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ON THE STABILITY AND INSTABILITY CRITERIA FOR CIRCULATORY SYSTEMS: A REVIEW

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Abstract. A survey of the selected published criteria – expressed by the properties of the system matrices – for the stability and instability of linear mechanical systems subjected to potential and circulatory forces is presented. In particular, recent generalizations of the well-known Merkin instability theorem are reported. Several simple numerical examples are used to illustrate the usefulness of the presented criteria and also to compare them.

1. Introduction

An interesting class of linear dynamical systems is associated with potential (conservative) and circulatory (positional non-conservative) forces and can be described by the equation

$$\tilde{M}\ddot{q} + \tilde{K}q + \tilde{N}q = 0, \tag{1.1}$$

where $\tilde{M}$, $\tilde{K}$ and $\tilde{N}$ are n by n constant real matrices, $\tilde{M}$ is symmetric and positive definite ($\tilde{M} = \tilde{M}^T > 0$), $\tilde{K}$ is symmetric, and $\tilde{N}$ is skew-symmetric ($\tilde{N} = -\tilde{N}^T$). $\tilde{M}$ is the mass matrix, $\tilde{K}$ describes the potential forces and $\tilde{N}$ the circulatory forces. The n-vector of generalized coordinates is denoted by q, and the dots indicate differentiation with respect to the time, t. Derivation of Equation (1.1) can be found in e.g. [1, 2]. Such a system is an important mathematical model in various areas of physics and engineering (elasticity, fluid dynamics, rotor dynamics, control motion, plasma physics, etc.).

Making the transformation $x = \tilde{M}^{1/2}q$, where the exponent $\frac{1}{2}$ indicates the unique positive definite square root of the matrix $\tilde{M}$, and premultiplying Equation (1.1) by $\tilde{M}^{-1/2}$, we get the following equation

$$\ddot{x} + Kx + Nx = 0, \tag{1.2}$$

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where the symmetric matrix $K = \tilde{M}^{-1/2}\tilde{K}\tilde{M}^{-1/2}$ and the skew-symmetric one $\tilde{N} = \tilde{M}^{-1/2}\tilde{N}\tilde{M}^{-1/2}$. Clearly, system (1.2) is equivalent to system (1.1), and we shall from here on consider system (1.2).

The system is said to be stable if every solution $x(t)$ for Equation (1.2) is bounded for all non-negative t. If $N = 0$ (pure potential system), according to the famous Lagrange theorem, the system is stable if the potential matrix K is positive definite ($K > 0$); otherwise the potential system is unstable. For $N \neq 0$, it is possible that non-conservative positional forces can destabilize a stable purely potential system, and that they can stabilize an unstable potential system [2, 3]. The study of the influence of circulatory forces on the stability of potential systems, including numerous specific problems, has a rich history (see, for example, monographs [2, 4–7]).

All solutions for Equation (1.2) can be characterized algebraically using properties of the quadratic matrix polynomial $L(\mu) = \mu^2 I + K + N$, where I is the identity matrix. The eigenvalues of system (1.2) are zeros of the characteristic polynomial $\Delta(\mu) = \det(L(\mu))$, and the multiplicity of an eigenvalue is the order of the corresponding zero in $\Delta(\mu)$. If μ is an eigenvalue, the nonzero vectors in the null space of $L(\mu)$ are the eigenvectors associated with μ. Since $\Delta(\mu) = \Delta(-\mu)$, all eigenvalues (spectrum of the system) are located symmetrically with respect to both the real and imaginary axes in the complex plane. This means that system (1.2) is stable only when every eigenvalue μ is on the imaginary axis and simple or semi-simple, i.e., if the eigenvalue has multiplicity k, there are k linearly independent associated eigenvectors.

Although the eigenvalue analysis (spectral analysis) of the system with the help of computer programs is in principle easy, the influence of forces and parameters in the system matrices on the stability becomes lost [8]. This is more or less also the case when applying the algebraic criteria [9, 10], which are similar to the well-known Routh–Hurwitz criterion for asymptotic stability. Therefore, alternative criteria, such as those that provide simpler conditions directly in terms of the matrices K and N, are more attractive, and numerous attempts have been made to establish such criteria. It should be noted that another approach to studying the stability of multi-parameter circulatory systems, based on the perturbation theory of eigenvalues, is presented in the monograph by Kirillov [6] (also, see [7, 11]), but it is not the subject of this review.

In the following sections we provide a survey of the selected published criteria – expressed by the properties of the system matrices K and N – for the stability (Section 2) and instability (Section 3) of the systems under consideration.

2. Stability criteria

There are only a few simple criteria that contain sufficient conditions for the stability of the systems under consideration.

It is frequently the case that the potential matrix is positive definite ($K > 0$). For this case, using Lyapunov's direct method, Agafonov [12, 13] proved the following result.

THEOREM 2.1. [12]. *System* (1.2) *is stable if* $K = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_1 > \lambda_2 > \ldots, \lambda_n > 0$, *and*

$$\|N\|_\infty < \frac{\{[(\lambda_1+\lambda_n)^2 + 2\lambda_n s]^{\frac{1}{2}} - (\lambda_1+\lambda_n)\}}{2}, \tag{2.1}$$

where $s = \min_{1 \leqslant i \neq j \leqslant n} |\lambda_i - \lambda_j|$ *and* $\|.\|_\infty$ *denotes the maximum absolute row sum norm.*

Recently, by means of the well-known Bauer–Fike localization theorems, the following assertion was proved, which also covers the case $K > 0$.

THEOREM 2.2. [14]. *Let* $\lambda_1, \ldots, \lambda_n$ *be the eigenvalues of the positive definite potential matrix* K. *If*

$$\|N\|_2 < \min_{1 \leqslant i \neq j \leqslant n} \frac{|\lambda_i - \lambda_j|}{2} \tag{2.2}$$

where $\|N\|_2$ *denotes the spectral norm of the matrix* N (*i.e., the square root of the maximum eigenvalue of* $N^T N$), *then system* (1.2) *is stable.*

It should be noted that condition (2.2) in Theorem 2.2 can be replaced by the cruder inequality

$$\|N\|_\infty < \min_{1 \leqslant i \neq j \leqslant n} \frac{|\lambda_i - \lambda_j|}{2} \tag{2.3}$$

which is easy to check because $\|N\|_2 \leqslant \|N\|_\infty$. Also, it is easy to see that whenever (2.1) is satisfied, condition (2.3), as well as (2.2), is also satisfied.

The following numerical example shows that Theorem 2.2 significantly improves Theorem 2.1.

EXAMPLE 2.1. Consider the system of three degrees of freedom with

$$N = \nu \begin{pmatrix} 0 & 3 & 4 \\ -3 & 0 & 0 \\ -4 & 0 & 0 \end{pmatrix}, \quad \nu \in \Re, \qquad K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

For this system, we calculate $\|N\|_2 = 5|\nu|$, $\|N\|_\infty = 7|\nu|$, and $s = 3$. Conditions (2.2) and (2.3) yield $\nu < 0.3$ and $\nu < 0.214\ldots$ respectively, while Theorem 2.1 (condition (2.1)) predicts that the system of this example is stable if $\nu < 0.023\ldots$.

When the potential matrix is not positive definite, the following condition sufficient for stability which was also obtained by Agafonov [12] can be used.

THEOREM 2.3. [12]. *System* (1.2) *is stable if* $K = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_1 > \lambda_2 > \ldots \lambda_{n-1} > 0$, $\lambda_n \leqslant 0$ *and*

$$a + b - [(a-b)^2 + 4s^{-2}\|N\|_\infty^4]^{\frac{1}{2}} > 64s^{-3}(8\lambda_1 + s)(s - 4\|N\|_\infty)^{-1}\|N\|_\infty^4,$$

with

$$a = \lambda_{n-1} + \sum_{k=1, k\neq n-1}^{n} \nu_{n-1k}^2/(\lambda_k - \lambda_{n-1}) \quad \textit{and} \quad b = \lambda_n + \sum_{k=1}^{n-1} \nu_{nk}^2/(\lambda_k - \lambda_n),$$

where ν_{ij} *denote elements of the circulatory matrix* N.

In the case $\lambda_n = 0$, this result can be improved as follows.

Theorem 2.4. [14]. *Let* $K = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, $\lambda_1 > \lambda_2 > \dots \lambda_{n-1} > 0$, *and* $\lambda_n = 0$. *If* $\|N\|_2 < \min_{1 \leqslant i \neq j \leqslant n} |\lambda_i - \lambda_j|/2$ *and* $\sum_{j=1}^{n-1} |\nu_{nj}| \neq 0$, *where* ν_{nj} *are the coefficients of the matrix* N, *then system* (1.2) *is stable.*

To compare Theorem 2.4 and Theorem 2.3, we consider a two degree of freedom system with

$$K = \lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad N = \nu \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $\lambda_1 > 0$ and $\nu \in \Re$. Theorem 2.4 gives $0 < 2|\nu|\lambda_1^{-1} < 1$, which is a sufficient and necessary condition for stability of this system [2]. On the other hand, Theorem 2.3 reduces to the much stronger condition $0 < 2|\nu|\lambda_1^{-1} < 0.103\dots$ [12].

Suppose that

$$[K, N^2] = 0 \quad \text{and} \quad [K, NKN] = 0, \tag{2.4}$$

where $[.\,,.]$ denotes the commutator of the two matrices. In particular, if $n = 2$, then conditions (2.4) are satisfied.

Theorem 2.5. [15]. *If the circulatory matrix* N *is non-singular, then system* (1.2), (2.4) *is stable if and only if*

$$A = NKN^{-1}K - N^2 > 0 \tag{2.5}$$

and

$$K + NKN^{-1}K - 2A^{1/2} > 0. \tag{2.6}$$

It should be noted that if $\det N \neq 0$, then n is necessarily even because the circulatory matrix N is skew-symmetric.

Example 2.2. Let

$$K = \operatorname{diag}(5, 5, -1, -1) \quad \text{and} \quad N = \nu \begin{pmatrix} 0 & 0 & 1 & 15 \\ 0 & 0 & 15 & 1 \\ -1 & -15 & 0 & 0 \\ -15 & -1 & 0 & 0 \end{pmatrix}, \quad 0 \neq \nu \in \Re. \tag{2.7}$$

Obviously, Theorems 2.1–2.4 are not applicable to this example. However, matrices (2.7) satisfy conditions (2.4) and, in addition, $\det N \neq 0$. Therefore, Theorem 2.5 is applicable. It is easy to see that conditions (2.5) and (2.6) reduce to the conditions $|\nu| > \sqrt{5}/14$ and $|\nu| < 3/16$, respectively. Thus, according to Theorem 2.5, system (1.2), (2.7) is stable only if $|\nu| \in (\sqrt{5}/14, 3/16)$.

The next criterion, related to the subclass of system (2.4) for which $K \geqslant 0$, is much simpler than Theorem 2.5, and it allows the possibility of $\det N = 0$ (for example, it is the case when n is odd).

Theorem 2.6. [14].

a) *If* $\det N \neq 0$ *and* $K \geqslant 0$, *then system* (1.2), (2.4) *is stable if and only if*

$$[K, N]^2 - 4N^4 > 0; \tag{2.8}$$

b) *If* $\det N = 0$ *and* $K \geqslant 0$, *then system* (1.2), (2.4) *is stable if and only if the following conditions are satisfied*

$$([K,N]^2 - 4N^4)\,|_{\mathrm{Im}\,N} > 0, \quad K\,|_{\mathrm{Ker}\,N} > 0 \tag{2.9}$$

where $\mathrm{Im}\,N$ *and* $\mathrm{Ker}\,N$ *stand for the image and null space of* N, *respectively.*

EXAMPLE 2.3. Let

$$K = \mathrm{diag}(3,1,1) \quad \text{and} \quad N = \nu \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \nu \in \Re. \tag{2.10}$$

The system of this example satisfies conditions (2.4), and, in addition, $\det N = 0$ and $K \geqslant 0$.

Therefore, part (b) of Theorem 2.6 can be applied. This assertion yields $1 - 2\nu^2 > 0$, i. e., system (1.2), with matrices K and N as in (2.10), is stable only when $|\nu| < \sqrt{2}/2$. On the other hand, it is easy to see that the characteristic equation of this system has the following roots

$$\pm i, \pm i\sqrt{2 \pm \sqrt{1 - 2\nu^2}},$$

which all are purely imaginary and simple if $1 - 2\nu^2 > 0$, which is in accordance with the prediction of Theorem 2.6.

3. Instability criteria and generalizations of the Merkin theorem

THEOREM 3.1. [4]. *System* (1.2) *is unstable if* $K = 0$.

This assertion is a special case of the following result.

THEOREM 3.2. [3]. *System* (1.2) *is unstable if the trace of matrix* K *is non-positive, i.e.,* $\mathrm{Tr}\,K \leqslant 0$.

We note that the above theorem is inapplicable to the case where the corresponding conservative system is stable, i.e., $K > 0$. This case is covered by the following criterion, which is easy to check.

THEOREM 3.3. [16]. *System* (1.2) *is unstable if*

$$\|N\|_F^2 > \|K\|_F^2 - \frac{1}{n}(\mathrm{Tr}\,K)^2, \tag{3.1}$$

where $\|.\|_F$ *denotes the Frobenius norm.*

Recall that the Frobenius norm of a real matrix is defined as the square root of the sum of the squares of its elements. Condition (3.1) gives an estimation of the lower bound for the intensity of circulatory forces (measured by the Frobenius norm of N) so that the introduction of arbitrary linear circulatory forces, the intensity of which is higher than this bound, into a stable potential system destroys its stability. To illustrate Theorem 3.3 we return to Example 2.3. We have $\|N\|_F^2 = 4\nu^2$, $\|K\|_F^2 = 11$, $\mathrm{Tr}\,K = 5$, and the instability condition (3.1) yields $|\nu| > \sqrt{2/3}$.

Notice that the system of this example is unstable if and only if $|\nu| \geqslant \sqrt{2}/2$, as shown in the previous section.

Theorem 3.3 was obtained afterwards in [17], along with the two following sufficient conditions for instability (also, see [10]).

THEOREM 3.4. [17]. *System* (1.2) *is unstable if one of the following inequalities holds*

(a) $$n(\|K^2\|_F^2 + \|N^2\|_F^2 - 4\|KN\|_F^2 + 2\,\mathrm{Tr}((KN)^2)) < (\|K\|_F^2 - \|N\|_F^2)^2;$$

(b) $$(\|K\|_F^2 - \|N\|_F^2) + n(\|K^2\|_F^2 + \|N^2\|_F^2 - 4\|KN\|_F^2 + 2\,\mathrm{Tr}((KN)^2)) < (\mathrm{Tr}(K^3) + 3\,\mathrm{Tr}(KN^2))^2$$

It is easy to see using appropriate examples that neither Theorem 3.3 nor Theorem 3.4 implies the other one.

A remarkable result concerning the destabilizing effect of circulatory forces states: "If linear non-conservative positional forces are introduced into a stable potential system that has equal natural frequencies of vibration, then the stability will be destroyed, irrespective of any nonlinear terms" [2]. This assertion is known as the Merkin theorem, which was first published in 1956 [4], and which can be viewed as a counterpart of one of the classical Kelvin–Tait–Chetayev stability theorems for circulatory forces (see [18]). For linear systems, this statement can be reformulated in terms of the matrices K and N as follows.

THEOREM 3.5. [4]. *System* (1.2) *is unstable if* $N \neq 0$ *and* $K = \lambda_0 I$, $\lambda_0 \in \Re^+$.

In other words, the addition of arbitrary circulatory forces, infinitesimal or finite, to a conservative system whose potential matrix has the same eigenvalues produces instability. If we additionally assume that $\det N \neq 0$, then the system will be completely unstable (i.e., every nonzero solution $x(t)$ for Equation (1.2) is unbounded). It follows from the following more general result which implicitly requires non-singularity of the circulatory matrix N.

THEOREM 3.6. [19]. *System* (1.2) *is completely unstable if* $2N^TN - [K, N] > 0$.

Let us consider Example 2.2 again. For this system, we have

$$2N^TN - [K,N] = 2\nu \begin{pmatrix} 226\nu & 30\nu & -3 & -45 \\ 30\nu & 226\nu & -45 & -3 \\ -3 & -45 & 226\nu & 30\nu \\ -45 & -3 & 30\nu & 226\nu \end{pmatrix}.$$

It is easy to see that the above matrix is positive definite if and only if $|\nu| > 3/14$, and, according to Theorem 3.6, under this condition system (1.2), (2.7) is completely unstable.

It should be observed that the right hand side of inequality (3.1) is equal to zero only in the degenerate case when all eigenvalues of the potential matrix K are identical, and, consequently, Theorem 3.3 is a generalization of the Merkin theorem. Another obvious generalization is given by the following statement.

THEOREM 3.7. [15, 20]. *System* (1.2) *is unstable if* $N \neq 0$ *and* $[K, N] = 0$.

This generalization was obtained in [15], and independently and, more recently, in [20], where it was also pointed out that in this case the potential matrix K has at least one repeated eigenvalue (i.e., the corresponding conservative system has at least two equal natural frequencies). Although there is an uncountable infinity of skew-symmetric matrices N that commute with the given potential matrix K having multiple eigenvalues, as shown in [20], the commutation condition is very restrictive and some attempts have been made recently to weaken this restriction [21–24].

THEOREM 3.8. [21]. *Let the potential matrix* K *have a single eigenvalue* λ_0 *with multiplicity* $m \geqslant 2$,*and let* $T = [T_p \mid T_r]$ *be an orthogonal matrix, where the* $n \times p$ *submatrix* T_p *contains any* $2 \leqslant p \leqslant m$ *eigenvectors* (*their order is immaterial*) *of* K *corresponding to the multiple eigenvalue, and the* $n \times r$ *submatrix* T_r *contains the remainder* (*i.e.,* $r = n - p$) *of the eigenvectors of* K. *Then, if the following conditions hold*

$$T_p^T N T_p \neq 0, \quad T_p^T N T_r = 0, \tag{3.2}$$

system (1.2) *is unstable.*

This criterion is a special case of a result that is related to general positional perturbations [21], and it was also obtained, in the case $p = m$, in [22]. It is obvious that under condition (3.2) equations (1.2) may be decoupled by using a coordinate transformation determined by the orthogonal matrix T into two subsystems, one of which assures instability. However, for the given matrix K with multiple eigenvalues, the orthogonal matrix T that diagonalizes K is not unique, making it difficult to apply this result. An alternative rank criterion which gives the same instability result as Theorem 3.8, but which avoids the non-uniqueness of the matrix T, as well as the calculations of its eigenvalues and eigenvectors, was developed in [23].

Introduce two semi-definite matrices

$$\Omega_m = \sum_{k,l=1}^{m-1} = [K^k, N^l]^T [K^k, N^l] \quad \text{and} \quad \Phi = \sum_{k=0}^{n-1} K^k N^2 K^k \tag{3.3}$$

as well as the following matrices

$$S_m = \begin{bmatrix} [K, N] \\ [K, N^2] \\ \vdots \\ [K, N^{m-1}] \\ [K^2, N] \\ \vdots \\ [K^{m-1}, N^{m-1}] \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} N \\ NK \\ \vdots \\ NK^{n-1} \end{bmatrix} \tag{3.4}$$

where m is a natural number such that $m \leqslant n$.

THEOREM 3.9. [23]. *Let a system* (1.2) *be given. If*

$$\operatorname{rank}\Phi \geqslant 2 + \operatorname{rank}\Omega_w, \tag{3.5}$$

where the matrices Φ *and* Ω_w *are determined by* (3.3) *and* $w = \operatorname{rank}\Phi$*, then the matrix* K *has at least one repeated eigenvalue and system* (1.2) *is unstable.*

In condition (3.5) the matrices Φ and Ω_w can be replaced by the matrices L and S_w, respectively, since $\operatorname{rank}\Phi = \operatorname{rank}L$ and $\operatorname{rank}\Omega_w = \operatorname{rank}S_w$.

The application of Theorem 3.8 and Theorem 3.9 is illustrated by the following example [23].

EXAMPLE 3.1. Consider system (1.2) with

$$K = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 2 \end{bmatrix} \quad \text{and} \quad N = \nu \begin{bmatrix} 0 & 2 & -1 & -2 \\ -2 & 0 & 2 & -1 \\ 1 & -2 & 0 & -2 \\ 2 & 1 & 2 & 0 \end{bmatrix}, \tag{3.6}$$

where ν is a nonzero real parameter.

The matrix K has the following eigenvalues and corresponding mutually orthogonal eigenvectors:

$$\lambda_{1,2,3} = 1, \quad t_1^{(1)} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}^T, \quad t_2^{(1)} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & -1 & 0 \end{bmatrix}^T,$$
$$t_3^{(1)} = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 & 1 & -1 \end{bmatrix}^T$$

$$\lambda_4 = 4, \quad t_4^{(1)} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1 & 0 & -1 \end{bmatrix}^T$$

For this system of eigenvectors, it is easy to see that the second condition of (3.2) is not satisfied. However, the 4 mutually orthogonal vectors in the following system

$$t_1^{(2)} = \frac{1}{\sqrt{15}} \begin{bmatrix} 1 & -2 & -1 & 3 \end{bmatrix}^T, \qquad t_2^{(2)} = \frac{1}{3\sqrt{15}} \begin{bmatrix} 9 & 7 & 1 & 2 \end{bmatrix}^T,$$
$$t_3^{(2)} = \frac{1}{3\sqrt{3}} \begin{bmatrix} 0 & -1 & 5 & 1 \end{bmatrix}^T, \qquad t_4^{(2)} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1 & 0 & -1 \end{bmatrix}^T$$

are also eigenvectors of the matrix K associated with the eigenvalues $\lambda_{1,2,3} = 0$ and $\lambda_4 = 4$, respectively, and we have

$$\begin{bmatrix} t_1^{(2)} & t_2^{(2)} \end{bmatrix}^T N \begin{bmatrix} t_1^{(2)} & t_2^{(2)} \end{bmatrix} = 3\nu \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and

$$\begin{bmatrix} t_1^{(2)} & t_2^{(2)} \end{bmatrix}^T N \begin{bmatrix} t_3^{(2)} & t_4^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, for this choice of eigenvectors of the matrix K conditions (3.2) are satisfied and, according to Theorem 3.8, system (1.2), (3.6) is unstable for any nonzero value of the parameter ν. We now apply Theorem 3.9. It is clear that for this example $w = \operatorname{rank}\Phi = 4$, since $\det N \neq 0$. Also, we have

$$\Omega_4 = \sum_{k,l=1}^{3} [K^k, N^l]^T [K^k, N^l] = 114882\nu^2 \begin{bmatrix} 9 & -9 & 0 & -9 \\ -9 & 10 & -5 & 8 \\ 0 & -5 & 25 & 5 \\ -9 & 8 & 5 & 10 \end{bmatrix},$$

which yields $\operatorname{rank}\Omega_4 = 2$, and, according to condition (3.5), instability follows.

For the systems for which $[K, N^2] = 0$, condition (3.5) can be replaced by a simpler one as follows.

Theorem 3.10. [23]. *If* $[K, N^2] = 0$ *and*

$$\operatorname{rank} N \geqslant 2 + \operatorname{rank} \sum_{k=1}^{r-1} [K^k, N]^T [K^k, N], \tag{3.7}$$

where $r = \operatorname{rank} N$, *then the matrix* K *has at least one repeated eigenvalue and system* (1.2) *is unstable.*

In this criterion, condition (3.7) can be replaced by the following one [23]

$$\operatorname{rank} N \geqslant 2 + \operatorname{rank} \begin{bmatrix} [K, N] \\ [K^2, N] \\ \vdots \\ [K^{r-1}, N] \end{bmatrix}.$$

For the systems confined by conditions (2.4) a much simpler criterion than the one given by Theorem 3.10 can be formulated as follows.

Theorem 3.11. [24]. *Suppose that conditions* (2.4) *are satisfied and*

$$\operatorname{rank}[K, N] < \operatorname{rank} N \tag{3.8}$$

Then the potential matrix K *has at least one repeated eigenvalue and system* (1.2) *is unstable.*

If $N \neq 0$ and the matrices K and N commute, then conditions (2.4), (3.8) obviously hold and Theorem 3.7 is a direct consequence of Theorem 3.11.

Let us now go back to Example 3.1 to illustrate the above criterion. We have $[K, N^2] = 0$, $[K, NKN] = 0$ (i.e., conditions (2.4) are satisfied), and

$$[K, N] = \nu \begin{pmatrix} 0 & 1 & -5 & 1 \\ 1 & -2 & 5 & 0 \\ -5 & 5 & 0 & 5 \\ -1 & 0 & 5 & 2 \end{pmatrix}.$$

Now, we calculate: $\operatorname{rank} N = 4$ and $\operatorname{rank}[K, N] = 2$. Thus, all conditions of Theorem 3.11 are satisfied and system (1.2), with matrices K and N as in (3.6), is unstable for every $\nu \neq 0$.

Finally, we remark that if the given matrices K and N satisfy the conditions of Theorems 3.7-3.11, then these conditions also hold when the matrix N is replaced by $N_\nu = \nu N$, where ν is an arbitrary real nonzero number. This means that the stable conservative system with the potential matrix K (which must have multiple eigenvalues) will become unstable through the addition of an arbitrarily small circulatory force determined by the matrix N_ν.

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ПРЕГЛЕД КРИТЕРИЈУМА СТАБИЛНОСТИ И НЕСТАБИЛНОСТИ ЦИРКУЛАЦИОНИХ СИСТЕМА

Резиме. Даје се преглед критеријума стабилности и нестабилности изражених преко својстава описних матрица механичких система са коначним бројем степена слободе, подвргнутих дејству потенцијалних и циркулационих сила. Посебна пажња је посвећена недавним уопштењима чувене Меркинове теореме о нестабилности. Неколико примјера је дато ради илустрације примјењивости одабраних критеријума и њиховог међусобног односа.

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