THEORETICAL AND APPLIED MECHANICS
Volume 47 (2020) Issue 1, 99–112 DOI: https://doi.org/10.2298/TAM191125001B

# SIZE EFFECTS ASSOCIATED WITH SKEW SYMMETRIC BURGERS TENSOR

**A. S. Borokinni, O. O. Fadodun, O. P. Layeni, A. P. Akinola, and B. A. Olokuntoye**

Abstract. This paper investigates size effect phenomena associated with the divergence of the transpose of plastic distortion in plastically deformed isotropic materials. The principle of virtual power, balance of energy, second law of thermodynamics, and codirectionality hypothesis are used to formulate the governing microforce balance and thermodynamically consistent constitutive relations for dissipative microscopic stresses associated with the plastic distortion and skew part of the Burgers tensor. It is obtained that the defect energy through the strictly skew Burgers tensor is converted to the defect energy via the divergence of the plastic distortion. The presence of material length scales in the obtained flow rule indicates that it is possible to apprehend size effects associated with the skew part of the Burgers tensor during the inhomogeneous plastic flow of solid material. Finally and amongst other things, it is shown that the dependency of the microscopic stress vector on the divergence of plastic distortion rate leads to weakening and strengthening effects in the flow rule.

## 1. Introduction

Studies have shown that, in the micron scale range of about 500 nanometers to 50 micrometers, the strength of the metallic component during inhomogeneous plastic flow is size-dependent [1]. This size effect is not captured by the classical plasticity theory due to its inability to accommodate intrinsic material length scales. Gradient plasticity theories [2] have been developed to circumvent these shortcomings.

There are many gradient plasticity theories in literature. For instance, Aifantis [3, 4] and Muhlhaus and Aifantis [5] developed gradient theories in which the Laplacian of an effective strain measure is incorporated into the classical yield criterion while Gudmundson [6], Gurtin [7, 8] and Gurtin and Anand [9] proposed gradient theories which resulted in size-dependent non-local flow rules. Other

2010 *Mathematics Subject Classification:* 74C10.
*Key words and phrases:* Burgers tensor, flow rule, plastic distortion, polycrystalline, size effects.

recent gradient theories associated with plastic size effects were proposed by Aifantis and his co-authors [10–14].

In 2008, Reddy et al. [15] showed the need to incorporate the divergence of plastic strain into the system of kinematic variables with a view to determining the qualitative properties of the solution to the flow rules arising in strain gradient plasticity. Borokinni et al. [16] considered the divergence of plastic strain rate in the gradient theory of plasticity and an associated power-conjugate known as the microscopic stress vector, obtaining thermodynamically consistent constitutive relations for the dissipative and energetic microscopic stresses. At the crystalline level, the flow of dislocation in the material structure is measured through plastic distortion and Burgers tensors [17]. Gurtin [8] used the Burgers tensor and its power-conjugate to establish a flow rule for rate-dependent processes [2, 18].

This work is concerned with a thermodynamically consistent formulation of plasticity theory involving size effects. The formulation is based on the basic laws of continuum mechanics, principle of virtual power, and the first and second laws of thermodynamics. However, the departure from other works in literature consists in the assumption that both the Burgers tensor and its internal microstress energy conjugate are strictly skew-symmetric. In this case, the energy through the Burgers tensor is converted to the energy via the divergence of plastic distortion tensor. This then allows the development of a divergence-based plasticity theory for investigating size effects in plastically deformed solids. The present theory has both an advantage and a limitation. The merit is that it is efficient in the sense that the internal microstress in plastically deformed solid bodies is measured by a vector quantity (rank-one object) whereas in the usual gradient plasticity theory, one uses a rank-three tensor (polar microstress tensor). The limitation is that the theory is restricted to problems where the Burgers tensor and its energy conjugate are strictly skew-symmetric.

Recently, Borokinni [19] investigated the difference between the well-known Aifantis [3] and Gurtin-Anand theories [9], both of which ignore the plastic spin. A point of departure is to remove the assumption of plastic irrotationality from the outset of the present article so that the plastic distortion rate tensor is considered as non-symmetric.

## 2. Notations

In component form, second-order tensor $\mathbf{A}$ is denoted by $A_{ij},\ i,j = 1,2,3$ and the product $\mathbf{Au}$ of $\mathbf{A}$ and vector $\mathbf{u}$ is denoted by $A_{ij}u_j$, where the repeated index indicates summation. The trace of $\mathbf{A}$ is denoted by $\operatorname{tr}\mathbf{A}$; the symmetric and skew parts of $\mathbf{A}$ are given by $\operatorname{sym}\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$ and $\operatorname{skw}\mathbf{A} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$ respectively, $\mathbf{A}^T$ being the transpose of $\mathbf{A}$; while the magnitude of $\mathbf{A}$ is given by $|\mathbf{A}| = (A_{ij}A_{ij})^{1/2}$. The inner-product of nonzero second-order tensors $\mathbf{A}$, $\mathbf{B}$ and that of third order tensors $\mathbb{A}$, $\mathbb{B}$ are defined by $\mathbf{A} : \mathbf{B} = A_{ij}B_{ij}$ and $\mathbb{A} \vdots \mathbb{B} = A_{ijk}B_{ijk}$ respectively. The deviatoric part of a second-order tensor $\mathbf{A}$ is defined by $\mathbf{A}_o = \operatorname{dev}\mathbf{A} = \mathbf{A} - \frac{1}{3}(\operatorname{tr}\mathbf{A})\mathbf{I}$, where $\mathbf{I}$ is the second-order unit tensor. Furthermore, given any nonzero vector $\mathbf{a}$, the component form of second-order tensor $(\mathbf{a}\times)$ is $(\mathbf{a}\times)_{ij} = \epsilon_{ikj}a_k$, where $\epsilon_{ikj}$ is the permutation symbol. The partial derivative of any quantity $(\,.\,)$ with respect to the spatial variable $x_i$ is defined by $(\,\cdot\,)_{,i} = \frac{\partial(\,\cdot\,)}{\partial x_i}$. Finally and in component forms, the gradients of a vector field $\mathbf{a}$ and tensor field $\mathbf{A}$ are defined by $(\nabla \mathbf{a})_{ij} = a_{i,j}$ and $(\nabla \mathbf{A})_{ijk} = A_{ij,k}$ respectively; the divergences of the same quantities are defined by $\operatorname{Div} \mathbf{a} = a_{k,k}$ and $(\operatorname{Div} \mathbf{A})_i = A_{ik,k}$ respectively; while the curl of a tensor $\mathbf{A}$ is given by $(\operatorname{Curl} \mathbf{A})_{ij} = (\nabla \times \mathbf{A})_{ij} = \epsilon_{ipq} A_{jq,p}$.

## 3. Kinematic relations

Let $\mathbf{u}$ denote the displacement vector of an arbitrary particle in a plastically deformed polycrystalline solid body $B$ undergoing infinitesimal deformation, then the gradient of displacement vector $\mathbf{u}$ admits the additive decomposition

$$\nabla \mathbf{u} = \mathbf{H}^e + \mathbf{H}^p, \quad \operatorname{tr} \mathbf{H}^p = 0,$$

where $\mathbf{H}^e$, $\mathbf{H}^p$ are the elastic and plastic components of $\nabla \mathbf{u}$ respectively, and $\operatorname{tr} \mathbf{H}^p$ is the trace of the tensor $\mathbf{H}^p$. The elastic component $\mathbf{H}^e = \mathbf{E}^e + \mathbf{W}^e$ characterizes both the stretching $\mathbf{E}^e$ and rotation $\mathbf{W}^e$ of the material lattice structure while the plastic component $\mathbf{H}^p = \mathbf{E}^p + \mathbf{W}^p$ accounts for an irreversible defect due to the formation and motion of dislocations in the material lattice structure; $\mathbf{E}^p$ is the plastic strain and $\mathbf{W}^p$ is the plastic rotation.

The Burgers tensor $\mathbf{G}$ is defined by

$$\mathbf{G} = \nabla \times \mathbf{H}^p = \operatorname{Curl} \mathbf{H}^p. \tag{3.1}$$

THEOREM 3.1. *Let the Burgers tensor* $\mathbf{G}$ *be a strictly skew-symmetric tensor, then* $\mathbf{G}$ *assumes the form* $\mathbf{G} = -\frac{1}{2}((\operatorname{Div} \mathbf{H}^{pT})\times)$, *where* $\mathbf{H}^p$ *is the plastic distortion tensor.*

PROOF. Since $\mathbf{G}$ is skew-symmetric, there exists an axial vector $\vec{g}$ such that

$$\mathbf{G} = \vec{g} \times . \tag{3.2}$$

The component form of Eq. (3.2) is

$$G_{ij} = \epsilon_{ikj} g_k,$$

Using Eq. (3.1) we have

$$\epsilon_{ipq} H^p_{jq,p} = \epsilon_{ikj} g_k. \tag{3.3}$$

Multiplying both sides of Eq. (3.3) by permutation symbol $\epsilon_{imj}$ gives

$$\epsilon_{imj} \epsilon_{ipq} H^p_{jq,p} = \epsilon_{imj} \epsilon_{ikj} g_k,$$

$$2\delta_{mk} g_k = -H^p_{qm,q} \tag{3.4}$$

where $\delta_{mn}$ is the Kronecker-delta symbol.

Equation (3.4) implies that the vector $\vec{g}$ is

$$\vec{g} = -\tfrac{1}{2}(\operatorname{Div} \mathbf{H}^{pT}). \tag{3.5}$$

Substituting Eq. (3.5) into Eq. (3.2) yields the desired result

$$\mathbf{G} = -\tfrac{1}{2}((\operatorname{Div} \mathbf{H}^{pT})\times). \tag{3.6}$$

This completes the proof. □

In view of Eq. (3.6); the defect energy $\mathbf{S} : \mathbf{G}$ due to the non-zero skew-symmetric microstress tensor $\mathbf{S} = -\vec{\chi}\times = (\vec{\chi}\times)^T$ via the Burgers tensor $\mathbf{G}$ takes the form

$$\mathbf{S} : \mathbf{G} = \vec{\chi} \cdot (\mathrm{Div}\, \mathbf{H}^{pT}), \tag{3.7}$$

where $-\vec{\chi}$ is an axial vector associated with the skew-symmetric microstress tensor $\mathbf{S}$.

The proof of relation in Eq. (3.7) is presented below.

Using Eqs. (3.2)

$$\mathbf{S} : \mathbf{G} = (-\vec{\chi}\times) : (\vec{g}\times) = -\epsilon_{ikj}\chi_k\epsilon_{imj}g_m,$$

$$\mathbf{S} : \mathbf{G} = -\epsilon_{ikj}\epsilon_{imj}\chi_k g_m = -2\delta_{mk}\chi_k g_m = -2\vec{\chi} \cdot \vec{g}. \tag{3.8}$$

Using Eq. (3.5) in Eq. (3.8) establishes the relation

$$\mathbf{S} : \mathbf{G} = \vec{\chi} \cdot (\mathrm{Div}\, \mathbf{H}^{pT}). \tag{3.9}$$

## 4. Internal Power Expenditure

The internal power expenditure of a system is the time derivative of the internal work expenditure. In view of Gurtin [8], let the elastic stress $\mathbf{T}$, plastic microstress $\mathbf{T}^p$, and skew symmetric microstress $\mathbf{S}$ be power-conjugates to elastic distortion rate $\dot{\mathbf{H}}^e$, plastic distortion rate $\dot{\mathbf{H}}^p$, and Burgers tensor rate $\dot{\mathbf{G}} = \mathrm{Curl}\, \dot{\mathbf{H}}^p$ respectively, then the net internal power expenditure $W_{\mathrm{int}}(P)$ within the subregion $P$ of the body is

$$W_{\mathrm{int}}(P) = \int_P (\mathbf{T} : \dot{\mathbf{H}}^e + \mathbf{T}^p : \dot{\mathbf{H}}^p + \mathbf{S} : \dot{\mathbf{G}}) dV.$$

Introducing $\mathbf{T} : \mathbf{H}^e = \mathbf{T} : (\mathbf{E}^e + \mathbf{W}^e) = \mathbf{T} : \mathbf{E}^e$ yields

$$W_{\mathrm{int}}(P) = \int_P (\mathbf{T} : \dot{\mathbf{E}}^e + \mathbf{T}^p : \dot{\mathbf{H}}^p + \mathbf{S} : \dot{\mathbf{G}}) dV. \tag{4.1}$$

Substituting Eq. (3.9) into Eq. (4.1) gives

$$W_{\mathrm{int}}(P) = \int_P [\mathbf{T} : \dot{\mathbf{E}}^e + \mathbf{T}^p : \dot{\mathbf{H}}^p + \vec{\chi} \cdot \mathrm{Div}\, \dot{\mathbf{H}}^{pT}] dV.$$

## 5. Microforce balance

The equilibrium equation (macroforce balance) at the points within the subregion $P$ of the body and the associated boundary condition (macrotraction condition) on the boundary $\partial P$ of $P$, governing the mechanical behavior of the body at macro scale, are [8, 9]

$$\mathrm{Div}\, \mathbf{T} + \mathbf{b} = \mathbf{0},$$

and

$$\mathbf{Tn} = \mathbf{t}(\mathbf{n}),$$

respectively, where $\mathbf{n}$ is the outward unit normal on the boundary $\partial P$ of $P$, $\mathbf{b}$ is the body force, and $\mathbf{t}(\mathbf{n})$ is the macrotraction.

Let the virtual velocities $\nu = (\tilde{\mathbf{u}}, \tilde{\mathbf{H}}^e, \tilde{\mathbf{H}}^p)$ satisfy the conditions

$$\nabla\tilde{\mathbf{u}} = \tilde{\mathbf{H}}^e + \tilde{\mathbf{H}}^p, \quad \operatorname{tr}\tilde{\mathbf{H}}^p = 0, \tag{5.1}$$

then the principle of virtual power implies that

$$\int_P [\mathbf{T} : \tilde{\mathbf{E}}^e + \mathbf{T}^p : \tilde{\mathbf{H}}^p + \vec{\chi}\cdot \operatorname{Div}\tilde{\mathbf{H}}^{pT}]dV = \int_{\partial P}[\mathbf{K}(\mathbf{n}) : \tilde{\mathbf{H}}^p + \mathbf{t}(\mathbf{n})\cdot\tilde{\mathbf{u}}]dA + \int_P \mathbf{b}\cdot\tilde{\mathbf{u}}dV, \tag{5.2}$$

where $\mathbf{K}(\mathbf{n})$ is the microtraction on boundary $\partial P$.

Since $\nu = (\tilde{\mathbf{u}}, \tilde{\mathbf{H}}^e, \tilde{\mathbf{H}}^p)$ are arbitrary; setting $\tilde{\mathbf{u}} = \mathbf{0}$ in Eq. (5.1) gives $\tilde{\mathbf{H}}^e = -\tilde{\mathbf{H}}^p$.

Substituting $\tilde{\mathbf{u}} = \mathbf{0}$ and $\tilde{\mathbf{H}}^e = -\tilde{\mathbf{H}}^p$ in Eq. (5.2) yields

$$\int_P [(\mathbf{T}^p - \mathbf{T}) : \tilde{\mathbf{H}}^p + \vec{\chi}\cdot \operatorname{Div}\tilde{\mathbf{H}}^{pT}]dV = \int_{\partial P}\mathbf{K}(\mathbf{n}) : \tilde{\mathbf{H}}^p dA. \tag{5.3}$$

Using the divergence theorem

$$\int_P \vec{\chi}\cdot \operatorname{Div}\tilde{\mathbf{H}}^{pT}dV = \int_{\partial P}(\vec{\chi}\otimes\mathbf{n})^T : \tilde{\mathbf{H}}^p dA - \int_P (\nabla\vec{\chi})^T : \tilde{\mathbf{H}}^p dV. \tag{5.4}$$

Substituting Eq. (5.4) into Eq. (5.3) gives

$$\int_P [\mathbf{T}^p - \mathbf{T} - (\nabla\vec{\chi})^T] : \tilde{\mathbf{H}}^p dV = \int_{\partial P}[\mathbf{K} - (\vec{\chi}\otimes\mathbf{n})^T] : \tilde{\mathbf{H}}^p dA. \tag{5.5}$$

Using the fundamental lemma of calculus of variation and invoking the constraint that $\mathbf{T}^p$ is deviatoric (since $\mathbf{H}^p$ is deviatoric), the solution to Eq. (5.5) gives both the microforce balance

$$\mathbf{T}_o = \mathbf{T}^p - (\nabla\vec{\chi})^T_o, \tag{5.6}$$

and its associated microtraction condition

$$\mathbf{K}(\mathbf{n}) = (\vec{\chi}\otimes\mathbf{n})^T_o. \tag{5.7}$$

Here $\mathbf{T}_o$ is the deviatoric part of the Cauchy stress. Within the framework of small deformation for isotropic materials, the *full* Cauchy stress is a linear function of the strain in the elastic regime. It is in fact written as

$$\mathbf{T} = 2\mu\mathbf{E} + \lambda\operatorname{tr}(\mathbf{E})\mathbf{I},$$

where $\mu$, $\lambda$ are the Lam$e$'s constants, and $\mathbf{E}$ is the total strain. In the plastic regime, $\mathbf{T}$ is constrained to account for yielding and irreversible restructuring in the plastic material. It is during this process that the deviatoric stress $\mathbf{T}_o$ is further required to satisfy the microscopic force balance (5.6) in addition to satisfying $\mathbf{T}_o = 2\mu\mathbf{E}^e_o$, where $\mathbf{E}^e_o$ is the deviatoric part of the elastic strain. In a plastically deformed body with the strictly skew-symmetric Burgers tensor, Eqs. (5.6) and (5.7) govern the interaction between elastic and plastic responses in the body.

## 6. Free energy imbalance and constitutive relations

Let $\psi$ be the free energy of the body measured per unit volume. The second law of thermodynamics, which states that the free energy cannot exceed power expended on the body, implies [9]

$$\dot{\psi} - \mathbf{T} : \dot{\mathbf{E}}^e - \mathbf{T}^p : \dot{\mathbf{H}}^p - \mathbf{S} : \dot{\mathbf{G}} \leqslant 0.$$

Introducing Eq. (3.9) into the above inequality gives

$$\dot{\psi} - \mathbf{T} : \dot{\mathbf{E}}^e - \mathbf{T}^p : \dot{\mathbf{H}}^p - \vec{\chi} \cdot \operatorname{Div} \dot{\mathbf{H}}^{pT} \leqslant 0. \tag{6.1}$$

In view of Eq. (3.6), the free energy takes the form

$$\psi = \hat{\psi}(\mathbf{E}^e, \mathbf{G}) = \check{\psi}(\mathbf{E}^e, \operatorname{Div} \mathbf{H}^{pT}). \tag{6.2}$$

Let the skew symmetric microstress $\mathbf{S}$ assume the form $\mathbf{S} = \mathbf{S}^{\text{en}} + \mathbf{S}^{\text{dis}}$; $\mathbf{S}^{\text{en}}$ and $\mathbf{S}^{\text{dis}}$ being energetic and dissipative components of $\mathbf{S}$ respectively, then

$$\mathbf{S}^{\text{en}} : \dot{\mathbf{G}} = \vec{\chi}^{\,\text{en}} \cdot \operatorname{Div} \dot{\mathbf{H}}^{pT} \tag{6.3}$$

and

$$\mathbf{S}^{\text{dis}} : \dot{\mathbf{G}} = \vec{\chi}^{\,\text{dis}} \cdot \operatorname{Div} \dot{\mathbf{H}}^{pT},$$

where $\vec{\chi}^{\,\text{en}}$ and $\vec{\chi}^{\,\text{dis}}$ are the respective corresponding energetic and dissipative components of $\vec{\chi}$.

The application of the chain rule on free energy in Eq. (6.2) gives

$$\dot{\psi} = \frac{\partial \hat{\psi}}{\partial \mathbf{E}^e} : \dot{\mathbf{E}}^e + \frac{\partial \hat{\psi}}{\partial \mathbf{G}} : \dot{\mathbf{G}}. \tag{6.4}$$

Using Coleman–Noll procedure, the energetic microstress $\mathbf{S}^{\text{en}}$ and the elastic stress $\mathbf{T}$ are given by

$$\mathbf{S}^{\text{en}} = \frac{\partial \hat{\psi}}{\partial \mathbf{G}} \quad \text{and} \quad \mathbf{T} = \frac{\partial \hat{\psi}}{\partial \mathbf{E}^e}. \tag{6.5}$$

Substituting Eqs. (6.5) and (6.3) into Eq. (6.4) gives

$$\dot{\psi} = \mathbf{T} : \dot{\mathbf{E}}^e + \vec{\chi}^{\,\text{en}} \cdot \operatorname{Div} \dot{\mathbf{H}}^{pT}. \tag{6.6}$$

Since $\mathbf{S} = -\vec{\chi}\times$, the microstress $\vec{\chi}$ assumes the form

$$\vec{\chi} = \vec{\chi}^{\,\text{en}} + \vec{\chi}^{\,\text{dis}}. \tag{6.7}$$

Using Eqs. (6.6) and (6.7) in Eq. (6.1) gives the dissipation inequality

$$\mathbf{T}^p : \dot{\mathbf{H}}^p + \vec{\chi}^{\,\text{dis}} \cdot \operatorname{Div} \dot{\mathbf{H}}^{pT} \geqslant 0. \tag{6.8}$$

The expression on the left-hand side of the inequality (6.8) is the local dissipation, which is nonnegative. The term $\mathbf{T} : \dot{\mathbf{H}}^p$ is the dissipation associated with plastic microstress $\mathbf{T}^p$, and $\vec{\chi}^{\,\text{dis}} \cdot \operatorname{Div} \dot{\mathbf{H}}^p$ is the local dissipation associated with size effects. The term prevailing in the dissipation inequality (6.8) will depend on the type of physical problems being considered.

In the present theory, the effective distortion-rate $d^p$ is defined by

$$d^p = \sqrt{|\dot{\mathbf{H}}^p|^2 + q^2 |\operatorname{Div} \dot{\mathbf{H}}^{pT}|^2}. \tag{6.9}$$

The expression in Eq. (6.9) is consistent with the result in Gurtin and Anand [9]. The effective distortion rate is introduced to ensure that the constitutive relations for the dissipative microscopic stresses are sufficiently consistent with the second law of thermodynamics. The length scale $q$ is introduced to ensure dimension consistency. An alternative form of Eq. (6.9), which is more general, is to replace the term $|q^2 \operatorname{Div} \dot{\mathbf{H}}^p|^2$ with $q^2|\nabla\dot{\mathbf{H}}^p|^2$. The parameter $q$ is the dissipative length scale associated with the divergence of transpose of plastic distortion.

Following Gurtin et. al [16] and Han and Reddy [20], we introduce a generalized plastic distortion rate $\dot{\mathbb{H}}^p$

$$\dot{\mathbb{H}}^p = (\dot{\mathbf{H}}^p, q \operatorname{Div} \dot{\mathbf{H}}^{pT})$$

and a generalized flow direction $\mathbb{N}^p$ involving the plastic spin

$$\mathbb{N}^p = \frac{\dot{\mathbb{H}}^p}{d^p}.$$

We define a generalized plastic stress $\mathbb{T}^p$ power-conjugate to $\dot{\mathbb{H}}^p$ by

$$\mathbb{T}^p = (\mathbf{T}^p, q^{-1}\vec{\chi}^{\,\mathrm{dis}}).$$

The dissipation inequality (6.8) takes the form

$$\delta = \mathbb{T}^p \bullet \dot{\mathbb{H}}^p \geqslant 0,$$

where the product $\bullet$ is defined by

$$\mathbb{T}^p \bullet \dot{\mathbb{H}}^p = \mathbf{T}^p : \dot{\mathbf{H}}^p + \vec{\chi}^{\,\mathrm{dis}} \cdot \operatorname{Div} \dot{\mathbf{H}}^{pT}.$$

Suppose the direction of the generalized plastic stress and the direction of the flow direction $\mathbb{N}^p$ coincide, then there exists a scalar function $Y(d^p)$ such that

$$\mathbb{T}^p = Y(d^p)\mathbb{N}^p.$$

Let $Y_1(d^p)$ assume the form $Y_1(d^p) = \frac{Y(d^p)}{d^p}$, then one has the following dissipative constitutive relations[1]

$$\mathbf{T}^p = Y_1(d^p)\dot{\mathbf{H}}^p, \quad \vec{\chi}^{\,\mathrm{dis}} = q^2 Y_1(d^p) \operatorname{Div} \dot{\mathbf{H}}^{pT}. \tag{6.10}$$

## 7. Quadratic free energy

In view of Gurtin [8], the quadratic, isotropic free energy $\psi = \hat{\psi}(\mathbf{E}^e, \mathbf{G})$ has the form

$$\hat{\psi}(\mathbf{E}^e, \mathbf{G}) = \mu|\mathbf{E}^e_o|^2 + \tfrac{1}{2}\kappa|\operatorname{tr}\mathbf{E}^e|^2 + \tfrac{1}{2}\mu Q^2|\mathbf{G}|^2, \tag{7.1}$$

where $Q$ is the energetic length scale. The energetic and dissipative length scales $Q$ and $q$ are presumably small lying in the micron scale but essentially enter the constitutive theory to ensure dimension consistency. Using Eq. $(6.5)_{\mathrm{a}}$ gives

$$\mathbf{S}^{\mathrm{en}} = \mu Q^2 \mathbf{G}.$$

[1] In the absence of plastic rotation, $Y_1(d^p)$ following Gurtin and Anand [9] can be chosen as $Y_1(d^p) = (\frac{d^p}{d_o})^m \frac{S}{d^p}$, where $d_o$ is the initial flow rate and $S$ is the flow resistance. By extension, this can be chosen to account for the plastic spin.

The energetic microstress vector $\vec{\chi}^{\text{en}}$ in terms of the Burgers tensor is obtained by[2]

$$\chi_r^{\text{en}} = \frac{\epsilon_{rij}}{4}\mu Q^2(G_{ij} - G_{ji}) = \frac{\mu Q^2}{4}(\epsilon_{ijr}\epsilon_{ipq}H^p_{jq,p} - \epsilon_{jri}\epsilon_{jpq}H^p_{iq,p})$$
$$= \frac{\mu Q^2}{4}(H^p_{pr,p} - H^p_{qq,r} - H^p_{qq,r} + H^p_{pr,p}) = \frac{\mu Q^2}{2}H^p_{pr,p}.$$

In invariant form, we have

$$\vec{\chi}^{\text{en}} = \frac{\mu Q^2}{2}\operatorname{Div}\mathbf{H}^{pT}. \tag{7.2}$$

Using Eq. (3.6)

$$|\mathbf{G}|^2 = \tfrac{1}{2}|\operatorname{Div}\mathbf{H}^{pT}|^2.$$

The alternative form of Eq. (7.1) reads

$$\hat{\psi}(\mathbf{E}^e, \mathbf{G}) = \check{\psi}(\mathbf{E}^e, \operatorname{Div}\mathbf{H}^{pT}) = \hat{\psi}^e(\mathbf{E}^e) + \hat{\psi}^{p*}(\operatorname{Div}\mathbf{H}^{pT}),$$

where

$$\hat{\psi}^e(\mathbf{E}^e) = \mu|\mathbf{E}^e_o|^2 + \frac{\kappa}{2}|\operatorname{tr}\mathbf{E}^e|^2,$$

and

$$\hat{\psi}^{p*}(\operatorname{Div}\mathbf{H}^{pT}) = \frac{\mu Q^2}{4}|\operatorname{Div}\mathbf{H}^{pT}|^2.$$

The combination of $\hat{\psi}^{p*}(\operatorname{Div}\mathbf{H}^{pT}) = \frac{\mu Q^2}{4}|\operatorname{Div}\mathbf{H}^{pT}|^2$ and Eq. (7.2) implies that

$$\vec{\chi}^{\text{en}} = \frac{\partial\hat{\psi}^{p*}}{\partial\operatorname{Div}\mathbf{H}^{pT}}.$$

## 8. Flow rule

The introduction of Eq. (6.7) into microforce balance Eq. (5.6) gives

$$\mathbf{T}_o + (\nabla\vec{\chi}^{\text{en}})^T_o = \mathbf{T}^p - (\nabla\vec{\chi}^{\text{dis}})^T_o, \tag{8.1}$$

and the gradient of $\vec{\chi}^{\text{en}}$ in Eq. (7.2) yields

$$\nabla\vec{\chi}^{\text{en}} = \frac{\mu Q^2}{2}\nabla\operatorname{Div}\mathbf{H}^{pT}. \tag{8.2}$$

Substituting Eqs. (6.10) and (8.2) into Eq. (8.1) gives the flow rule

$$\mathbf{T}_o + \frac{\mu Q^2}{2}(\nabla\operatorname{Div}\mathbf{H}^{pT})^T_o = Y_1(d^p)\dot{\mathbf{H}}^p - q^2(\nabla(Y_1(d^p)\operatorname{Div}\dot{\mathbf{H}}^{pT}))^T_o. \tag{8.3}$$

The symmetric deviatoric components of Eq. (8.3) give[3]

$$\mathbf{T}_o + \frac{\mu Q^2}{2}\operatorname{sym}_o(\nabla\operatorname{Div}\mathbf{H}^{pT}) = Y_1(d^p)\dot{\mathbf{E}}^p - q^2\operatorname{sym}_o\nabla(Y_1(d^p)\operatorname{Div}\dot{\mathbf{H}}^{pT}).$$

while the skew component of Eq. (8.3) is

$$-\frac{\mu Q^2}{2}\operatorname{skw}(\nabla\operatorname{Div}\mathbf{H}^{pT}) = Y_1\dot{\mathbf{W}}^p + q^2\operatorname{skw}\nabla(Y_1(d^p)\operatorname{Div}\mathbf{H}^{pT}).$$

---

[2] Recall that $\mathbf{S}^{\text{en}} = (\vec{\chi}^{\text{en}}\times)^T = -\vec{\chi}^{\text{en}}\times$; so that $\chi^{\text{en}}_r = \frac{1}{2}\epsilon_{rij}S^{\text{en}}_{ij}$.

[3] where we the denote $\operatorname{sym}_o\mathbf{A}$ as the symmetric-deviatoric part of a second order tensor $\mathbf{A}$ and defined as $\operatorname{sym}_o\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) - \frac{1}{3}\operatorname{tr}(\mathbf{A})\mathbf{I}$.

REMARK 8.1. In special problems where plastic rotation $\mathbf{W}^p = \mathbf{0}$, the symmetric component of the flow rule reduces to

$$\mathbf{T}_o + \frac{\mu Q^2}{2}\,\mathrm{sym}_o(\nabla\,\mathrm{Div}\,\mathbf{E}^p) = Y_1(d^p)\dot{\mathbf{E}}^p - q^2\,\mathrm{sym}_o(\nabla(Y_1(d^p)\,\mathrm{Div}\,\dot{\mathbf{E}}^p))$$

and the corresponding skew component takes the form

$$-\frac{\mu Q^2}{2}\,\mathrm{skw}(\nabla\,\mathrm{Div}\,\mathbf{E}^p) = q^2\,\mathrm{skw}(\nabla(Y_1(d^p)\,\mathrm{Div}\,\dot{\mathbf{E}}^p)). \tag{8.4}$$

Furthermore, when $\nabla\,\mathrm{Div}\,\mathbf{E}^p$ is symmetric and $q \neq 0$, Eq. (8.4) vanishes.

## 9. Microscopic simple boundary conditions and variational formulation

In view of Eq. (5.7), the null-power expenditure on the boundary $\partial B$ of $B$ gives

$$(\vec{\chi}\otimes\mathbf{n})_o^T) : \dot{\mathbf{H}}^p = 0.$$

Thus, we assume simple boundary conditions

$$\dot{\mathbf{H}}^p = \mathbf{0} \quad \text{on} \quad \Gamma_{\text{hard}} \quad \text{and} \quad (\vec{\chi}\otimes\mathbf{n})_o^T = \mathbf{0} \quad \text{on} \quad \Gamma_{\text{free}},$$

where $\Gamma_{\text{hard}}$ and $\Gamma_{\text{free}}$ are complementary subsurfaces called microscopically hard and free boundaries respectively.

To obtain the variational formulation of the flow rule, we assume null power expenditure on the boundary $\partial B$ of the body $B$ and that a portion of the boundary is microscopically hard while the remaining is microscopically free so that

$$\dot{\mathbf{H}}^p = \mathbf{0} \quad \text{on} \quad \Gamma_{\text{hard}}. \tag{9.1}$$

Let test field $\mathbf{F}$ satisfy Eq. (9.1). Invoking the constraint of null expenditure of power, the weak form of the flow rule reads

$$\int_B [(\mathbf{T}^p - \mathbf{T}_o) : \mathbf{F} + \vec{\chi}\cdot\mathrm{Div}\,\mathbf{F}^T]dV = 0. \tag{9.2}$$

Using the divergence theorem, Eq. (9.2) becomes

$$\int_B (\mathbf{T}^p - \mathbf{T}_o - (\nabla\vec{\chi})^T) : \mathbf{F}dV + \int_{\Gamma_{\text{free}}} [(\vec{\chi}\otimes\mathbf{n})_o^T] : \mathbf{F}dA = 0.$$

## 10. Plastic free-energy balance

The time derivative of the defect energy $\hat{\psi}^p(\mathrm{Div}\,\mathbf{H}^{pT})$ is

$$\dot{\hat{\psi}}^p(\mathrm{Div}\,\mathbf{H}^{pT}) = \vec{\chi}^{\,\text{en}}\cdot\mathrm{Div}\,\dot{\mathbf{H}}^{pT}. \tag{10.1}$$

Integrating Eq. (10.1) over the body gives

$$\frac{d}{dt}\int_B \hat{\psi}^p dV = \int_B [\vec{\chi}^{\,\text{en}}\cdot\mathrm{Div}\,\dot{\mathbf{H}}^{pT}]dV.$$

In view of the null power expenditure on the boundary $\partial B$ of the body $B$, and the expression for internal power expenditure, we have

$$\int_B [\vec{\chi}^{\,\text{en}}\cdot\mathrm{Div}\,\dot{\mathbf{H}}^{pT}]dV = \int_B (\mathbf{T}_o - \mathbf{T}^p) : \dot{\mathbf{H}}^p dV - \int_B [\vec{\chi}^{\,\text{dis}}\cdot\mathrm{Div}\,\dot{\mathbf{H}}^{pT}]dV.$$

Thus, the plastic free-energy balance for the problem under consideration takes the form

$$\frac{d}{dt}\int_B \hat{\psi}^p dV = \int_B \mathbf{T}_o : \dot{\mathbf{H}}^p dV - \int_B [\mathbf{T}^p : \dot{\mathbf{H}}^p + \vec{\chi}^{\,\mathrm{dis}} \cdot \operatorname{Div} \dot{\mathbf{H}}^{pT}] dV. \tag{10.2}$$

Equation (10.2) confirms the well-known results that the rate of increase in the defect energy cannot exceed the plastic working represented by $\int_B \mathbf{T}_o : \dot{\mathbf{H}}^p dV$ (that is, $\frac{d}{dt}\int_B \hat{\psi}^p dV \leqslant \int_B \mathbf{T}_o : \dot{\mathbf{H}}^p dV$).

## 11. Strengthening and weakening of the material

Here, we follow the style adopted by Gurtin and Anand [9] in determining the effect of the flow stress $\mathbf{T}_o$ on the strengthening and weakening of the material. We assume that:

(1) the material is rate independent and

$$Y_1(d^p) = \frac{Y_o}{d^p},$$

where $Y_o$ is the constant coarse grain yield strength;

(2) $Q = 0$, so that $\hat{\psi}^p(\mathbf{G}) = 0$ and $\vec{\chi}^{\,\mathrm{en}} \equiv \mathbf{0}$;

(3) at no time is the plastic distortion rate homogeneous; and

(4) boundary conditions are microscopically simple.

From plastic free-energy balance Eq. (10.2), we have

$$\int_B \mathbf{T}_o : \dot{\mathbf{H}}^p dV = Y_o \int_B \sqrt{|\dot{\mathbf{H}}^p|^2 + q^2 |\operatorname{Div} \dot{\mathbf{H}}^{pT}|^2}\, dV. \tag{11.1}$$

For a fixed time, let $\max|\mathbf{T}_o|$ be the maximum value of $|\mathbf{T}_o|$ over $B$, then by the Cauchy–Schwarz inequality we have

$$\left|\int_B \mathbf{T}_o : \dot{\mathbf{H}}^p dV\right| \leqslant \int_B |\mathbf{T}_o||\dot{\mathbf{H}}^p| dV \leqslant \max|\mathbf{T}_o| \int_B |\dot{\mathbf{H}}^p| dV.$$

Using Eqs. (6.9) and (11.1)

$$\begin{aligned} Y_o \int_B |\dot{\mathbf{H}}^p| dV &< Y_o \int_B \sqrt{|\dot{\mathbf{H}}^p|^2 + q^2 |\operatorname{Div} \dot{\mathbf{H}}^{pT}|^2}\, dV \\ &\leqslant \max|\mathbf{T}_o| \int_B |\dot{\mathbf{H}}^p| dV. \end{aligned}$$

Hence we have the standard inequality

$$\max|\mathbf{T}_o| > Y_o. \tag{11.2}$$

Equation (11.2) implies that at each time, there exists some nontrivial region of the body which is strengthened by the plastic flow.

On the other hand, integrating both sides of the microforce balance Eq. (5.6) over $B$ we have

$$\int_B \mathbf{T}_o dV = \int_B \mathbf{T}^p dV - \int_B (\nabla \vec{\chi})_o^T dV.$$

Using a microscopically simple boundary condition gives

$$\int_B \mathbf{T}_o dV = \int_B \mathbf{T}^p dV - \int_{\Gamma_{\text{hard}}} (\vec{\chi} \otimes \mathbf{n})_o^T dA. \tag{11.3}$$

In view of assumption 2, we have

$$\mathbf{T}^p = Y_o \frac{\dot{\mathbf{H}}^p}{d^p}; \quad \vec{\chi} = q^2 Y_o \frac{\operatorname{Div} \dot{\mathbf{H}}^{pT}}{d^p}. \tag{11.4}$$

Using Eq. (11.4)$_\text{a}$ we have

$$\left| \int_B \mathbf{T}^p dV \right| = Y_o \left| \int_B \frac{\dot{\mathbf{H}}^p}{d^p} dV \right| < Y_o \left| \int_B \frac{\dot{\mathbf{H}}^p}{|\dot{\mathbf{H}}^p|} dV \right| \leqslant Y_o \text{Vol}(B),$$

where $\text{Vol}(B)$ is the volume of the body $B$.

Also, (noting that $\dot{\mathbf{H}}^p = \mathbf{0}$ on $\Gamma_{\text{hard}}$) using Eq. (11.4) we have

$$\begin{aligned} \left| \int_{\Gamma_{\text{hard}}} (\vec{\chi} \otimes \mathbf{n})_o^T dA \right| &= Y_o \left| \int_{\Gamma_{\text{hard}}} \frac{q^2 (\operatorname{Div} \dot{\mathbf{H}}^{pT} \otimes \mathbf{n})_o^T}{\sqrt{|\dot{\mathbf{H}}^p|^2 + q^2 |\operatorname{Div} \dot{\mathbf{H}}^{pT}|^2}} dA \right| \\ &< Y_o q \int_{\Gamma_{\text{hard}}} \frac{|(\mathbf{n} \otimes \operatorname{Div} \dot{\mathbf{H}}^{pT})_o|}{|\operatorname{Div} \dot{\mathbf{H}}^{pT}|} dA \leqslant Y_o q \operatorname{Area}(\Gamma_{\text{hard}}), \end{aligned}$$

where $\operatorname{Area}(\Gamma_{\text{hard}})$ is the area of the microscopically hard boundary $\Gamma_{\text{hard}}$. In obtaining this inequality, we have made use of the fact that given any second order tensor $\mathbf{A}$ and vectors $\mathbf{a}$ and $\mathbf{b}$ we have

$$\begin{aligned} &|\mathbf{A}_o| \leqslant |\mathbf{A}| \quad \text{and} \\ &|\mathbf{a} \otimes \mathbf{b}| = |\mathbf{a}||\mathbf{b}|. \end{aligned}$$

Thus we have

$$\left| \int_{\Gamma_{\text{hard}}} (\vec{\chi} \otimes \mathbf{n})_o^T dA \right| < Y_o q \operatorname{Area}(\Gamma_{\text{hard}}).$$

From Eq. (11.3), it is clear that

$$\begin{aligned} \left| \int_B \mathbf{T}_o dV \right| &\leqslant \left| \int_B \mathbf{T}^p dV \right| + \left| \int_{\Gamma_{\text{hard}}} (\vec{\chi} \otimes \mathbf{n})_o^T dA \right| \\ &< Y_o \operatorname{Vol}(B) + Y_o q \operatorname{Area}(\Gamma_{\text{hard}}) = Y_o (1 + \epsilon_{\text{hard}}) \operatorname{Vol}(B), \end{aligned}$$

where

$$\epsilon_{\text{hard}} = \frac{q \operatorname{Area}(\Gamma_{\text{hard}})}{\operatorname{Vol}(B)}.$$

Clearly

$$\left| \frac{1}{\operatorname{Vol}(B)} \int_B \mathbf{T}_o dV \right| < Y_o (1 + \epsilon_{\text{hard}}).$$

Using the microscopically free boundary condition, with $\epsilon_{\text{hard}} = 0$, we have

$$\left| \frac{1}{\operatorname{Vol}(B)} \int_B \mathbf{T}_o dV \right| < Y_o, \tag{11.5}$$

where $|\frac{1}{\operatorname{Vol}(B)} \int_B \mathbf{T}_o dV|$ is known as the average flow stress.

Equation (11.5) implies that if the boundary is microscopically free, then at each time, there exists a nontrivial region of the body that is weakened by the plastic flow.

## 12. Concluding remarks

This study shows that the defect energy via the strictly skew-symmetric Burgers tensor is converted to defect energy via the divergence of plastic distortion in solid bodies undergoing inhomogeneous plastic flow. It is obtained that the dependency of the microscopic stress vector on the divergence of transpose of plastic distortion rate leads to weakening and strengthening effects in the flow rule. Furthermore, it is shown that, when the term $\nabla \operatorname{Div} \mathbf{E}^p$ is symmetric and the dissipative length scale $q \neq 0$, the plastic strain rate is governed by the symmetric deviatoric flow rule. The results in this work find numerous applications in the analysis of size effects in plastically deformed solids for cases where the Burgers tensor is strictly skew-symmetric. For future study and investigation, the present theory can be extended to include surface/interface energy terms [21, 22].

## References


1. J. W. Hutchinson, *Plasticity at the micron scale*, Int. J. Solids Struct. **37** (2000), 225–238.
2. A. Panteghini, L. Bardella, *On the finite element implementation of higher-order gradient plasticity, with focus on theories based on plastic distortion incompatibility*, Comput. Methods Appl. Mech. Eng. **310** (2016), 840–865.
3. E. C. Aifantis, *On the microstructural origin of certain inelastic models*, Trans. ASME J. Eng. Mater. Tech. **106** (1984), 326–330.
4. E. C. Aifantis, *The physics of plastic deformation*, Int. J. Plast. **3** (1987), 211–247.
5. H. B. Mualhaus, E. C. Aifantis, *A variational principle for gradient plasticity*, Int. J. Solids Struct. **28** (1991), 845–857.
6. P. Gudmundson, *A unified treatment of strain gradient plasticity*, J. Mech. Phys. Solids **52** (2004), 1379–1406.
7. M. E. Gurtin, *On framework for small-deformation viscoplasticity: free energy, microforces, strain gradients*, Int. J. Plast. **9** (2003), 47–90.
8. M. E. Gurtin, *A gradient theory of small-deformation isotropic plasticity that accounts for Burgers vector and dissipation due to plastic spin*, J. Mech. Phys. Solids **52** (2004), 2545–2568.
9. M. E. Gurtin, L. Anand, *A theory of strain-gradient plasticity for isotropic, plastically irrotational materials. Part I: Small deformations*, J. Mech. Phys. Solids **53** (2005), 1624–1649.
10. E. C. Aifantis, *Gradient plasticity: Handbook of Materials Behaviour Models*, J. Lemaitre ed., Academic Press, 281-297, 2001.
11. E. C. Aifantis, *On the gradient approach-relation to Eringen's nonlocal theory*, Int. J. Eng. Sci. **49** (2011), 1367–1377.
12. G. Mokios, E. C. Aifantis, *Gradient effects in micro-/nano-identation*, Materials Science and Technology **28** (2012), 1072–1078.
13. E. C. Aifantis, *Material mechanics: Perspectives and prospects*, Acta Mech. **225** (2014), 999–1012.
14. E. C. Aifantis *Internal length gradient (ILG) material mechanics across scales and disciplines*, Adv. Appl. Mech. **49** (2016), 1–110.
15. B. D. Reddy, F. Ebobisse, A. McBride, *Well-posedness of a model of strain gradient plasticity for plastically irrotational materials*, Int. J. Plast. **24** (2008), 55–73.
16. A. Borokinni, A. Akinola, O. Layeni, *A theory of strain-gradient plasticity with effect of internal microforce*, Theor. Appl. Mech. (Belgrade) **44** (2017), 1–13.

17. M. E. Gurtin, E. Fried, L. Anand, *The Mechanics and Thermodynamics of Continua*, Cambridge University Press, Cambridge, 2010.
18. L. H. Poh, R. H. J. Peerlings, *The plastic rotation effect in an isotropic gradient plasticity model for applications at the meso scale*, Int. J. Solids Struct. **78–79** (2016), 57–69.
19. A. S. Borokinni, *A gradient theory based on the Aifantis theory using the Gurtin-Anand strain gradient plasticity approach*, Journal of the Mechanical Behavior of Materials **27** (2018), 20180012.
20. W. Han, B. D. Reddy, *Plasticity: Mathematical Theory and Numerical Analysis*, Springer-Velag, New York, 2013.
21. K. E. Aifantis, J. R. Willis, *The role of interfaces in enhancing the yield strength of composites and polycrystals*, J. Mech. Phys. Solids **53** (2005), 1047–1070.
22. K. E. Aifantis, W. A. Soer, J. De Hosson, J. R. Willis, *Interfaces within strain gradient plasticity: Theory and Experiments*, Acta Materialia **54** (2006), 5077–5085.

## ЕФЕКТИ ВЕЛИЧИНЕ ПОВЕЗАНИ СА КОСОСИМЕТРИЧНИМ БУРГЕРСОВИМ ТЕНЗОРОМ

Резиме. У овом раду истражују се појаве ефеката величине повезаних са дивергенцијом транспонованог пластичних изобличења у пластично деформисаним изотропним материјалима. Принцип виртуелне снаге, равнотеже енергије и други закон термодинамике се користе за формулисање једначина равнотеже микро-сила и термодинамичких конзистентних конститутивних односа за дисипативне микроскопске напоне повезане са пластичним изобличењем и кососиметричним делом Бургерсовог тензора. Добијено је да се енергија дефекта кроз строго коси-Бургерсов тензор претвара у енергију дефекта кроз дивергенцију пластичног изобличења. Присуство скала дужине материјала у добијеним једначинама тока указује да је могуће опазити ефекте величине који су повезани са кососиметричним делом Бургерсовог тензора током нехомогеног пластичног струјања чврстог материјала. Коначно и између осталог, показало се да зависност микроскопског вектора напрезања од дивергенције брзине пластичног изобличења доводи до слабљења и јачања ефекта у једначинама тока.

Department of Mathematics
Obafemi Awolowo University
Ile-Ife
Nigeria
borokinniadebowale@gmail.com; aborokini@unilag.edu.ng (corresponding author)

ofadodun@oauife.edu.ng

olawanle.layeni@gmail.com

aakinola@oauife.edu.ng

toyeolokun@oauife.edu.ng

(Received 25.11.2019.)
(Available online 06.05.2020.)