right);$$

$$\left(\mathfrak{g}_{4,10}, S_1 = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & -\epsilon & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & -\lambda_1 \end{pmatrix}\right); \quad \left(\mathfrak{g}_{4,10}, S_1 = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & -\epsilon c^2 & 0 \\ 0 & c & 0 & 2\epsilon c^2 \end{pmatrix}\right).$$

Remark 3.1. The results obtained by Karki and Thompson in [15] for the solvable case can be derived from Corollary 3.2 by restricting the metrics to the Riemannian signature. Furthermore, this corollary shows that by switching from the Riemannian to the pseudo-Riemannian signature, the list of groups admitting the non-trivial Einstein metric is longer than the one obtained by Jensen in [14]. Additionally, the algebra $\mathfrak{g}_{4,10}$ appears for the first time as the underlining algebra for the Einstein space, since both observed inner products have a neutral signature and hence were not previously investigated.

Finally, let us conclude with a couple of examples.

Example 3.1. In [14] it was proven that all 4-dimensional, simply connected, homogeneous Riemannian Einstein manifolds are necessarily symmetric. The same does not hold for the pseudo-Riemannian setting. However, most of the non-trivial Einstein metric algebras are locally symmetric. The only non-symmetric examples from Corollary 3.2 are $(\mathfrak{g}^{-\frac{1}{5},\frac{2}{5}}_{4,5}, S^A)$ and $(\mathfrak{g}^{-\frac{1}{2},\frac{1}{4}}_{4,5}, S^A)$.

This can be compared with the Lorentz classification from [9, Theorem 3.1.], where the non-symmetric, non-flat Einstein examples occur only in the following classes:

(6) $\quad [f_1, f_4] = -2Af_1,\ [f_2, f_4] = -5Af_2 + 6\epsilon Af_3,\ [f_3, f_4] = Af_3, \quad \text{with } A \neq 0;$

(8) $\quad [f_1, f_4] = \epsilon\dfrac{A+B}{3}f_1,\ [f_2, f_4] = \epsilon\dfrac{5B-A}{6}f_2 + Bf_3,$

$\quad [f_3, f_4] = Af_2 + \epsilon\dfrac{5A-B}{6}f_3, \quad \text{with } A \neq \pm B;$

(9) $\quad [f_1, f_4] = 5\dfrac{A}{2}f_1 + 3\epsilon Af_3,\ [f_2, f_4] = Af_2,\ [f_3, f_4] = -\dfrac{A}{2}f_3, \quad \text{with } A \neq 0;$

$$(11) \qquad [f_1, f_4] = -\frac{2\epsilon\sqrt{2}A}{3} f_1 + \delta A f_3,\ [f_2, f_4] = \frac{\epsilon\sqrt{2}A}{3} f_2,$$
$$[f_3, f_4] = A f_2 - \frac{\epsilon\sqrt{2}A}{6} f_3, \quad \text{with } A \neq 0.$$

With the appropriate change of basis, it can be seen that classes (6), (8) and (9) give rise to the algebra $\mathfrak{g}_{4,5}^{-\frac{1}{5},\frac{2}{5}}$, while class (11) corresponds to the algebra $\mathfrak{g}_{4,5}^{-\frac{1}{2},\frac{1}{4}}$. The corresponding inner products (expressed in the basis from Proposition 1.2) in all cases represent a special form of the appropriate inner product S^A from the corollary above.

Example 3.2. Observe the algebra $\mathfrak{g}_{4,9}^{0}$ with the inner product $S_2^0 :\ \langle e_1, e_1\rangle = \langle e_3, e_4\rangle = 1$, $\langle e_2, e_2\rangle = a > 0$ of the Lorentz signature. From Table 1, it follows that this metric algebra is Ricci-parallel for every $a > 0$ and that for $a = 2$ it is Ricci-flat (hence the space is trivially Einstein), but it is not locally symmetric.

For $a = 2$ this metric algebra corresponds to the following class from [9, Theorem 3.1.]:

$$(14) \qquad [f_1, f_2] = \sqrt{(A+D)^2 + 4B^2}\, f_3,\ [f_1, f_4] = -Bf_1 + Df_2 + Ef_3,$$
$$[f_2, f_4] = Af_1 + Bf_2 + Cf_3,$$

with $A = \frac{1}{\sqrt{2}}$, $B = C = 0$, $D = -\sqrt{2}$. From [9, Table 1], it follows that this is not a flat/locally symmetric example, since $A + D \neq B = 0$ and $AD + B^2 \neq 0$, but it is confirmed that the algebra is Ricci-flat.

If $a \neq 2$, then the metric algebra belongs to the class:

$$(18) \qquad [f_1, f_2] = Af_3,\ [f_1, f_4] = -Bf_1 + Cf_2 + Df_3,\ [f_2, f_4] = Ef_1 + Bf_2 + Ff_3,$$
$$\text{with } A^2 \neq 4B^2 + (C+E)^2,$$

from [9, Theorem 4.1.], where $A = E = \frac{1}{\sqrt{a}}$, $B = D = F = 0$, $C = -\sqrt{a}$. This is one of the two classes where examples of non-locally symmetric metrics occur. The considered metric is not locally symmetric since none of the conditions $E = A + C$ $B = C + E = 0$ are satisfied, but it is a Lorentz Walker manifold (see [9, Theorem 4.2.]) with the parallel vector field $e_4 = f_3$.

Future research should primarily focus on completing the classification for the non-solvable Lie groups. Additionally, these results can be used for the classification of sub-Riemannian and nonholonomic Riemannian structures on 4-dimensional Lie groups (for the classification in the 3-dimensional case one can refer to [1, 5, 6]).

Another open question is the geodesical completeness of classified metrics. The geodesic of an arbitrary left invariant metric on a Lie group G can be considered as a motion of a "generalized rigid body" with a configuration space G. All Riemannian geodesics are complete. In the pseudo-Riemannian case, only partial results are known: all metrics on 2-step nilpotent Lie groups are geodesically complete [13], on the real hyperbolic space $\mathbb{R}H^n$ the only complete metrics are the Riemannian ones [21], etc. However, a more general result, or even the classification of groups that admit the pseudo-Riemannian complete metrics, is yet to come.

TABLE 1. Curvature properties of metric algebras

algebra	inner product	
	$R=0$	$\nabla R=0$
$4\mathfrak{g}_1$	S	$\times$
$\mathfrak{g}_2+2\mathfrak{g}_1$	S_4^0	$S^A\ (\kappa_1=0)$ $S_1^0(\kappa_1=0, \kappa_2=\epsilon_2, c=0, d=-\epsilon_2 b^2/a^2)$ $S_1^0(\kappa_1=\epsilon_1, \kappa_2=\epsilon_2, a=b=c=d=0)$ $S_1^0(\kappa_1=\kappa_2=0, a=1, b=d=0)$
$2\mathfrak{g}_2$	$S_6\ (a=b=c=d=0)$ $S_6\ (a=b=d=0, c=-2)$ $S_6\ (a=c=d=0, b=-2)$	$S_1\ (a=d=e=f=0)$ $S_1\ (\epsilon_1=\epsilon_2=d,\ e=f, \lambda_1=\lambda_2=-a=-\epsilon_1 f^2)$ $S_4,\ S_5$ $S_7\ (a=c=d=e=0)$ $S_7\ (a=d=e=0, c=-2)$
$\mathfrak{g}_{3,1}+\mathfrak{g}_1$	$S_{01}^\epsilon,\ S_0^0$	S^ϵ
$\mathfrak{g}_{3,2}+\mathfrak{g}_1$	S_6^0	$\times$
$\mathfrak{g}_{3,3}+\mathfrak{g}_1$	$S_1^0\ (\kappa_1=\epsilon_1, \kappa_2=\epsilon_2, a=b=c=d=0)$	$S^A\ (a=b=c=0),\ S_4^0$
$\mathfrak{g}_{3,4}^\alpha+\mathfrak{g}_1$	$\alpha=-1:\quad S^A\ (\kappa_1=\epsilon_1, \kappa_2=\kappa_3=a=b=0)$ $\alpha=-1:\quad S_4^0\ (\kappa_1=\kappa_2=0)$ $\alpha=\frac{1}{2}:\quad S_1^0\ (\kappa_1=\kappa_2=b=c=0)$	$\alpha=\frac{1}{2}:\quad S^A\ (\kappa_1=\kappa_2=b=c=0, \kappa_3=\epsilon_3)$ $\alpha=\frac{1}{2}:\quad S_1^0\ (\kappa_1=\kappa_2=b=c=0)$ $\alpha\neq-1:\quad S^A\ (\kappa_1=\epsilon_1, \kappa_2=\kappa_3=a=b=0)$ $\alpha\neq-1:\quad S_4^0\ (\kappa_1=\kappa_2=0)$ $-1\leqslant\alpha<1:\quad S_4^0\ (a=0)$
$\mathfrak{g}_{3,5}^\alpha+\mathfrak{g}_1$	$\alpha=0:\quad S^A\ (a=b=0, \kappa_1=\epsilon_1, \kappa_2=c=\epsilon_2)$ $\alpha=0:\quad S_2^0\ (d=\epsilon_1)$	$S^A\ (a=b=0, \kappa_1=\epsilon_1, \kappa_2=c=\epsilon_2)$ $S_2^0\ (d=-\epsilon_1), S_3^0\ (a=c=0)$
$\mathfrak{g}_{4,1}$	$S_{1,\mu}^\epsilon\ (\mu=-1),\ S_1^0$	$\times$
$\mathfrak{g}_{4,2}^\alpha$	$\alpha=1:\ S_4^0, S_6^0$	$\times$
$\mathfrak{g}_{4,3}$	$\times$	S_6^0
$\mathfrak{g}_{4,4}$	$\times$	$S^{A_3}\ (a=0, d=-\epsilon_1)$
$\mathfrak{g}_{4,5}^{\alpha,\beta}$	$\alpha=2\beta-1<1:\quad S_1^0\ (a=1, \kappa_1=\kappa_2=b=c=0)$ $\alpha=\frac{1}{3}, \beta=\frac{2}{3}:\quad S_1^0\ (a=1, \kappa_2=\epsilon_2, \kappa_1=b=c=0)$ $\alpha=2\beta-1<1:\quad S_3^0\ (\kappa_1=\kappa_2=0)$ $\alpha=\beta=1:\quad S_4^0$ $\alpha=-\beta:\quad S_4^0\ (\kappa_1=\kappa_2=0)$ $\alpha=-\beta=-\frac{1}{2}:\quad S_4^0\ (\kappa_1=0)$	$\alpha=2\beta-1:\quad S^A\ (\kappa_1=\kappa_2=b=c=0, \kappa_3=\epsilon_3)$ $\alpha=-\beta:\quad S^A\ (\kappa_1=\epsilon_1, a=b=\kappa_2=\kappa_3=0)$ $\alpha=\beta=1:\quad S^A$
$\mathfrak{g}_{4,6}^{\alpha,\beta}$	$\alpha=\beta:\quad S_2^0\ (d=\epsilon_1)$ $\alpha=0:\quad S_2^0\ (d=\epsilon_1)$	$\alpha=\beta:\quad S^A\ (a=b=0, \kappa_1=\epsilon_1, \kappa_2=c=\epsilon_2)$ $\alpha=0:\quad S^A\ (a=b=0, \kappa_1=\epsilon_1, \kappa_2=c=\epsilon_2)$
$\mathfrak{g}_{4,7}$	$\times$	$\times$
$\mathfrak{g}_{4,8}^\alpha$	$\alpha=-\frac{1}{2}:\quad S_2^0\ (\kappa_1=\epsilon_1, a=b=c=0)$ $\alpha=-\frac{1}{2}:\quad S_5^0\ (\kappa_1=0)$ $\alpha=0:\quad S_2^0\ (\kappa_1=b=0, a=\epsilon_1=1)$ $\alpha=0:\quad S_2^0\ (a=\epsilon_1=-1), S_3^0$ $\alpha=1:\quad S_2^0\ (a=-\frac{\epsilon_1}{4})$	$\|\alpha\|<1, \alpha\neq0:\quad S^\epsilon\ (\kappa_1=b=c=d=0, e=-\epsilon_1(1+\alpha)^2a^2)$ $\alpha=\frac{1}{2}:\quad S_2^0\ (\kappa_1=c=0)$ $\alpha=-\frac{1}{2}:\quad S_5^0$ $\alpha=-1:\quad S_3^0\ (a=-\frac{1}{2}),\ S_5^0\ (\kappa_1=0)$ $\alpha=0:\quad S^\epsilon\ (\kappa_1=b=c=d=0, e=-\epsilon_1 a^2)$ $\alpha=0:\quad S_0^0\ (\kappa_1=a=d=0)$ $\alpha=1:\quad S_1^\epsilon\ (b=4\epsilon_1\epsilon_2 a, c=0)$ $\alpha=1:\quad S_2^\epsilon\ (\kappa_1=0, b=-4\epsilon_2 c^2)$
$\mathfrak{g}_{4,9}^\alpha$	$\times$	$\alpha=0:\ S_1^0\ (a=\epsilon_1-b\neq0 \text{ or } a=b)$ $\alpha>0:\ S^\epsilon\ (a=b, c=4\epsilon_1\alpha^2a^2, d=e=0)$
$\mathfrak{g}_{4,10}$	$S_2\ (a=-1, c=\lambda_2=0)$	$S_1\ (a=\epsilon, b=c=0)$ $S_1\ (a=-\epsilon, b=c=0, \lambda_2=-\lambda_1)$ $S_1\ (a=b=d=0, \lambda_2=-2\lambda_1=2\epsilon c^2)$ $S_2\ (a=1, c=0)$

TABLE 1. Curvature properties of metric algebras (continued)

algebra	inner product	
	$\rho=0$	$\nabla\rho=0$
$4\mathfrak{g}_1$	$\times$	$\times$
$\mathfrak{g}_2+2\mathfrak{g}_1$	$S_1^0\ (\kappa_1=\kappa_2=\epsilon_2, a=\sqrt{2}, b=c=d=0)$	$S^A\ (\kappa_1=0)$ $S_1^0(\kappa_1=0, \kappa_2=\epsilon_2, c=0, d=-\epsilon_2 b^2/a^2)$ $S_1^0(\kappa_1=\epsilon_1, \kappa_2=\epsilon_2, b=c==d=0)$ $S_1^0(\kappa_1=\kappa_2=0, a=1, b=d=0)$ $S_1^0(\kappa_2=0, a=1, b=c=d=0)$
$2\mathfrak{g}_2$	$S_6\ (b=c=d=0)$ $S_6\ (b=d=0, c=-2)$ $S_6\ (c=d=0, b=-2)$	$S_1\ (a=d=e=f=0)$ $S_1\ (\epsilon_1=\epsilon_2=d,\ e=f, \lambda_1=\lambda_2=-a=-\epsilon_1 f^2)$ $S_4,\ S_5$ $S_7\ (a=c=d=e=0)$ $S_7\ (a=d=e=0, c=-2)$
$\mathfrak{g}_{3,1}+\mathfrak{g}_1$	$\times$	$S^\epsilon,\ S^\epsilon_{02}$
$\mathfrak{g}_{3,2}+\mathfrak{g}_1$	$S_3^0\ (d=-2\epsilon_1)$	$S_3^0,\ S_5^0\ (b=0)$
$\mathfrak{g}_{3,3}+\mathfrak{g}_1$	$\times$	$S^A\ (a=b=c=0),\ S_4^0$
$\mathfrak{g}_{3,4}^{\alpha}+\mathfrak{g}_1$	$\alpha=\frac{1}{2}:\quad S_1^0\ (\kappa_1\kappa_2=b=c=0)$ $\alpha\neq\frac{1}{2}:\quad S_1^0\ (\kappa_1=\epsilon_1, \kappa_2=\epsilon_2, a=\sqrt{\frac{2\epsilon_1\epsilon_2(\alpha-1)}{2\alpha-1}}, b=c=0)$ $\alpha=-1:\quad S_4^0\ (\kappa_1=0 \text{ or } \kappa_2=0)$ $\alpha\neq-1:\quad S_4^0\ (\kappa_1=\kappa_2=\epsilon_1, a=\frac{\sqrt{2(\alpha^2+1)}}{\alpha+1})$	$\alpha=\frac{1}{2}:\quad S^A\ (\kappa_1=\kappa_2=b=c=0, \kappa_3=\epsilon_3)$ $\alpha\neq-1:\quad S^A\ (\kappa_1=\epsilon_1, \kappa_2=\kappa_3=a=b=0)$ $-1\leqslant\alpha<1:\quad S_4^0$
$\mathfrak{g}_{3,5}^{\alpha}+\mathfrak{g}_1$	$\alpha>1:\quad S_2^0\ (d=(1-2\alpha^2)\epsilon_1 \pm 2\alpha\sqrt{\alpha^2-1}$ $\alpha\geqslant1:\quad S_3^0\ (a=0, c=\pm 2\sqrt{\alpha^2-1})$	$S^A\ (a=b=0, \kappa_1=\epsilon_1, \kappa_2=c=\epsilon_2)$ $S_2^0, S_3^0\ (a=0)$
$\mathfrak{g}_{4,1}$	$S^\epsilon_{1,\mu}\ (\mu=1)$	$S^\epsilon_{1,\mu},\ S^\epsilon_{2,\mu},\ S_2^0$
$\mathfrak{g}_{4,2}^{\alpha}$	$\alpha=1:\ S_5^0$ $\alpha\neq1:\ S_3^0\left(d=\frac{\epsilon_1}{4(\alpha-1)}\right)$	$\times$
$\mathfrak{g}_{4,3}$	$S_5^0\ (b=0)$	$S_5^0,\ S_6^0$
$\mathfrak{g}_{4,4}$	S_6^0	$S^{A_3}\ (a=0, d=-\epsilon_1)$
$\mathfrak{g}_{4,5}^{\alpha,\beta}$	$\alpha\neq2\beta-1:\quad S_1^0\ (\kappa_1=\epsilon_1, \kappa_2=\epsilon_2,$ $a=\sqrt{\frac{2\epsilon_1\epsilon_2(\alpha^2-(\alpha+1)\beta+1)}{(\alpha+1)(\alpha-2\beta+1)}}, b=c=0)$ $\alpha\neq2\beta-1:\quad S_3^0\ (\kappa_1=\kappa_2=\epsilon_1, a=\sqrt{\frac{2(\alpha\beta+\alpha-\beta^2-1)}{(\beta+1)(2\alpha-\beta-1)}})$ $\alpha=2\beta-1<1:\quad S_1^0\ (\kappa_1\kappa_2=b=c=0)$ $\alpha=-\beta:\quad S_4^0\ (\kappa_1\kappa_2=0)$ $\alpha\neq-\beta:\quad S_4^0\ (\kappa_1=\epsilon_1, \kappa_2=\epsilon_2,$ $a=\sqrt{\frac{2\epsilon_1\epsilon_2((\alpha-1)\alpha+(\beta-1)\beta)}{(\alpha+\beta-2)(\alpha+\beta)}})$	$\alpha=2\beta-1:\quad S^A\ (\kappa_1=\kappa_2=b=c=0, \kappa_3=\epsilon_1)$ $\alpha=-\frac{1}{5}, \beta=\frac{2}{5}:\quad S^A\ (\kappa_1=b=c=0, \kappa_2=\epsilon_2, \kappa_3=\epsilon_1)$ $\alpha=-\frac{1}{2},\ \beta=\frac{1}{4}:\quad S^A\ (\kappa_1=b=0, \kappa_2=\kappa_3=\epsilon_2, c=\epsilon_1\sqrt{2})$ $\alpha=-\beta:\quad S^A\ (\kappa_1=\epsilon_1, a=b=\kappa_2=\kappa_3=0)$ $\alpha=\beta=1:\quad S^A$
$\mathfrak{g}_{4,6}^{\alpha,\beta}$	$\beta=-2\alpha=-\frac{2}{\sqrt{3}}:\ S^A\ (a=b=0, \kappa_1=\epsilon_1, \kappa_2=-c=\epsilon_2)$ $S_2^0\ (d=\epsilon_1(2\alpha(\beta-\alpha)+1) \pm 2\sqrt{\alpha(\alpha-\beta)(\alpha^2-\alpha\beta-1})$ $S_3^0\ (a=0, c=\pm 2\sqrt{\alpha^2-\alpha\beta-1})$	$\alpha=\beta:\quad S^A\ (a=b=0, \kappa_1=\epsilon_1, \kappa_2=c=\epsilon_2)$ $\alpha=0:\quad S^A\ (a=b=0, \kappa_1=\epsilon_1, \kappa_2=c=\epsilon_2)$
$\mathfrak{g}_{4,7}$	$S_2^0\ \left(b=\frac{a^2-c^2}{4a}\right)$	$\times$
$\mathfrak{g}_{4,8}^{\alpha}$	$\lvert\alpha\rvert<1, \alpha\neq0:\quad S_3^0\ (b=\frac{((1+\alpha)^2a^2-1)\epsilon_1}{4\alpha})$ $\lvert\alpha\rvert<1, \alpha\neq0:\quad S_6^0\ (a=\frac{3}{\alpha-1})$ $\alpha=-\frac{1}{2}:\quad S_2^0\ (\kappa_1=\epsilon_1, a=c=0),\ S_5^0$ $\alpha=-1:\quad S_3^0\ (b=\frac{\epsilon_1}{4}), S_6^0\ (a=-\frac{3}{2})$ $\alpha=0:\quad S_2^0\ (a=\epsilon_1=1)$	$\lvert\alpha\rvert<1, \alpha\neq0:\quad S^\epsilon\ (\kappa_1=b=c=d=0, e=-\epsilon_1(1+\alpha)^2a^2)$ $\alpha=-\frac{1}{3}:\quad S^\epsilon\ (\kappa_1=c=d=0, e=-\frac{4}{9}\epsilon_1 a^2)$ $\alpha=\frac{1}{2}:\quad S_2^0\ (\kappa_1=c=0)$ $\alpha=-1:\quad S_3^0, S_5^0, S_6^0$ $\alpha=0:\quad S^\epsilon\ (\kappa_1=b=c=d=0, e=-\epsilon_1 a^2)$ $\alpha=0:\quad S_0^0\ (\kappa_1=a=d=0)$ $\alpha=1:\quad S_1^\epsilon\ (b=4\epsilon_1\epsilon_2 a, c=0)$ $\alpha=1:\quad S_2^\epsilon\ (\kappa_1=0, b=-4\epsilon_2 c^2)$
$\mathfrak{g}_{4,9}^{\alpha}$	$\alpha\geqslant0:\quad S_2^0\ (a=\frac{\epsilon_1}{2}(3+\sqrt{\frac{9\alpha^2+1}{\alpha^2+1}}))$ $\alpha\geqslant0:\quad S_1^0\ (a=b(1+2\alpha^2)\pm\sqrt{1+4b^2\alpha^2(1+\alpha^2)})$	$\alpha=0:\ S_1^0, S_2^0$ $\alpha>0:\ S^\epsilon\ (a=b, c=4\epsilon_1\alpha^2a^2, d=e=0)$
$\mathfrak{g}_{4,10}$	$S_2\ (a=-1, c=0)$	$S_1\ (a=\epsilon, b=c=0)$ $S_1\ (a=-\epsilon, b=c=0, \lambda_2=-\lambda_1)$ $S_1\ (a=b=d=0, \lambda_2=-2\lambda_1=2\epsilon c^2)$ $S_2\ (a=1, c=0)$

Acknowledgments. Partially supported by the Serbian Ministry of Education, Science and Technological Development through Faculty of Mathematics, University of Belgrade.

References

1. A. Agrachev, D. Barilari, *Sub-Riemannian structures on* 3*D Lie groups*, J. Dyn. Control Syst. **18**(1) (2012), 21–44.
2. D. V. Alekseevskiĭ, *Homogeneous Riemannian spaces of negative curvature*, Math. USSR, Sb. **25**(1) (1975), 87–109.
3. A. Andrada, M. L. Barberis, I. G. Dotti, G. P. Ovando, *Product structures on four dimensional solvable Lie algebras*, Homology Homotopy Appl. **7**(1) (2005), 9–37.
4. T. Arias-Marco, O. Kowalski, *Classification of four-dimensional homogeneous D'Atri spaces*, Czech. Math. J. **58**(1) (2008), 203–239.
5. D. Barrett, R. Biggs, C. Remsing, O. Rossi, *Invariant nonholonomic Riemannian structures on three-dimensional Lie groups*, J. Geom. Mech. **8** (2016), 139–167.
6. D. I. Barrett, C. C. Remsing, *A note on flat nonholonomic Riemannian structures on three-dimensional Lie groups*, Beitr. Algebra Geom. **60**(3) (2019), 419–36.
7. L. Bérard-Bérgery, *Homogeneous Riemannian Spaces of Dimension Four*, Seminar A. Besse, Four-dimensional Riemannian geometry, 1985.
8. N. Bokan, T. Šukilović, S, Vukmirović, *Lorentz geometry of* 4*-dimensional nilpotent Lie groups*, Geom. Dedicata. **177**(1) (2015), 83–102.
9. G. Calvaruso, A. Zaeim, *Four-dimensional Lorentzian Lie groups*, Differ. Geom. Appl. **31**(4) (2013), 496–509.
10. T. Christodoulakis, G. O. Papadopoulos, A. Dimakis, *Automorphisms of real four-dimensional Lie algebras and the invariant characterization of homogeneous* 4*-spaces*, J. Phys. A, Math. Gen. **36**(2) (2002), 427–441.
11. L. A. Cordero, P. E. Parker, *Left-invariant Lorentz metrics on* 3*-dimensional Lie groups*, Rend. Mat. Appl. **17** (1997), 129–155.
12. C. S. Gordon, E. N. Wilson, *Isometry groups of Riemannian solvmanifolds*, Trans. Am. Math. Soc. **307**(1) (1988), 245–269.
13. M. Guediri, *Sur la complétude des pseudo-métriques invariantes à gauche sur les groupes de Lie nilpotents*, Rend. Sem. Mat. Univ. Pol. Torino. **52** (1994), 371–376.
14. G. R. Jensen, *Homogeneous Einstein spaces of dimension four*, J. Differ. Geom. **3**(3–4) (1969), 309–49.
15. M. B. Karki, G. Thompson, *Four-dimensional Einstein Lie groups*, Differ. Geom. Dyn. Syst. **18** (2016), 43–57.
16. J. Lauret, *Homogeneous nilmanifolds of dimension* 3 *and* 4, Geom. Dedicata. **68** (1997), 145—155.
17. J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Adv. Math. **21**(3) (1976), 293–329.
18. G. M. Mubarakzyanov. *On solvable Lie algebras*, Izv. Vyssh. Uchebn. Zaved., Mat **32**(1) (1963), 114–123. (Russian)
19. J. Patera, P. Winternitz, *Subalgebras of real three- and four-dimensional Lie algebras*, J. Math. Phys. 18(7) (1977), 1449–1455.
20. T. Šukilović, *Geometric properties of neutral signature metrics on* 4*-dimensional nilpotent Lie groups*, Rev. Unión Mat. Argent. **57**(1) (2016), 23–47.
21. S. Vukmirović, T. Šukilović, *Geodesic completeness of the left-invariant metrics on* $\mathbb{R}H^n$, Ukr. Mat. Zhurn. **72**(5) (2020), 611–619.

Appendix A.

Table A.1. The list of 4-dimensional solvable Lie algebras in different notations

not [18]	$\mathfrak{g}_2+2\mathfrak{g}_1$	$2\mathfrak{g}_2$		$\mathfrak{g}_{3,1}+\mathfrak{g}_1$	$\mathfrak{g}_{3,2}+\mathfrak{g}_1$	$\mathfrak{g}_{3,3}+\mathfrak{g}_1$
not [3]	$\mathfrak{rr}_{3,0}$	$\mathfrak{r}_2\mathfrak{r}_2$		$\mathfrak{rh}_3$	$\mathfrak{rr}_3$	$\mathfrak{rr}_{3,1}$
not [19]	$A_2\oplus 2A_1$	$2A_2$		$A_{3,1}\oplus A_1$	$A_{3,2}\oplus A_1$	$A_{3,3}\oplus A_1$
not [18]	$\mathfrak{g}_{3,4}^{\alpha}+\mathfrak{g}_1$		$\mathfrak{g}_{3,5}^{\alpha}+\mathfrak{g}_1$			$\mathfrak{g}_{4,1}$
not [3]	$\mathfrak{rr}_{3,-1}$	$\mathfrak{rr}_{3,\lambda}, \lvert\lambda\rvert<1, \lambda\neq 0$	$\mathfrak{r}'_{3,\gamma}$			$\mathfrak{n}_4$
not [19]	$A_{3,4}\oplus A_1$	$A_{3,5}^{\alpha}\oplus A_1, \lvert\alpha\rvert<1$	$A_{3,6}\oplus A_1$	$A_{3,7}^{\alpha}\oplus A_1, \alpha>0$		$A_{4,1}$
not [18]	$\mathfrak{g}_{4,2}^{\alpha}$		$\mathfrak{g}_{4,3}$	$\mathfrak{g}_{4,4}$	$\mathfrak{g}_{4,5}^{\alpha,\beta}$	$\mathfrak{g}_{4,6}^{\alpha,\beta}$
not [3]	$\mathfrak{r}_{4,\lambda}, \lambda\neq 0$		$\mathfrak{r}_{4,0}$	$\mathfrak{r}_4$	$\mathfrak{r}_{4,\alpha,\beta}$	$\mathfrak{r}'_{4,\gamma,\delta}$
not [19]	$A_{4,2}^{\alpha}, \alpha\neq 0$		$A_{4,3}$	$A_{4,4}$	$A_{4,5}^{\alpha,\beta}$	$A_{4,6}^{\alpha,\beta}$
not [18]	$\mathfrak{g}_{4,7}$	$\mathfrak{g}_{4,8}^{\alpha}$		$\mathfrak{g}_{4,9}^{\alpha}$		$\mathfrak{g}_{4,10}$
not [3]	$\mathfrak{h}_4$	$\mathfrak{d}_4$	$\mathfrak{d}_{4,\lambda}$	$\mathfrak{d}'_{4,\delta}$		$\mathfrak{r}'_2$
not [19]	$A_{4,7}$	$A_{4,8}$	$A_{4,9}^{\beta}$	$A_{4,10}$	$A_{4,11}^{\delta}$	$A_{4,12}$

Table A.2. The automorphisms of 4-dimensional solvable Lie algebras

algebra	automorphisms
$\mathfrak{g}_2+2\mathfrak{g}_1$	$\begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad a_{11}, a_{22}a_{33}-a_{23}a_{32}\neq 0$
$2\mathfrak{g}_2$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & a_{23} & a_{24} \\ 1 & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix}, \quad a_{22},\ a_{44}, a_{24},\ a_{42}\neq 0$
$\mathfrak{g}_{3,1}+\mathfrak{g}_1$	$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{11}a_{22}-a_{12}a_{21} & a_{34} \\ a_{41} & a_{42} & 0 & a_{44} \end{pmatrix}, a_{11}a_{22}-a_{12}a_{21}, a_{44}\neq 0$

$\mathfrak{g}_{3,2}+\mathfrak{g}_1$		$\begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{22} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $a_{11},\, a_{22} \neq 0$
$\mathfrak{g}_{3,3}+\mathfrak{g}_1$		$\begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $a_{11}, a_{22}a_{33}-a_{23}a_{32} \neq 0$
$\mathfrak{g}^{\alpha}_{3,4}+\mathfrak{g}_1$	$0<\lvert\alpha\rvert<1$	$\begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & 0 & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $a_{11},\, a_{22},\, a_{33} \neq 0$
	$\alpha=-1$	$\begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & 0 & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $a_{11},\, a_{22},\, a_{33} \neq 0$ or $\begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & 0 & a_{34} \\ 0 & 0 & 0 & -1 \end{pmatrix}$, $a_{11},\, a_{23},\, a_{32} \neq 0$
$\mathfrak{g}^{\alpha}_{3,5}+\mathfrak{g}_1$	$\alpha>0$	$\begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & -a_{23} & a_{22} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $a_{11}, a_{22}^2+a_{23}^2 \neq 0$
	$\alpha=0$	$\begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & -a_{23} & a_{22} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{23} & -a_{22} & a_{34} \\ 0 & 0 & 0 & -1 \end{pmatrix}$, $a_{11}, a_{22}^2+a_{23}^2 \neq 0$
$\mathfrak{g}_{4,1}$		$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{11}a_{22} & 0 \\ a_{41} & a_{42} & a_{11}a_{32} & a_{11}^2a_{22} \end{pmatrix}$, $a_{11}, a_{22} \neq 0$
$\mathfrak{g}^{\alpha}_{4,2}$	$\alpha\neq 1$	$\begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{22} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $a_{11},\, a_{22} \neq 0$

	$\alpha=1$	$\begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{12} & a_{24} \\ 0 & 0 & a_{11} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$a_{11},\, a_{22}\neq 0$
$\mathfrak{g}_{4,3}$		$\begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{22} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$a_{11},\, a_{22}\neq 0$
$\mathfrak{g}_{4,4}$		$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{11} & a_{12} & a_{24} \\ 0 & 0 & a_{11} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$a_{11}\neq 0$
$\mathfrak{g}_{4,5}^{\alpha,\beta}$	$\alpha\neq\beta, \beta<1$	$\begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & 0 & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$a_{11},\, a_{22},\, a_{33}\neq 0$
	$\alpha<1$, $\beta=1$	$\begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} \\ 0 & a_{22} & 0 & a_{24} \\ a_{31} & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$a_{22}, a_{11}a_{33}-a_{13}a_{31}\neq 0$
	$\alpha=\beta<1$	$\begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$a_{11}, a_{22}a_{33}-a_{23}a_{32}\neq 0$
	$\alpha=\beta=1$	$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	
$\mathfrak{g}_{4,6}^{\alpha,\beta}$		$\begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & -a_{23} & a_{22} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$a_{11}, a_{22}^2+a_{23}^2\neq 0$
$\mathfrak{g}_{4,7}$		$\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ 0 & a_{11} & a_{23} & 0 \\ 0 & 0 & 1 & 0 \\ -a_{11}a_{23} & a_{13}a_{11}-a_{23}a_{11}-a_{12}a_{23} & a_{43} & a_{11}^2 \end{pmatrix}$	$a_{11}\neq 0$
$\mathfrak{g}_{4,8}^{\alpha}$	$\lvert\alpha\rvert<1$, $\alpha\neq 0$	$\begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & a_{22} & a_{23} & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{a_{11}a_{23}}{\alpha} & a_{13}a_{22} & a_{43} & a_{11}a_{22} \end{pmatrix}$	$a_{11},\, a_{22}\neq 0$

	$\alpha=-1$	$\begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & a_{22} & a_{23} & 0 \\ 0 & 0 & 1 & 0 \\ a_{11}a_{23} & a_{13}a_{22} & a_{43} & a_{11}a_{22} \end{pmatrix}$, $a_{11}, a_{22} \neq 0$ or $\begin{pmatrix} 0 & a_{12} & a_{13} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ 0 & 0 & -1 & 0 \\ -a_{13}a_{21} & -a_{12}a_{23} & a_{43} & -a_{12}a_{21} \end{pmatrix}$, $a_{12}, a_{21} \neq 0$
	$\alpha=0$	$\begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_{41} & a_{13}a_{22} & a_{43} & a_{11}a_{22} \end{pmatrix}$, $a_{11}, a_{22} \neq 0$
	$\alpha=1$	$\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & 0 & 1 & 0 \\ d_1 & d_2 & a_{43} & d_3 \end{pmatrix}$, $d_1=a_{13}a_{21}-a_{11}a_{23}$, $d_2=a_{13}a_{22}-a_{12}a_{23}$, $d_3=a_{11}a_{22}-a_{12}a_{21}\neq 0$
$\mathfrak{g}_{4,9}^{\alpha}$	$\alpha=0$	$\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ -a_{12} & a_{11} & a_{23} & 0 \\ 0 & 0 & 1 & 0 \\ d_1 & d_2 & a_{43} & d_3 \end{pmatrix}$, $d_1=a_{12}a_{23}-a_{11}a_{13}$, $d_2=-a_{12}a_{13}-a_{11}a_{23}$, $d_3=a_{11}^2+a_{12}^2\neq 0$ or $\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{12} & -a_{11} & a_{23} & 0 \\ 0 & 0 & -1 & 0 \\ d_1 & d_2 & a_{43} & d_3 \end{pmatrix}$, $d_1=a_{12}a_{23}+a_{11}a_{13}$, $d_2=a_{12}a_{13}-a_{11}a_{23}$, $d_3=-a_{11}^2-a_{12}^2\neq 0$
	$\alpha>0$	$\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ -a_{12} & a_{11} & a_{23} & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\Delta_2-\alpha\Delta_1}{1+\alpha^2} & \frac{-\Delta_1-\alpha\Delta_2}{1+\alpha^2} & a_{43} & a_{11}^2+a_{12}^2 \end{pmatrix}$, $a_{11}^2+a_{12}^2\neq 0$, $\Delta_1=a_{11}a_{23}+a_{12}a_{13}$, $\Delta_2=a_{12}a_{23}-a_{11}a_{13}$
$\mathfrak{g}_{4,10}$		$\begin{pmatrix} a_{11} & -a_{12} & a_{13} & a_{23} \\ a_{12} & a_{11} & a_{23} & -a_{13} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} a_{11} & a_{12} & a_{13} & -a_{23} \\ a_{12} & -a_{11} & a_{23} & a_{13} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, $a_{11}^2+a_{12}^2\neq 0$

КЛАСИФИКАЦИЈА ЛЕВО ИНВАРИЈАНТНИХ МЕТРИКА НА ЧЕТВОРОДИМЕНЗИОНИМ РЕШИВИМ ЛИЈЕВИМ ГРУПАМА

Резиме. У овом раду изложена је потпуна класификација лево инваријантних метрика произвољне сигнатуре на четвородимензионим Лијевим групама. Ако се Лијева алгебра идентификује са алгебром лево инваријантних векторских поља на придруженој Лијевој групи G, тада постоји јединствена "1-1" коресподенција између скаларног производа $\langle \cdot\,, \cdot \rangle$ на алгебри $\mathfrak{g} = \mathrm{Lie}\, G$ и лево инваријантне метрике g на Лијевој групи. Стога, проблем класификације је сведен на проблем класификације парова $(\mathfrak{g}, \langle \cdot\,, \cdot \rangle)$ које називамо метричким Лијевим алгебрама. Иако две метричке алгебре могу бити изометричне чак и ако њихове одговарајуће Лијеве алгебре нису изоморфне, у раду ће бити показано да су у четвородимензионом решивом случају ова два појма еквивалентна.

На крају се разматрају кривинска својства добијених метричких алгебри и, као последица, класификују се равне, локално симетричне, Ричи-равне, Ричи-паралелне и Ајнштајнове метрике.

Faculty of Mathematics
University of Belgrade
Belgrade
Serbia
tijana@matf.bg.ac.rs

(Received 26.08.2020.)
(Available online 07.12.2020.)