THEORETICAL AND APPLIED MECHANICS
Volume 47 (2020) Issue 2, 155–168 DOI: https://doi.org/10.2298/TAM200120009B

# ON RHEONOMIC NONHOLONOMIC DEFORMATIONS OF THE EULER EQUATIONS PROPOSED BY BILIMOVICH

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ABSTRACT. In 1913 A. D. Bilimovich observed that rheonomic constraints which are linear and homogeneous in generalized velocities are ideal. As a typical example, he considered rheonomic nonholonomic deformation of the Euler equations whose scleronomic version is equivalent to the nonholonomic Suslov system. For the Bilimovich system, equations of motion are reduced to quadrature, which is discussed in rheonomic and scleronomic cases.

## 1. Introduction

Let $q = (q_1 \ldots, q_n)$ be generalized coordinates on the configuration space $Q$ of the system. The Lagrange equations describing the motion of the system may be written as

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q_i} - \frac{\partial T}{\partial q_i} = Q_i, \quad i = 1, \ldots, n,$$

where $T$ denotes the kinetic energy and $Q$ is a force. Assume now that the system is subject to additional independent constraints

$$f_j(q, \dot q, t) = 0, \quad j = 1, \ldots, k < n$$

and we have a constrained Lagrangian system with the number of degrees of freedom $\mathrm{dim}Q - k = n - k$. These constraints may be thought of as an addition of constraint forces to the original Lagrange equations.

The constraints are called integrable if they can be written in the form

$$f_j = \frac{d}{dt} g_j(q, t) = 0$$

for some functions $g_j$. Otherwise the constraints are called nonintegrable. After 1917, according to Hertz, nonintegrable constraints have been called nonholonomic.

2010 *Mathematics Subject Classification:* 37J60; 70F25.

*Key words and phrases:* rheonomic Lagrangian systems, nonholonomic mechanics, integrability by quadratures.

Similarly to the Lagrangian function, constraints may be time-dependent (rheonomic) or time-independent (scleronomic). Thus, we say that the constrained Lagrangian system is scleronomic (rheonomic) if the constraints and the Lagrangian are time-independent (time-dependent).

A nonholonomic constraint $f_j = 0$ is said to be ideal if the infinitesimal work of the constraint force vanishes for any admissible infinitesimal virtual displacement

$$\sum_{i=1}^{k} \frac{\partial f_j}{\partial \dot{q}_i} \delta q_i = 0. \tag{1.1}$$

This equation is the so-called Chetaev condition (see the paper [8], published in 1932). Some simple examples show that Chetaev's rule cannot be used in general (see [7, 19]).

Equations of motion of the nonholonomic system are deduced using the Lagrange–d'Alembert principle, Gauss and Appel principles, Hamilton–Suslov principle and so on. The general theory of linear and nonlinear, rheonomic and scleronomic, ideal and nonideal constraints and the corresponding nonholonomic systems has been discussed in many recent papers and textbooks. From the existing extensive list of literature, we have chosen publications particularly close to the work of Bilimovich [2, 3], see [1, 6, 9–11, 14, 15, 17, 18, 20–24] and references therein.

In 1903 A.D. Bilimovich graduated from Kiev University with the gold medal for his work "Application of geometric derivatives to the theory of curves and surfaces". He was a student of famous mechanicians K. G. Suslov and P.V. Voronets. After graduation, he was appointed a teaching assistant at the Department of Theoretical and Applied Mechanics. In 1907 he received the title of Privatdocent of the Department of Theoretical and Applied Mechanics of Kiev University, where in 1912 he defended his master's thesis "Equations of motion for conservative systems and their applications", which was published later in two papers [2, 3].

After the death of A. M. Lyapunov on November 3, 1918, he headed the commission for the preservation, processing and preparation for printing of the academician's works, which saved his manuscript "On Some Equilibrium Figures of a Rotating Fluid".

In January 1920, he left Odessa and soon found refuge in Serbia, where he created a large scientific school of analytical mechanics. Much credit also goes to him for creating a number of scientific associations and institutes in Serbia (in the 1920–1960s), and participating in the publishing activities of his immigrant compatriots: the Russian Academic Circle (April), Russian Scientific Institute, two editions of "Materials for the Bibliography of Russian Scientific Works Abroad", and the Mathematical Institute of the Serbian Academy of Sciences, the opening of which took place in May 1946. In 1949, the first volume of "Transactions of the Mathematical Institute of the Serbian Academy of Sciences" was published. It was in this edition that he published his works for several years, including the memoirs of Lyapunov in Odessa (1956). In addition, A. D. Bilimovich was one of the founders of the Yugoslav Society of Mechanics. His scientific activity was marked by his election on February 18, 1925 as a corresponding member, and February 17, 1936 as a full member of the Serbian Academy of Sciences and Arts [4].

So, in 1913–1914 Bilimovich published two papers in which he discussed the "commonly known fact" that scleronomic constraints

$$f_j = \sum_{i=1}^{n} b_{ij}(q)\dot{q}_j + b_j = 0, \quad j = 1, \dots, k < n,$$

are ideal if $b_j = 0$. Bilimovich noted that rheonomic constraints which are linear and homogeneous in generalized velocities

$$f_j = \sum_{i=1}^{n} b_{ij}(q,t)\dot{q}_j = 0, \quad j = 1, \dots, k < n,$$

are also ideal constraints. Bilimovich's first paper [2] was submitted to Comptes rendus de l'Académie des Sciences by Appel, so Appel also knew this "commonly known fact (1.1). Thus, Appel, Suslov and Voronets also knew this "commonly known fact" (1.1) about linear and homogeneous constraints in generalized velocities, but we cannot find the original source of this fact.

A typical example of rheonomic constraints is a rod of varying length, see Bilimovich's paper [3]. In this note we discuss another typical example associated with a rotating rod, which allows us to study rheonomic nonholonomic deformations of the Euler equations [2]. This rheonomic Bilimovich system is a generalization of the scleronomic nonholonomic Suslov system up to the choice of coordinates. The physical realization of the corresponding constraint which was proposed by Bilimovich is not realistic, in contrast to the nonholonomic pendulum [3], but it is also a typical rheonomic nonholonomic system, which we will integrate by quadratures below.

**1.1. On constraints that are linear in velocities and imposed on a conservative dynamical system.** Consider Lagrange's equations of the second kind

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i, \quad i = 1, \dots, n, \tag{1.2}$$

in which the kinetic energy is a second-degree polynomial in the generalized velocities and can be represented as

$$T = T_2 + T_1 + T_0,$$

where

$$T_2 = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}\dot{q}_i\dot{q}_j, \quad T_1 = \sum_{k=1}^{n} a_k\dot{q}_k, \quad T_0 = a_0,$$

and the coefficients $a_{ij}$ and $a_k$ are functions of the generalized coordinates $q_1, \dots, q_n$ and time $t$.

If one divides the forces $Q_i$ into potential and nonpotential ones

$$Q_i = -\frac{\partial V}{\partial q_i} + \widetilde{Q}_i,$$

then one can formulate a theorem of the change in the total mechanical energy $E = T + V$ of the holonomic system

$$\frac{d}{dt}E = \sum_{i=1}^{n} \tilde{Q}_i \dot{q}_i + \frac{d}{dt}(T_1 + 2T_0) - \frac{\partial L}{\partial t}, \tag{1.3}$$

where the Lagrangian is $L = T - V$. The imposition of $k < n$ nonholonomic constraints

$$f_j(q, \dot{q}) = 0, \quad j = 1, \ldots, k < n$$

changes Lagrange's equations (1.2)

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i + \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial \dot{q}_i}, \quad i = 1, \ldots, n,$$

and the theorem of the change in the total mechanical energy (1.3)

$$\frac{d}{dt}E = \sum_{i=1}^{n} (\tilde{Q}_i + Q_i^*)\dot{q}_i + \frac{d}{dt}(T_1 + 2T_0) - \frac{\partial L}{\partial t}, \quad Q_i^* = \sum_{j=1}^{k} \lambda_j \sum_{i=1}^{n} \frac{\partial f_j}{\partial \dot{q}_i} \dot{q}_i,$$

where $Q_i^*$ are the reaction forces of the nonholonomic constraints.

The homogeneous linear functions of the generalized velocities

$$f_j = \sum_{i=1}^{n} b_{ji}(q, t)\dot{q}_i$$

are distinguished by the fact that in this case the imposition of nonholonomic constraints does not change the theorem of the change in the total mechanical energy, since the work done by these corresponding reaction forces is zero

$$\sum_{i=1}^{n} Q_i^* \dot{q}_i = \sum_{j=1}^{k} \lambda_j \sum_{i=1}^{n} \frac{\partial f_j}{\partial \dot{q}_i} \dot{q}_i = \sum_{j=1}^{k} \lambda_j \sum_{i=1}^{n} b_{ji} \dot{q}_i = \sum_{j=1}^{k} \lambda_j f_j = 0.$$

This fact follows from Euler's theorem of homogeneous functions, which is also used for a standard derivation of the initial theorem of the change in the total mechanical energy of the holonomic system.

In discussing this well-known fact, in [2], Bilimovich noted that the coefficients $b_{ij}$ can explicitly depend on time $t$. Thus, imposing rheonomic constraints which are linear in the velocities in the velocities on the conservative system whose total mechanical energy does not change during the motion of the system, we obtain a rheonomic nonholonomic system for which the total mechanical energy is conserved as well.

## 2. Rheonomic deformations of the Euler equations

The main example in the work of Bilimovich [2] is related to the nonholonomic deformation of Euler's equations

$$I\dot{\omega} = I\omega \times \omega \quad \Longleftrightarrow \quad \frac{d}{dt}\Big(\frac{\partial T}{\partial \omega_i}\Big) = (I\omega \times \omega)_i,$$

where $\omega$ is the angular velocity vector of a rigid body and $I$ is the inertia tensor of the body. For this conservative system the total mechanical energy coincides with the kinetic energy

$$T = \tfrac{1}{2}(I\omega, \omega),$$

which remains unchanged during motion, as does the squared angular momentum of the body, $M^2$.

Solving two equations $T = E$ and $M^2 = m$ for $\omega_1$ and $\omega_2$ and substituting the resulting solutions into the equation of motion for the third angular velocity component, we obtain the well-known autonomous differential equation with separable variables

$$\frac{d\omega_3}{dt} = \sqrt{P_4(\omega_3, E, m)} \quad \text{or} \quad \Big(\frac{d\omega_3}{dt}\Big)^2 = P_4(\omega_3, E, m),$$

which contains a fourth-degree polynomial $P_4$ in $\omega_3$. An explicit form of this polynomial and a solution to this quadrature in terms of elliptic functions can be found, for example, in the textbooks [5, 10].

Imposing on this integrable system the nonholonomic rheonomic constraint

$$f = \omega_1 - g(t)\omega_2 = 0, \tag{2.1}$$

where $g(t)$ is an arbitrary function of time, changes the equations of motion

$$I\dot{\omega} = I\omega \times \omega + \lambda\frac{\partial f}{\partial \omega}, \quad \Longleftrightarrow \quad \frac{d}{dt}\Big(\frac{\partial T}{\partial \omega_i}\Big) = (I\omega \times \omega)_i + \lambda\frac{\partial f}{\partial \omega_i}, \tag{2.2}$$

in which, however, the total mechanical energy of the system remains unchanged:

$$T = \tfrac{1}{2}(I\omega, \omega) = E.$$

The undetermined Lagrange multiplier $\lambda$ appearing in these equations can be found by differentiating the constraint

$$\frac{df}{dt} = \sum_{i=1}^{3} \frac{\partial f}{\partial \omega_i}\dot{w}_i + \frac{\partial f}{\partial t} = 0.$$

Solving the equations $T = E$ and $f = 0$ for $\omega_1$ and $\omega_2$, we obtain

$$\omega_1 = \frac{\sqrt{A} - (g(t)I_{13} + I_{23})\omega_3}{C}\, g(t), \quad \omega_2 = \frac{\sqrt{A} - (g(t)I_{13} + I_{23})\omega_3}{C},$$

where

$$A = 2EC - B\omega_3^2, \qquad C = g(t)^2 I_{11} + 2g(t)I_{12} + I_{22},$$
$$B = g(t)^2(I_{11}I_{33} - I_{13}^2) + 2g(t)(I_{12}I_{33} - I_{13}I_{23}) + I_{22}I_{33} - I_{23}^2.$$

Substituting these solutions into the equation of motion for $\omega_3$, we obtain a nonautonomous differential equation with separable variables

$$\begin{aligned}\frac{d\omega_3}{dt} &= \frac{((1-g^2)I_{12} + g(I_{11} - I_{22}))A}{BC} \\ &- \frac{(((1+g^2)\omega_3 - g')\sqrt{A} - g'\omega_3(gI_{13} + I_{23}))(g(I_{11}I_{23} - I_{12}I_{13}) + I_{12}I_{23} - I_{13}I_{22})}{BC},\end{aligned} \tag{2.3}$$

where $g = g(t)$ and $g' = dg(t)/dt$. Similarly to the Euler case, it can be rewritten in a square-free form

$$(2.4)\quad \Big(\frac{d\omega_3}{dt} - \frac{((1-g^2)I_{12} + g(I_{11}-I_{22}))A}{BC} - \frac{(g'\omega_3(gI_{13}+I_{23}))(g(I_{11}I_{23}-I_{12}I_{13}) + I_{12}I_{23} - I_{13}I_{22})}{BC}\Big)^2 = \frac{((1+g^2)\omega_3 - g')^2(g(I_{11}I_{23}-I_{12}I_{13}) + I_{12}I_{23} - I_{13}I_{22})^2 A}{BC}.$$

Bilimovich reduced a similar equation for rheonomic nonholonomic pendulum to elliptic quadrature after a suitable change of time [3]. We suppose that this equation (2.4) could also be reduced to elliptic quadrature. If one integrates this equation, then the remaining components of the angular velocity vector, $\omega_1(t)$ and $\omega_2(t)$, can be found from the equations $T = E$ and $f = 0$.

Thus, we obtain formal quadrature (2.4) for the abstract rheonomic Lagrangian system with constraint (2.1). Of course, in general we have to define more exactly the physical properties of the constraints, which impose restrictions to the set of possible values of the constraint forces and to the admissible path. In order to do it, we need a realistic physical realization of the constraints, which is absent for the Bilimovich system.

In [2] Bilimovich studied tensor of inertia

$$(2.5)\qquad I = \begin{pmatrix} \mathrm{A} & 0 & 0 \\ 0 & \mathrm{A} & 0 \\ 0 & 0 & \mathrm{C} \end{pmatrix},$$

and proved that integration of the equations of motion (2.2) is reduced to the simultaneous integration of the following equations in terms of the Euler angles

$$\dot\psi \sin\phi[-\cos\theta + g(t)\sin\theta] + \dot\phi[\sin\theta + g(t)\cos\theta] = 0,$$
$$\dot\psi\cos\phi + \dot\theta = \Gamma,$$
$$\mathrm{A}(\dot\psi^2\sin^2\phi + \dot\phi^2) = 2E - \mathrm{C}\Gamma^2,$$

where $\Gamma$ is an arbitrary constant. Of course, we can also rewrite our quadrature (2.3) in terms of the Euler angles using change of variables

$$\omega_1 = p = -\dot\psi\sin\phi\cos\theta + \dot\phi\sin\theta,$$
$$\omega_2 = q = \dot\psi\sin\phi\sin\theta + \dot\phi\cos\theta,$$
$$\omega_3 = r\dot\psi\cos\phi + \dot\theta$$

from the Bilimovich paper [2]. If $g(t) = \alpha \in \mathbb{R}$ the constraint (2.1) can be written in the form

$$f = (a, \omega) = 0, \quad a = (1, -\alpha, 0),$$

where $a$ is the eigenvector of tensor $I$ given by (2.5) and at the level of the angular velocity we have $\dot\omega = 0$. Thus, we actually deal with the Suslov problem [25].

Let us briefly discuss solutions to (2.3) in other partial cases. For example, if the tensor of inertia has the form

$$I = \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{12} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix},$$

then the second term in (2.3) disappears and the separated equation becomes square root free

$$\frac{d\omega_3}{dt} = \frac{(2E - I_{33}\omega_3^2)(I_{12}(1-g(t)^2) + (I_{11}-I_{22})g(t))}{I_{33}(I_{11}g(t)^2 + 2I_{12}g(t) + I_{22})}.$$

The general solution to this equation is

$$\omega_3(t) = \sqrt{\frac{2E}{I_{33}}}\tan\left(\sqrt{\frac{2E}{I_{33}}}\left(c - \int \frac{(I_{12}(1-g(t)^2) + (I_{11}-I_{22})g(t))}{I_{11}g(t)^2 + 2I_{12}g(t) + I_{22}}dt\right)\right),$$

where $c$ is the constant of integration.

For example, at $E = 1, I_{33} = 4, I_{11} = 2, I_{22} = 1$ and $I_{12} = I_{13} = I_{23} = 0$, solutions to the separated equation (2.3) with $g(t) = \cos(t)$ and $g(t) = \alpha$ are presented in Figure 1.

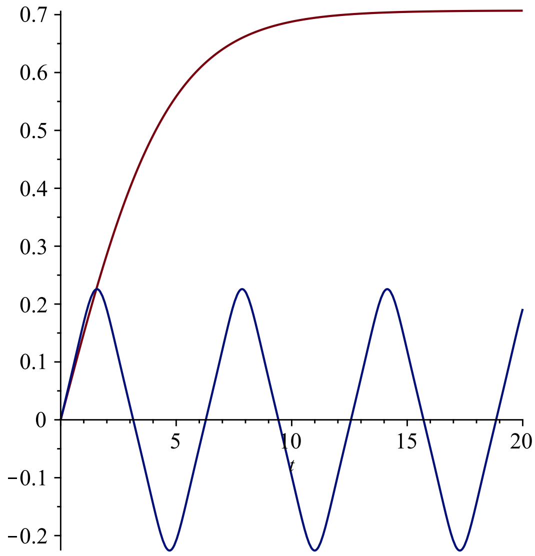


FIGURE 1. Graphs of $\omega_3(t)$ at $I_{12} = I_{13} = I_{23} = 0$.

The red curve denotes a graph for the scleronomic constraint $g(t) = 0.4$, and the blue curve is a solution graph for the rheonomic constraint with the periodic function $g(t) = \cos t$.

If at the same parameter values the off-diagonal moment of inertia $I_{12}$ is not zero, for example, $I_{12} = 0.05$, then the solutions to the separated equation (2.3) with $g(t) = \cos(t)$ and $g(t) = \alpha$ have the following form:

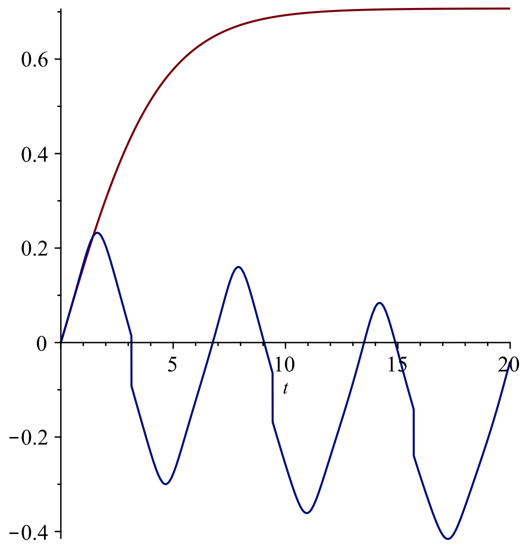


FIGURE 2. Graphs of $\omega_3(t)$ at $I_{12} = 0.05$, $I_{13} = I_{23} = 0$.

For the rheonomic constraint we have the following corresponding phase portrait

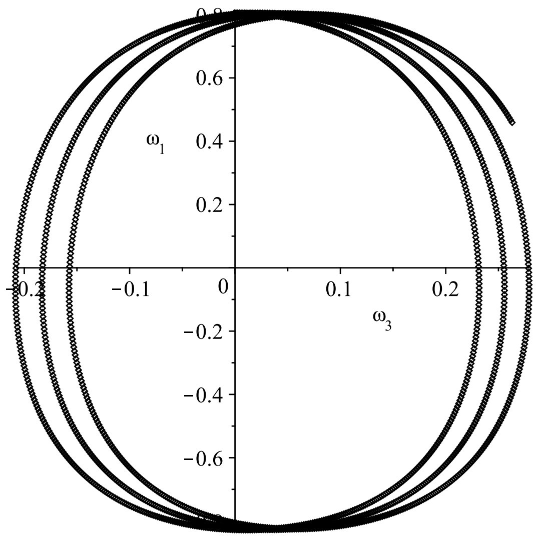


FIGURE 3. Dependence $\omega_1$ on $\omega_3$ at $I_{12} = 0.05$, $I_{13} = I_{23} = 0$.

If $I_{13} \neq 0$ and $I_{23} \neq 0$, the solutions to the separated equations (2.3) or (2.4) can be obtained numerically. For instance, if $I_{12} = 0.05$ and $I_{13} = -1$, then the numerical solutions to the separated equation (2.3) with $g(t) = \cos(t)$ and $g(t) = \alpha$ have the following form at $A > 0$

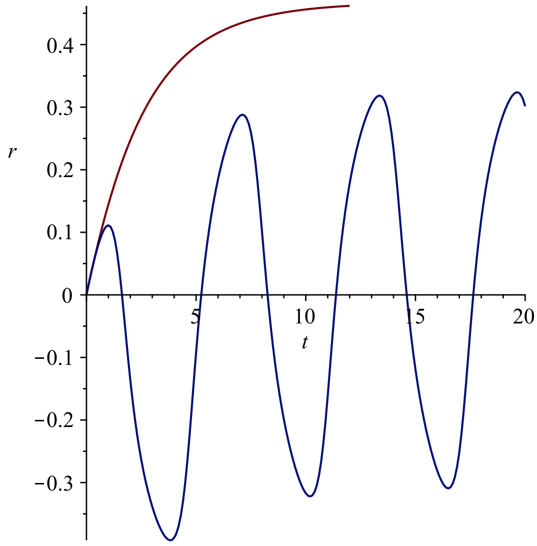


FIGURE 4. Graphs of $\omega_3(t)$ at $I_{12} = 0.05$, $I_{13} = -1$ and $I_{23} = 0$.

For the rheonomic constraint we have the following corresponding phase portrait

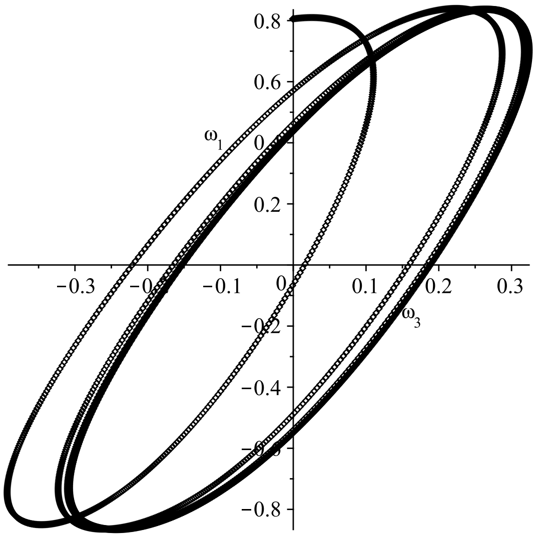


FIGURE 5. Dependence $\omega_1$ on $\omega_3$ at $I_{12} = 0.05$, $I_{13} = -1$ and $I_{23} = 0$.

After some time the motion becomes almost periodic. Here $I_{13} \neq 0$, but equation (2.3) can be integrated numerically because $A > 0$ under selected initial conditions. At the other values of parameters and initial conditions we have to solve second-order nonlinear equation (2.4), which cannot be converted to an explicit first-order system.

The off-diagonal moments of inertia almost do not change the motion pattern in the case of the scleronomic constraint. In the case of the rheonomic constraint, the motion pattern depends on the values of the off-diagonal moments of inertia rather strongly. In what follows, we will show how in the scleronomic case the off-diagonal moments of inertia are related to the existence of an invariant measure and to the Hamiltonian property of equations of motion.

**2.1. Scleronomic deformations of the Euler equations.** If we assume that the function $g(t)$ is constant, $g(t) = \alpha$, then the system of three differential equations (2.2)

$$X = \begin{pmatrix} \dot\omega_1 \\ \dot\omega_2 \\ \dot\omega_3 \end{pmatrix} = I^{-1}(I\omega \times \omega) + \lambda I^{-1} \begin{pmatrix} 1 \\ -\alpha \\ 0 \end{pmatrix}$$

defines a dynamical system on the plane

$$D = \{ f = (a, \omega) = \omega_1 - \alpha\omega_2 = 0 \} \tag{2.6}$$

with the integral $T$. This is Suslov problem and the solution can be found in [11] for the case $a = (0, 0, 1)$. After a suitable choice of coordinate system, the solution from [11] can be directly used for $a = (1, -\alpha, 0)$.

This makes it possible to reduce the order of this system by two and to obtain one differential equation (2.3), which in this case has the form

$$\begin{aligned} \frac{d\omega_3}{dt} &= F(\omega_3, E, \alpha) \\ &= \frac{(\alpha^2 I_{12} - \alpha(I_{11} - I_{22}) - I_{12})A}{BC} \\ &\quad + \frac{(\alpha^2+1)(\alpha(I_{11}I_{23} - I_{12}I_{13}) + I_{12}I_{23} - I_{13}I_{22})\omega_3\sqrt{A}}{BC}, \end{aligned} \tag{2.7}$$

where

$$A = B\omega_3^2 + 2EC, \qquad C = (\alpha^2 I_{11} + 2\alpha I_{12} + I_{22}),$$
$$B = \alpha^2(I_{13}^2 - I_{11}I_{33} -) + 2\alpha(I_{13}I_{23} - I_{12}I_{33}) - I_{22}I_{33} + I_{23}^2.$$

Note that calculations simplify considerably at

$$\alpha(I_{11}I_{23} - I_{12}I_{13}) + I_{12}I_{23} - I_{13}I_{22} = 0 \tag{2.8}$$

when in the definition of the function $F$ (2.7) the root $\sqrt{A}$ is absent and

$$F = \frac{(\alpha^2 I_{12} - \alpha(I_{11} - I_{22}) - I_{12})A}{BC}.$$

Under the condition $\det I \neq 0$ this simplification occurs in the unique case $I_{13} = I_{23} = 0$, which we will consider below in more detail.

When $I_{13} = I_{23} = 0$, the Lagrange multiplier is

$$\lambda = -\frac{\omega_2\omega_3\big((I_{11}^2 - I_{11}I_{33} + I_{12}^2)\omega_1^2 + 2\omega_1\omega_2 I_{12}(I_{11} + I_{22} - I_{33})\omega_2 + (I_{12}^2 + I_{22}^2 - I_{22}I_{33})\omega_2^2\big)}{I_{11}\omega_1^2 + 2I_{12}\omega_1\omega_2 + I_{22}\omega_2^2}.$$

Substituting $\lambda$ into the equations of motion (2.2), we obtain the vector field in $\mathbb{R}^3$

$$X = \begin{pmatrix} \dfrac{\big(\omega_1^2 I_{12} - \omega_2(I_{11} - I_{22})\omega_1 - \omega_2^2 I_{12}\big)\omega_1\omega_3}{\omega_1^2 I_{11} + 2\omega_1\omega_2 I_{12} + \omega_2^2 I_{22}} \\[2ex] \dfrac{\big(\omega_1^2 I_{12} - \omega_2(I_{11} - I_{22})\omega_1 - \omega_2^2 I_{12}\big)\omega_2\omega_3}{\omega_1^2 I_{11} + 2\omega_1\omega_2 I_{12} + \omega_2^2 I_{22}} \\[2ex] \dfrac{-\omega_1^2 I_{12} + \omega_2(I_{11} - I_{22})\omega_1 + \omega_2^2 I_{12}}{I_{33}} \end{pmatrix}$$

which possesses first integrals $T(\omega)$, $h(\omega) = \omega_1/\omega_2$ and invariant multiplier

$$\rho(\omega) = \frac{1}{\omega_1\omega_2},$$

which satisfies the standard Jacobi equation in $\mathbb{R}^3$

$$\frac{\partial(\rho X_1)}{\partial\omega_1} + \frac{\partial(\rho X_2)}{\partial\omega_2} + \frac{\partial(\rho X_3)}{\partial\omega_3} = 0.$$

This invariant multiplier defines the invariant singular measure in $\mathbb{R}^3$

$$\mu = \rho(\omega)\, d\omega_1 \wedge d\omega_2 \wedge d\omega_3.$$

The Suslov problem is defined on the plane $D \subset \mathbb{R}^3$ (2.6) that corresponds to the invariants set of first integral $h(\omega) = \alpha$.

According to the referee comment, it is interesting since there are various extensions. For example, if we extend the system such that $f(\omega)$ is the integral, then the kinetic energy $T$ is no more the first integral of the system, and the system cannot be written in the Hamiltonian form with the Hamiltonian $T$ (the corresponding vector field slightly differs from $X$ although it coincides with $X$ on $D$ and one can easily be confused). In the current extension $T$ is the first integral and the Hamiltonian form exists. Indeed, it is easy to see that the vector field $X$ is Hamiltonian

$$X = PdT$$

with respect to the Hamiltonian $T$ and the Poisson bivector

$$P = \omega_1\omega_2 \begin{pmatrix} 0 & I_3\omega_3 & -I_{12}\omega_1 - I_2\omega_2 \\ -I_3\omega_3 & 0 & I_{12}\omega_1 + I_1\omega_1 \\ I_{12}\omega_1 + I_2\omega_2 & -I_{12}\omega_1 - I_1\omega_1 & 0 \end{pmatrix} - \frac{\omega_1^2 I_{12} - \omega_1\omega_2 I_{11} + \omega_1\omega_2 I_{22} - \omega_2^2 I_{12}}{(\omega_1^2 I_{11} + 2\omega_1\omega_2 I_{12} + \omega_2^2 I_{22})I_{33}} \begin{pmatrix} 0 & 0 & \omega_1 \\ 0 & 0 & \omega_2 \\ -\omega_1 & -\omega_2 & 0 \end{pmatrix}.$$

Function $h(\omega)$ is the Casimir function of this Poisson bivector $P$, i.e. $Pdh(\omega) = 0$.

For nonphysical solutions to equation (2.8), for example,

$$I = \begin{pmatrix} I_{11} & 0 & I_{13} \\ 0 & 0 & 0 \\ I_{13} & 0 & I_{33} \end{pmatrix},$$

an invariant measure also exists, and the equations of motion are Hamiltonian.

If vector $a$ in $f = (a, \omega)$ is not the eigenvector of tensor $I$, then the system has no smooth invariant measure. The general problem is studied in [12, 16]. The next derivation of a singular measure for the Suslov problem was proposed by the referee. Let us take vector field $X$ on $\mathbb{R}^3$ and restrict $X$ on a plane $D$ (2.6). The singular measure $\mu$ should be restricted to $D$ in order to have the measure for the Suslov problem. We have

$$\mu = \rho d\omega_1 \wedge d\omega_2 \wedge d\omega_3 = \rho df \wedge \Omega,$$

where

$$\Omega = \frac{1}{1+\alpha^2}(\alpha d\omega_1 + d\omega_2) \wedge d\omega_3.$$

Thus, the singular invariant measure of the Suslov problem is $\rho\Omega|_D$.

## 3. Conclusion

In [3] Bilimovich found solutions for the rheonomic constrained Lagrangian system in terms of elliptic quadratures. It is the first example of explicit integration of equations of motion of nonholonomic systems with rheonomic constraints.

In [2] Bilimovich studied rheonomic nonholonomic deformations of the Euler equations and very briefly discussed explicit integration of this system in the partial axially symmetric case. In this note we present explicit integration of these equations of motion in generic cases.

It will be interesting to study inhomogeneous Bilimovich systems with constraint

$$f = \omega_1 - g(t)\omega_2 = a, \quad a \in \mathbb{R}.$$

It is a nonideal constraint and the corresponding constrained equations of motion do not preserve total energy $T$. However, the corresponding two-dimensional flow could preserve some integral in partial cases similar to the inhomogeneous Suslov problem [13].

**Acknowledgments.** We are very grateful to the referees for a thorough analysis of the manuscript, constructive suggestions and proposed corrections, which have certainly led to a more profound discussion of the results.

This work was supported by the Russian Science Foundation (project no. 19-71-30012) and performed at the Steklov Mathematical Institute of the Russian Academy of Sciences. The authors declare that they have no conflicts of interest.

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## БИЛИМОВИЋЕВЕ РЕОНОМНЕ НЕХОЛОНОМНЕ ДЕФОРМАЦИЈЕ ОЈЛЕРОВИХ ЈЕДНАЧИНА

Резиме. А. Д. Билимовић је 1913. приметио да су реономне хомогене линеарне везе по генералисаним брзанама идеалне. Као типичан пример, разматрао је реономну нехолономну деформацију Ојерових једначина, чија је скрелономна верзија еквивалентна Сусловљевом проблему. Једначине Билимовићевог система су сведене на квадратуре, при чему је дискутован реономни и склерономни случај.

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(Received 20.01.2020.)
(Revised 25.06.2020.)
(Available online 26.08.2020.)

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