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EVOLUTION EQUATIONS OF TRANSLATIONAL-ROTATIONAL MOTION OF A NON-STATIONARY TRIAXIAL BODY IN A CENTRAL GRAVITATIONAL FIELD

Mukhtar Minglibayev, Alexander Prokopenya, and Oralkhan Baisbayeva

Abstract. The translational-rotational motion of a triaxial body with constant dynamic shape and variable size and mass in a non-stationary Newtonian central gravitational field is investigated. Differential equations of motion of the triaxial body in the relative coordinate system with the origin at the center of a non-stationary spherical body are obtained. The axes of the Cartesian coordinate system fixed to the non-stationary triaxial body are coincident with its principal axes and their relative orientation is assumed to remain unchanged in the course of evolution. An analytical expression for the force function of the Newtonian interaction of the triaxial body of variable mass and size with a spherical body of variable size and mass is obtained. Differential equations of translational-rotational motion of the non-stationary triaxial body are derived in Jacobi osculating variables and are studied with the perturbation theory methods. The perturbing function is expanded in power series in terms of the Delaunay–Andoyer elements up to the second harmonic element inclusive. The evolution equations of the translational-rotational motion of the non-stationary triaxial body are obtained in the osculating elements of Delaunay–Andoyer.

1. Introduction

In classical Celestial Mechanics, real celestial bodies are usually modelled by points of constant mass (or by spherically symmetric bodies). In the case when such description does not adequately describe a real physical problem, celestial bodies are considered as rigid bodies of constant size, mass and internal structure [1–4]. The model becomes much more complicated because three additional degrees of freedom are required to describe the rotational motion of the rigid body about its center of mass. Besides, three translational degrees of freedom are used to describe the orbital motion of the body. The resulting system of differential equations of motion

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is not analytically solvable and application of special methods for its investigation is required. The modern theory of translational-rotational motion of rigid celestial bodies is one of the basic elements of classical and celestial mechanics, as well as cosmodynamics [5–9].

Observational astronomy indicates that real celestial bodies are not point and rigid. They are non-stationary, and their masses, sizes, shapes and structures change in the course of their evolution [10–13]. These processes occur especially intensively in double and multiple systems [14–16]. In this regard, it becomes actual to develop mathematical models of the motion of celestial bodies with variable mass, size, shape and structure. The aim of this work is to investigate the translational-rotational motion of two non-stationary bodies, one of which is spherical and the other one triaxial. It is assumed that the initial dynamic shapes of the bodies are preserved, whereas their masses and sizes change in time [13].

The paper is organized as follows. In Section 2 we formulate the physical problem and obtain the equations of motion in different coordinate systems. Section 3 is devoted to integrating the unperturbed equations of motion by the Hamilton–Jacobi method. Then the equations of motion are rewritten in the osculating analogues of the Jacobi elements. In Section 4 we obtain the equations of translational-rotational motion of the non-stationary triaxial body in a central non-stationary gravitational field of a spherical body in the osculating elements of Delaunay–Andoyer. Then we calculate the expansion of the perturbing functions in terms of the Delaunay–Andoyer elements in Section 5. Finally, we obtain the evolution equations in Section 6 and summarise the obtained results in the Conclusions.

2. Formulation of the problem and equations of motion

2.1. Model description. We consider a special case of translational-rotational motion of two non-stationary bodies attracting each other in accordance with Newton's law of gravitation [13]. We assume that the first body has a spherical mass distribution; its mass and radius change with time but its dynamic shape is preserved. In other words, the body retains its spherical symmetry all the time.

The second (triaxial) body is assumed to have an arbitrary dynamic structure and its principal moments of inertia are different. Besides, the mass and size of the second body are variable, but its initial dynamic shape is preserved. Such a case takes place, for example, if the body has three mutually perpendicular planes of symmetry. Then the initial orientation of the principal axes of inertia and location of the center of mass in the triaxial body remain unchanged in the course of its evolution.

In the model under consideration, we exploit the following assumptions:

1. The first body is a sphere with a spherical mass distribution (see Fig. 1), with variable mass $m_1 = m_1(t)$ and variable radius $l_1^* = l_1^*(t)$. Its principal moments of inertia are the same functions of time $A_1(t) = B_1(t) = C_1(t)$.
2. The second body is a triaxial body of variable mass $m_2 = m_2(t) = m_2(t_0)m(t)$ which has an arbitrary dynamic structure and a characteristic linear size $l_2^* = l_2^*(t) = l(t_0)\chi(t)$, where t_0 is an initial instant of time. Its principal central moments

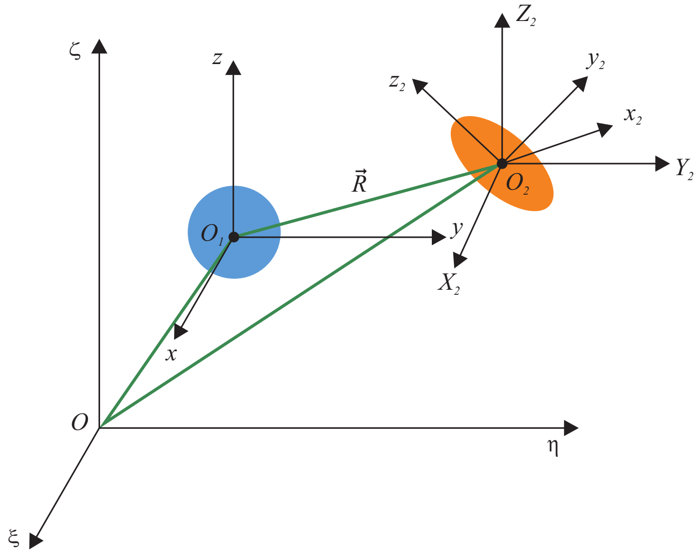

FIGURE 1. The sphere and triaxial body

of inertia are different and vary with time:

$$A_2 = A_2(t), \quad B_2 = B_2(t), \quad C_2 = C_2(t), \quad A_2 \geqslant B_2 \geqslant C_2, \quad A_2 \neq C_2.$$

3. The second body retains its dynamic shape. Its principal moments of inertia change at the same rate

$$\frac{A_2(t)}{A_2(t_0)} = \frac{B_2(t)}{B_2(t_0)} = \frac{C_2(t)}{C_2(t_0)} = m\chi^2,$$

where $m = m(t)$, $\chi = \chi(t)$ are given functions of time. This means that the original shape of the second body remains the same, whereas its size and mass change with time.

4. The axes of the Cartesian coordinate system fixed to the second body coincide with its principal axes of inertia and their mutual orientation is preserved all the time.

5. The mass and characteristic size of the bodies change at different specific rates

$$\frac{\dot{m}_1(t)}{m_1(t)} \neq \frac{\dot{m}_2(t)}{m_2(t)}, \quad \frac{\dot{l}_1^*(t)}{l_1^*(t)} \neq \frac{\dot{l}_2^*(t)}{l_2^*(t)}.$$

Note that the dot over a symbol denotes the total derivative of the corresponding function with respect to time.

6. The relative velocities of the particles leaving the bodies or settled at the bodies are equal to zero [11]. Therefore, total reactive forces and their torques are equal to zero. Besides, additional torques arising due to the fact that the bodies have a variable structure are equal to zero:

$$\vec{F}_{tot.react.} = 0, \quad \vec{M}^{(add)} = 0.$$

7. We use an expansion of the force function of the Newtonian interaction of the bodies accurate up to the second zonal harmonics (see [1,3]):

$$U \approx U_1 + U_2.$$

2.2. Equations of motion in the absolute coordinate system. Taking into account assumptions 1–7, the equations of translational-rotational motion of the two bodies in the absolute coordinate system may be written in the form [1,8,13]

$$m_1\ddot{\xi}_1 = \frac{\partial U}{\partial \xi_1}, \quad m_1\ddot{\eta}_1 = \frac{\partial U}{\partial \eta_1}, \quad m_1\ddot{\zeta}_1 = \frac{\partial U}{\partial \zeta_1}, \tag{2.1}$$

$$\frac{d}{dt}(A_1p_1) = 0, \quad \frac{d}{dt}(B_1q_1) = 0, \quad \frac{d}{dt}(C_1r_1) = 0, \quad A_1 = B_1 = C_1, \tag{2.2}$$

$$\begin{aligned} &m_2\ddot{\xi}_2 = \frac{\partial U}{\partial \xi_2}, \quad m_2\ddot{\eta}_2 = \frac{\partial U}{\partial \eta_2}, \quad m_2\ddot{\zeta}_2 = \frac{\partial U}{\partial \zeta_2}, \\ &\frac{d}{dt}(A_2p_2) - (B_2 - C_2)q_2r_2 = \frac{\sin\varphi_2}{\sin\theta_2}\Big[\frac{\partial U}{\partial \psi_2} - \cos\theta_2\frac{\partial U}{\partial \varphi_2}\Big] + \cos\varphi_2\frac{\partial U}{\partial \theta_2}, \\ &\frac{d}{dt}(B_2q_2) - (C_2 - A_2)r_2p_2 = \frac{\cos\varphi_2}{\sin\theta_2}\Big[\frac{\partial U}{\partial \psi_2} - \cos\theta_2\frac{\partial U}{\partial \varphi_2}\Big] - \sin\varphi_2\frac{\partial U}{\partial \theta_2}, \\ &\frac{d}{dt}(C_2r_2) - (A_2 - B_2)p_2q_2 = \frac{\partial U}{\partial \varphi_2}. \end{aligned} \tag{2.3}$$

Here p_1, q_1, r_1 and p_2, q_2, r_2 are the projections of angular velocity of the bodies on the axes of coordinate systems fixed to them, which are defined by the kinematic Euler equations

$$\begin{aligned} &p_i = \dot{\psi}_i \sin\theta_i \sin\varphi_i + \dot{\theta}_i \cos\varphi_i, \quad q_i = \dot{\psi}_i \sin\theta_i \cos\varphi_i - \dot{\theta}_i \sin\varphi_i, \\ &r_i = \dot{\psi}_i \cos\theta_i + \dot{\varphi}_i, \qquad i = 1, 2, \end{aligned} \tag{2.4}$$

where $\psi_i, \varphi_i, \theta_i$ are the corresponding Euler angles (Fig. 2).

Equations (2.1)–(2.4) completely characterise the translational-rotational motion of the two non-stationary bodies in the absolute coordinate system in the considered formulation. The equations of rotational motion (2.2) of the spherical body are easily integrated. However, the force function U depends on the distance R between the centers of inertia of the bodies and the equations of translational motion (2.1) should therefore be integrated together with the corresponding equations (2.3) describing translational motion of the second body. To find the distance R as a function of time, it is expedient to use a relative coordinate system.

2.3. Equations of motion in the relative coordinate system. Let us consider a relative coordinate system O_1xyz with the origin at the center of the spherical body. Assume that the coordinate axes of the relative coordinate system are parallel to the corresponding axes of the absolute coordinate system $O\xi\eta\zeta$.

Denote the relative coordinates of the center of inertia of the triaxial body by

$$x = \xi_2 - \xi_1, \quad y = \eta_2 - \eta_1, \quad z = \zeta_2 - \zeta_1.$$

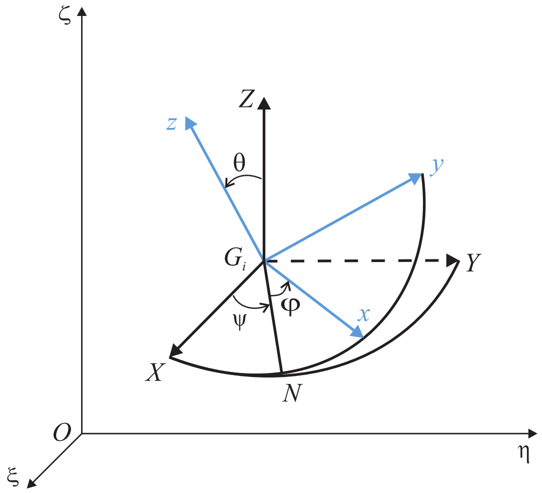

FIGURE 2. The Euler angles

Based on the equations of motion (2.1)–(2.4), we obtain the equations of translational-rotational motion of the triaxial non-stationary body in the relative coordinate system O_1xyz in the form

$$\mu(t)\ddot{x} = \frac{\partial U}{\partial x}, \quad \mu(t)\ddot{y} = \frac{\partial U}{\partial y} \quad \mu(t)\ddot{z} = \frac{\partial U}{\partial z} \tag{2.5}$$

$$\begin{aligned} &\frac{d}{dt}(A(t)p) - (B(t) - C(t))qr = \frac{\sin\varphi}{\sin\theta}\left[\frac{\partial U}{\partial \psi} - \cos\theta\frac{\partial U}{\partial \varphi}\right] + \cos\varphi\frac{\partial U}{\partial \theta} \\ &\frac{d}{dt}(B(t)q) - (C(t) - A(t))rp = \frac{\cos\varphi}{\sin\theta}\left[\frac{\partial U}{\partial \psi} - \cos\theta\frac{\partial U}{\partial \varphi}\right] - \sin\varphi\frac{\partial U}{\partial \theta} \\ &\frac{d}{dt}(C(t)r) - (A(t) - B(t))pq = \frac{\partial U}{\partial \varphi}. \end{aligned} \tag{2.6}$$

Here $\mu(t) = m_1(t)m_2(t)/(m_1(t) + m_2(t))$ is a reduced mass,

$$U \approx U_1 + U_2, \quad U_1 = \frac{fm_1m_2}{R}, \quad R^2 = x^2 + y^2 + z^2,$$

$$U_2 = fm_1\frac{A + B + C - 3I}{2R^3}, \quad A = A_2, \quad B = B_2, \quad C = C_2, \tag{2.7}$$

$$I = A\alpha^2 + B\beta^2 + C\gamma^2,$$

f is the gravitational constant, I is the moment of inertia of the triaxial body relative to the axis given by the vector $\vec{R} = \overrightarrow{O_1O_2}$ connecting the centers of mass of the two bodies, α, β, γ are cosines of the angles formed by vector $\vec{R}$ and the axes of the coordinate system $O_2x_2y_2z_2$ fixed to the triaxial body (see Fig. 1),

$$p = \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi, \quad q = \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi, \quad r = \dot{\psi}\cos\theta + \dot{\varphi}, \tag{2.8}$$

$\varphi = \varphi_2$, $\psi = \psi_2$, $\theta = \theta_2$ are the corresponding Euler angles (see [1, 4, 6, 19]). The obtained equations (2.5)–(2.8) completely characterise the translational-rotational motion of the non-stationary triaxial body in the gravitational field of the spherical body in the relative coordinate system.

3. Equations of motion in the osculating analogues of the elements of Jacobi

Since equations of motion (2.5)–(2.6) are not integrable, we use the perturbation theory to investigate the dynamics of the system. This means that equations (2.5)–(2.6) are reduced to two perturbed problems, each of which is integrable in the case when there are no perturbations.

3.1. Integration of unperturbed translational-rotational motion by the Hamilton-Jacobi method. Let us rewrite equations (2.5) in the form

$$\begin{aligned}\ddot{x} &= -f\frac{m_1+m_2}{R^3}x + bx + \frac{\partial V}{\partial x}, \quad \ddot{y} = -f\frac{m_1+m_2}{R^3}y + by + \frac{\partial V}{\partial y},\\ \ddot{z} &= -f\frac{m_1+m_2}{R^3}z + bz + \frac{\partial V}{\partial z},\end{aligned} \tag{3.1}$$

where

$$V = \frac{m_1+m_2}{m_1 m_2}U_2 - \frac{1}{2}bR^2, \quad b = (m_1+m_2)\frac{d^2}{dt^2}\Big(\frac{1}{m_1+m_2}\Big).$$

Such a definition of the force function V is expedient because it enables us to define the unperturbed translational motion by setting $V = 0$. Indeed, for $V = 0$ equations (3.1) are integrated by the Hamilton–Jacobi method (see [13]). To demonstrate this, let us change to new variables ρ, δ, λ given by

$$x = \nu\rho\cos\delta\cos\lambda, \quad y = \nu\rho\cos\delta\sin\lambda, \quad z = \nu\rho\sin\delta,$$

where

$$\nu = \frac{m_1(t_0)+m_2(t_0)}{m_1(t)+m_2(t)} = \nu(t).$$

Defining the momenta that are canonically conjugate to the coordinates ρ, δ, λ as

$$P_\rho = \frac{\partial K^{trans}}{\partial\dot{\rho}}, \quad P_\delta = \frac{\partial K^{trans}}{\partial\dot{\delta}}, \quad P_\lambda = \frac{\partial K^{trans}}{\partial\dot{\lambda}}, \quad K^{trans} = \frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2),$$

we write out the equations (3.1) in the canonical form (see [13])

$$\begin{aligned}\dot{\rho} &= \frac{\partial H^{trans}}{\partial P_\rho}, & \dot{\delta} &= \frac{\partial H^{trans}}{\partial P_\delta}, & \dot{\lambda} &= \frac{\partial H^{trans}}{\partial P_\lambda},\\ \dot{P}_\rho &= -\frac{\partial H^{trans}}{\partial \rho}, & \dot{P}_\delta &= -\frac{\partial H^{trans}}{\partial \delta}, & \dot{P}_\lambda &= -\frac{\partial H^{trans}}{\partial \lambda},\end{aligned} \tag{3.2}$$

where the Hamiltonian is given by

$$H^{trans} = H_0^{trans} + H_1^{trans},$$

$$H_0^{trans} = \frac{1}{2\nu^2}\Big[(P_\rho - \nu\dot{\nu}\rho)^2 + \frac{P_\delta^2}{\rho^2} + \frac{P_\lambda^2}{\rho^2\cos^2\delta}\Big] - \Big[f\frac{m_1+m_2}{\nu\rho} + \frac{1}{2}(\dot{\nu}^2+\nu\ddot{\nu})\rho^2\Big],$$

$$H_1^{trans} = -\frac{m_1 + m_2}{m_1 m_2} U_2 + \frac{1}{2}\nu\ddot{\nu}\rho^2. \tag{3.3}$$

Defining the momenta P_ψ, P_φ, P_θ that are canonically conjugate to the coordinates ψ, φ, θ as

$$P_\psi = \frac{\partial K^{rot}}{\partial \dot{\psi}}, \quad P_\varphi = \frac{\partial K^{rot}}{\partial \dot{\varphi}}, \quad P_\theta = \frac{\partial K^{rot}}{\partial \dot{\theta}}, \quad K^{rot} = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2),$$

we can also rewrite equations (2.6) in the canonical form (see [6, 8, 13, 18])

$$\begin{aligned} \dot{\psi} &= \frac{\partial H^{rot}}{\partial P_\psi}, & \dot{\varphi} &= \frac{\partial H^{rot}}{\partial P_\varphi}, & \dot{\theta} &= \frac{\partial H^{rot}}{\partial P_\theta}, \\ \dot{P}_\psi &= -\frac{\partial H^{rot}}{\partial \psi}, & \dot{P}_\varphi &= -\frac{\partial H^{rot}}{\partial \varphi}, & \dot{P}_\theta &= -\frac{\partial H^{rot}}{\partial \theta}, \end{aligned} \tag{3.4}$$

where

$$H^{rot} = H_0^{rot} + H_1^{rot},$$

$$H_0^{rot} = \frac{1}{2A}\Big[\frac{1}{\sin^2\theta}(P_\psi - P_\varphi\cos\theta)^2 + P_\theta^2\Big] + \frac{P_\varphi^2}{2C},$$

$$H_1^{rot} = \frac{1}{2}\Big(\frac{1}{B} - \frac{1}{A}\Big)\frac{[(P_\psi - P_\varphi\cos\theta)\cos\varphi - P_\theta\sin\varphi\sin\theta]^2}{\sin^2\theta} - \Big(U_2 - \frac{1}{2}\nu\ddot{\nu}\rho^2\Big). \tag{3.5}$$

Such a definition of the perturbing functions H_1^{trans} and H_1^{rot} enables us to introduce two independent integrable problems describing unperturbed translational and rotational motion by setting $H_1^{trans} = 0$ and $H_1^{rot} = 0$. Actually, the simplified system

$$\begin{aligned} \dot{\rho} &= \frac{\partial H_0^{trans}}{\partial P_\rho}, & \dot{\delta} &= \frac{\partial H_0^{trans}}{\partial P_\delta}, & \dot{\lambda} &= \frac{\partial H_0^{trans}}{\partial P_\lambda}, \\ \dot{P}_\rho &= -\frac{\partial H_0^{trans}}{\partial \rho}, & \dot{P}_\delta &= -\frac{\partial H_0^{trans}}{\partial \delta}, & \dot{P}_\lambda &= -\frac{\partial H_0^{trans}}{\partial \lambda}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \dot{\psi} &= \frac{\partial H_0^{rot}}{\partial P_\psi}, & \dot{\varphi} &= \frac{\partial H_0^{rot}}{\partial P_\varphi}, & \dot{\theta} &= \frac{\partial H_0^{rot}}{\partial P_\theta}, \\ \dot{P}_\psi &= -\frac{\partial H_0^{rot}}{\partial \psi}, & \dot{P}_\varphi &= -\frac{\partial H_0^{rot}}{\partial \varphi}, & \dot{P}_\theta &= -\frac{\partial H_0^{rot}}{\partial \theta}, \end{aligned} \tag{3.7}$$

is integrated by the Hamilton–Jacobi method (see [13]). Omitting the calculations, we write out a general integral of the system in the form

$$
\begin{gathered}
\int_{\rho_0}^{\rho} \frac{d\rho}{\sqrt{2\alpha_1 + \frac{2M_0}{\rho} - \frac{\alpha_2^2}{\rho^2}}} = \int_{t_0}^{t} \frac{dt}{\nu^2} + \beta_1, \\
\alpha_2 \int_0^{\delta} \frac{d\delta}{\sqrt{\alpha_2^2 - \frac{\alpha_3^2}{\cos^2\delta}}} - \alpha_2 \int_{\rho_0}^{\rho} \frac{d\rho}{\rho^2\sqrt{2\alpha_1 + \frac{2M_0}{\rho} - \frac{\alpha_2^2}{\rho^2}}} = \beta_2, \\
\lambda - \alpha_3 \int_0^{\delta} \frac{d\delta}{\cos^2\delta\sqrt{\alpha_2^2 - \frac{\alpha_3^2}{\cos^2\delta}}} = \beta_3, \quad M_0 = f(m_1(t_0) + m_2(t_0)), \\
P_\rho = \nu\dot{\nu}\rho + \sqrt{2\alpha_1 + \frac{2M_0}{\rho} - \frac{\alpha_2^2}{\rho^2}}, \quad P_\delta = \sqrt{\alpha_2^2 - \frac{\alpha_3^2}{\cos^2\delta}}, \quad P_\lambda = \alpha_3,
\end{gathered} \tag{3.8}
$$

$$
\begin{gathered}
A_0 \int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\Omega(\theta)}} = \int_{t_0}^{t} \frac{dt}{m\chi^2} + \beta_1', \quad A_0 = A(t_0), \quad C_0 = C(t_0), \\
\psi - \int_{\theta_0}^{\theta} \left\{ \frac{1}{\sqrt{\Omega(\theta)}} \Big(\frac{\alpha_3' - \alpha_2' \cos\theta}{\sin^2\theta} \Big) \right\} d\theta = \beta_3', \\
\varphi - \int_{\theta_0}^{\theta} \left\{ \frac{1}{\sqrt{\Omega(\theta)}} \Big(\frac{A_0}{C_0}\alpha_2' - \frac{(\alpha_3' - \alpha_2' \cos\theta)\cos\theta}{\sin^2\theta} \Big) \right\} d\theta = \beta_2', \\
P_\psi = \alpha_3', \quad P_\varphi = \alpha_2', \quad P_\theta = \sqrt{\Omega(\theta)},
\end{gathered} \tag{3.9}
$$

where

$$
\Omega(\theta) = 2A_0\alpha_1' - \frac{A_0}{C_0}\alpha_2'^2 - \frac{(\alpha_3' - \alpha_2'\cos\theta)^2}{\sin^2\theta}.
$$

The constants of integration

$$
\alpha_k, \quad \beta_k, \quad \alpha_k', \quad \beta_k', \qquad k = 1, 2, 3 \tag{3.10}
$$

in equations (3.8) and (3.9) are analogues of Jacobi elements in the corresponding stationary problem (see [1,6,18]). Equations (3.8) describe the unperturbed aperiodic motion on a quasi-canonical section (see [13]) while equations (3.9) describe the unperturbed rotational motion of the axisymmetrical non-stationary body about its center of mass.

3.2. Equations of translation-rotational motion in osculating analogues of the Jacobi elements. Simplified equations (3.6) and (3.7) describe the unperturbed translational-rotational motion. In case of $H_1^{trans} \neq 0$ and $H_1^{rot} \neq 0$, equations of translational-rotational motion (3.2) and (3.4) of the problem under consideration reduce to the following differential equations in the osculating analogues of Jacobi elements (3.10)

$$
\dot{\alpha}_k = -\frac{\partial H_1^{trans}}{\partial \beta_k}, \quad \dot{\beta}_k = \frac{\partial H_1^{trans}}{\partial \alpha_k}, \qquad k = 1, 2, 3, \tag{3.11}
$$

$$\dot{\alpha'}_k = -\frac{\partial H_1^{rot}}{\partial \beta'_k}, \quad \dot{\beta'}_k = \frac{\partial H_1^{rot}}{\partial \alpha'_k}, \qquad k = 1, 2, 3, \tag{3.12}$$

where H_1^{trans} and H_1^{rot} defined in (3.3) and (3.5), respectively, should be rewritten in the new canonical variables (3.10).

4. Equations of translational-rotational motion of the non-stationary triaxial body in the osculating Delaunay–Andoyer elements

4.1. Equations of translational-rotational motion in osculating analogues of the Delaunay–Andoyer elements. Equations (3.11) describe the perturbed motion of the center of inertia of the triaxial body in osculating analogues of the Jacobi elements. Accordingly, equations (3.12) describe the perturbed rotational motion of this body about its center of inertia in analogues of Jacobi elements, as well. Further, it is preferable (see [6,17,19]) to introduce the analogues of the Delaunay–Andoyer elements

$$L, \quad G, \quad H, \quad l, \quad g, \quad h, \tag{4.1}$$

$$L', \quad G', \quad H', \quad l', \quad g', \quad h', \tag{4.2}$$

according to the following relationships

$$-\alpha_1 = \frac{M_0^2}{2L^2}, \quad \alpha_2 = G, \quad \alpha_3 = H, \tag{4.3}$$

$$\beta_1 = \frac{L^3}{M_0^2} l - \Phi_1(t), \quad \beta_2 = g, \qquad \beta_3 = h, \tag{4.4}$$

$$L' = \alpha'_2, \quad G' = \left(2\alpha'_1 A_0 + \frac{C_0 - A_0}{C_0}\alpha_2'^2\right)^{1/2}, \quad H' = \alpha'_3,$$

$$g' = \frac{G'}{A_0}[\Phi_2(t) + \beta'_1], \quad h' = \beta'_3,$$

$$l' = \frac{A_0 - C_0}{A_0 C_0}\alpha_2[\Phi_2(t) + \beta'_1] + \beta'_2,$$

where

$$\Phi_1(t) = \int_{t_0}^{t} \frac{dt}{\nu^2(t)}, \quad \Phi_2(t) = \int_{t_0}^{t} \frac{dt}{m(t)\chi^2(t)}.$$

Equations (3.11) and (3.12) of motion in osculating elements (4.1) and (4.2) have the form

$$\begin{aligned} \dot{L} &= \frac{\partial F}{\partial l}, \quad \dot{G} = \frac{\partial F}{\partial g}, \quad \dot{H} = \frac{\partial F}{\partial h}, \\ \dot{l} &= -\frac{\partial F}{\partial L}, \quad \dot{g} = -\frac{\partial F}{\partial G}, \quad \dot{h} = -\frac{\partial F}{\partial H}, \end{aligned} \tag{4.5}$$

$$F = \frac{1}{\nu^2}\frac{M_0^2}{2L^2} - H_1^{trans}, \tag{4.6}$$

$$\dot{L}' = \frac{\partial F'}{\partial l'}, \qquad \dot{G}' = \frac{\partial F'}{\partial g'}, \qquad \dot{H}' = \frac{\partial F'}{\partial h'},$$
$$\dot{l}' = -\frac{\partial F'}{\partial L'}, \quad \dot{g}' = -\frac{\partial F'}{\partial G'}, \quad \dot{h}' = -\frac{\partial F'}{\partial H'}, \tag{4.7}$$

$$F' = \frac{1}{2}\Big(\frac{1}{m\chi^2}\Big[\frac{G'^2}{A_0} + \frac{A_0 - C_0}{A_0 C_0} L'^2\Big]\Big) - H_1^{rot}. \tag{4.8}$$

Note that the perturbing functions H_1^{trans} and H_1^{rot} in (4.6) and (4.8) defined in (3.3) and (3.5), respectively, should be rewritten in the osculating elements (4.1) and (4.2).

4.2. Equations of unperturbed translational-rotational motion in osculating analogues of the Delaunay–Andoyer elements.

4.2.1. *Unperturbed translational motion.* By setting $H_1^{trans} = 0$ in (4.6), we obtain the unperturbed Hamiltonian as

$$F = F_{unpert} = \frac{1}{\nu^2(t)} \cdot \frac{M_0^2}{2L^2}.$$

Equations of unperturbed translational motion of the center of inertia of the triaxial non-stationary body in analogues of Delaunay elements (4.1) have the form

$$\dot{l} = -\frac{\partial F_{unpert}}{\partial L}, \quad \dot{g} = 0, \quad \dot{h} = 0,$$
$$\dot{L} = 0, \qquad \dot{G} = 0, \quad \dot{H} = 0. \tag{4.9}$$

Integrals of the system (4.9) can be written as follows

$$L = L_0 = const, \quad G = G_0 = const, \quad H = H_0 = const,$$
$$l = \frac{M_0^2}{L^3}\int_{t_0}^{t} \frac{dt}{\nu^2(t)} + l_0, \quad l_0 = const, \quad g = g_0 = const, \quad h = h_0 = const.$$

4.2.2. *Unperturbed rotational motion.* By setting $H_{1pert}^{rot} = 0$ in (4.8), we obtain the Hamiltonian determining the unperturbed rotational motion as

$$F' = \frac{1}{2m\chi^2}\Big[\frac{G'^2}{A_0} + \frac{A_0 - C_0}{A_0 C_0} L'^2\Big].$$

Equations of unperturbed rotational motion in analogues of Andoyer variables (4.2) have the form

$$\dot{L}' = 0, \qquad \dot{G}' = 0, \qquad \dot{H}' = 0,$$
$$\dot{l}' = -\frac{\partial F'}{\partial L'}, \quad \dot{g}' = -\frac{\partial F'}{\partial G'}, \quad \dot{h}' = 0. \tag{4.10}$$

From equation (4.10) it follows

$$L' = L_0' = const, \quad G' = G_0' = const, \quad H' = H_0' = const,$$
$$l' = \frac{A_0 - C_0}{A_0 C_0} L' \int_{t_0}^{t} \frac{dt}{m\chi^2} + l_0', \quad l_0' = const,$$
$$g' = \frac{G'}{A_0}\int \frac{dt}{m\chi^2} + g_0', \quad g_0' = const, \quad h' = h_0' = const.$$

The geometric meaning of the analogues of Andoyer variables

$$L', \quad G', \quad H', \quad l', \quad g', \quad h',$$

is shown in Fig. 3.

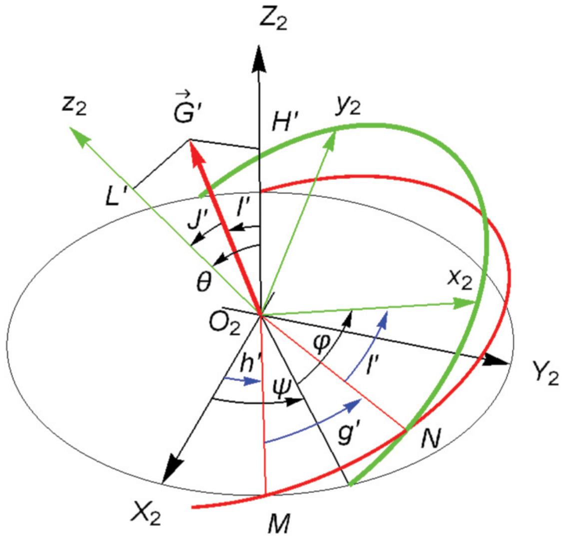

FIGURE 3. Andoyer variables

The following notations are accepted in Fig. 3:

1. $O_2X_2Y_2Z_2$ is the non-rotating König coordinate system whose axes are parallel to the axes of the absolute coordinate system $O\xi\eta\zeta$;
2. $O_2x_2y_2z_2$ is the Cartesian coordinate system fixed to the triaxial body, the axes of which are coincident with the principal axes of this body;
3. O_2NM is the plane normal to the angular momentum $\vec{G}'$, passing through the origin (barycenter);
4. O_2M is the line of intersection of the planes $O_2X_2Y_2$ and O_2MN;
5. O_2N is the line of intersection of the plane $O_2x_2y_2$ and the plane O_2MN normal to the angular momentum $\vec{G}'$;
6. $\vec{G}'$ is the angular momentum of the triaxial body;
7. L' is the component of the angular momentum vector $\vec{G}'$ along the axis O_2z_2;
8. H' is the component of the angular momentum vector $\vec{G}'$ along the axis O_2Z_2;
9. l' is the angle between the line O_2N and the axis O_2x_2;
10. g' is the angle between the lines O_2M and O_2N;
11. h' is the angle between the axis O_2X_2 and the line O_2M.

5. Expansion of the perturbing function in terms of the Delaney–Andoyer elements

In order to obtain the equations of perturbed motion in an explicit form, it is necessary to rewrite the perturbing functions (4.6) and (4.8) in terms of the Delaunay–Andoyer elements. Taking into account the relationship between different systems of canonical variables (see [17,19]), we can write

$$\frac{[(P_\psi - P_\varphi\cos\theta)\cos\varphi - P_\theta\sin\varphi\sin\theta]^2}{\sin^2\theta} = (Bq)^2 = B^2\Big(\frac{\sqrt{G'^2-L'^2}}{B}\cos l\Big)^2 = (G'^2-L'^2)\cos^2 l',$$

$$-\frac{1}{2}\Big(\frac{1}{A}-\frac{1}{B}\Big)\frac{[(P_\psi - P_\varphi\cos\theta)\cos\varphi - P_\theta\sin\varphi\sin\theta]^2}{\sin^2\theta} = -\frac{1}{2}\Big(\frac{1}{A}-\frac{1}{B}\Big)(G'^2-L'^2)\cos^2 l'.$$

Therefore, the perturbing function (3.5) takes the form

$$H_1^{rot} = -\frac{1}{2}\Big(\frac{1}{A}-\frac{1}{B}\Big)(G'^2-L'^2)\cos^2 l' - \Big\{U_2 - \frac{1}{2}bR^2\Big\}. \tag{5.1}$$

Note that the expression in curly brackets in (5.1) as well as expression (3.3) should be written in terms of Delaunay–Andoyer elements (4.1) and (4.2). Observe that the force function U_2 defined in (2.7) depends on the cosines α, β, γ of the angles formed by vector $\vec{R}$ and axes of the coordinate system $O_2x_2y_2z_2$ fixed to the triaxial body. These cosines may be represented in the form

$$\begin{gathered}\alpha = a_{11}\frac{x}{R} + a_{21}\frac{y}{R} + a_{31}\frac{z}{R}, \quad \beta = a_{12}\frac{x}{R} + a_{22}\frac{y}{R} + a_{32}\frac{z}{R},\\ \gamma = a_{13}\frac{x}{R} + a_{23}\frac{y}{R} + a_{33}\frac{z}{R},\end{gathered} \tag{5.2}$$

where $\frac{x}{R}$, $\frac{y}{R}$, $\frac{z}{R}$ are the direction cosines of the vector $\vec{R}$ with respect to the axes of the non-rotating coordinate system $O_2X_2Y_2Z_2$ which have the form (see [13])

$$\begin{gathered}\frac{x}{R} = \cos u\cos\Omega - \sin u\sin\Omega\cos i, \quad \frac{y}{R} = \cos u\sin\Omega + \sin u\cos\Omega\cos i,\\ \frac{z}{R} = \sin u\sin i, \quad R = \nu\rho, \quad u = \upsilon + \omega, \quad \rho = \frac{a(1-e^2)}{1+e\cos\upsilon}.\end{gathered} \tag{5.3}$$

Parameters a, e, i, Ω, ω in (5.3) correspond to the Kepler orbital elements known from the classical two-body problem; they are analogues of the major semi-axis, eccentricity, inclination, longitude of the ascending node, and the longitude of the peri-center of the unperturbed quasi-elliptic orbit of the triaxial body (see [13]). The variable υ is a true anomaly determining position of this body on the quasi-elliptic orbit.

Coefficients a_{ij} in (5.2) are the direction cosines of the axes of the coordinate system $O_2x_2y_2z_2$ fixed to the triaxial body with respect to the non-rotating coordinate systems $O_2X_2Y_2Z_2$. In the analogues of the Andoyer variables, the direction cosines a_{ij} are expressed as follows (see [17, 19])

$$
\begin{aligned}
a_{11} &= \cos h' \cos g' \cos l' - \cos h' \cos J' \sin g' \sin l' - \sin h' \cos I' \cos l' \sin g' \\
&\quad - \sin h' \cos I' \cos g' \cos J' \sin l' + \sin h' \sin I' \sin l' \sin J', \\
a_{21} &= \sin h' \cos g' \cos l' - \sin h' \cos J' \sin g' \sin l' + \cos h' \cos I' \cos l' \sin g' \\
&\quad + \cos h' \cos I' \cos g' \cos J' \sin l' - \cos h' \sin I' \sin l' \sin J', \\
a_{31} &= \sin I' \sin g' \cos l' + \sin I' \cos J' \cos g' \sin l' + \cos I' \sin l' \sin J',
\end{aligned}
\tag{5.4}
$$

$$
\begin{aligned}
a_{12} &= -\cos h' \cos J' \cos l' \sin g' - \cos h' \cos g' \sin l' \\
&\quad - \sin h' \cos I' \cos g' \cos l' \cos J' + \sin h' \cos I' \sin g' \sin l' \\
&\quad + \sin h' \sin I' \cos l' \sin J', \\
a_{22} &= -\sin h' \cos J' \cos l' \sin g' - \sin h' \cos g' \sin l' \\
&\quad + \cos h' \cos I' \cos g' \cos l' \cos J' - \cos h' \cos I' \sin g' \sin l' \\
&\quad - \cos h' \sin I' \cos l' \sin J', \\
a_{32} &= \sin I' \cos J' \cos l' \cos g' - \sin I' \sin g' \sin l' + \cos I' \cos l' \sin J',
\end{aligned}
\tag{5.5}
$$

$$
\begin{aligned}
a_{13} &= \cos h' \sin J' \sin g' + \sin h' \sin I' \cos J' + \sin h' \cos I' \cos g' \sin J', \\
a_{23} &= \sin h' \sin J' \sin g' - \cos h' \sin I' \cos J' - \cos h' \cos I' \cos g' \sin J', \\
a_{33} &= \cos I' \cos J' - \sin I' \cos g' \sin J',
\end{aligned}
\tag{5.6}
$$

where

$$
\cos I' = \frac{H'}{G'}, \quad \sin I' = \sqrt{1 - \frac{H'^2}{G'^2}}, \quad \cos J' = \frac{L'}{G'}, \quad \sin J' = \sqrt{1 - \frac{L'^2}{G'^2}}.
$$

Note that the direction cosines (5.6) do not depend on the angle l'. To simplify further computation, it is convenient to rewrite expressions (5.4) and (5.5) in the form

$$
\begin{aligned}
a_{11} &= \kappa_{11} \cos l' + \kappa_{12} \sin l', \\
a_{21} &= \kappa_{21} \cos l' + \kappa_{22} \sin l', \\
a_{31} &= \kappa_{31} \cos l' + \kappa_{32} \sin l', \\
a_{12} &= \kappa_{12} \cos l' - \kappa_{11} \sin l', \\
a_{22} &= \kappa_{22} \cos l' - \kappa_{21} \sin l', \\
a_{32} &= \kappa_{32} \cos l' - \kappa_{31} \sin l',
\end{aligned}
\tag{5.7}
$$

where

$$
\begin{aligned}
\kappa_{11} &= \cos h' \cos g' - \sin h' \sin g' \frac{H'}{G'}, \\
\kappa_{12} &= \sin h' \sqrt{\left(1 - \frac{H'^2}{G'^2}\right)\left(1 - \frac{L'^2}{G'^2}\right)} - \sin h' \cos g' \frac{H'L'}{G'^2} - \cos h' \sin g' \frac{L'}{G'}, \\
\kappa_{21} &= \sin h' \cos g' + \cos h' \sin g' \frac{H'}{G'},
\end{aligned}
$$

$$\kappa_{22} = -\cos h' \sqrt{\Big(1 - \frac{H'^2}{G'^2}\Big)\Big(1 - \frac{L'^2}{G'^2}\Big)} + \cos h' \cos g' \frac{H'L'}{G'^2} - \sin h' \sin g' \frac{L'}{G'},$$
$$\kappa_{31} = \sin g' \sqrt{\Big(1 - \frac{H'^2}{G'^2}\Big)},$$
$$\kappa_{32} = \cos g' \frac{L'}{G'} \sqrt{\Big(1 - \frac{H'^2}{G'^2}\Big)} + \frac{H'}{G'} \sqrt{\Big(1 - \frac{L'^2}{G'^2}\Big)}.$$

Using the obvious relation $\alpha^2 + \beta^2 + \gamma^2 = 1$, we can rewrite expression (2.7) in the form

$$U_2 = \Big[\frac{fm_1(A+B+C)}{2} - \frac{3fm_1A}{2}\Big]\Big\{\frac{1}{R^3}\Big\} + \Big[\frac{3fm_1A}{2} - \frac{3fm_1B}{2}\Big]\Big\{\frac{\beta^2}{R^3}\Big\} + \Big[\frac{3fm_1A}{2} - \frac{3fm_1C}{2}\Big]\Big\{\frac{\gamma^2}{R^3}\Big\}.$$

Then we have

$$-\frac{1}{2}bR^2 + U_2 = -\frac{1}{2}b\{R^2\} + \Big[\frac{1}{2}fm_1(B+C-2A)\Big]\Big\{\frac{1}{R^3}\Big\} + \Big[\frac{3}{2}fm_1(A-B)\Big]\Big\{\frac{\beta^2}{R^3}\Big\} + \Big[\frac{3}{2}fm_1(A-C)\Big]\Big\{\frac{\gamma^2}{R^3}\Big\}. \tag{5.8}$$

The expressions in curly brackets in equation (5.8) should be expressed through the Delaunay–Andoyer variables. Denote

$$\begin{aligned} W_1 &= R^2 = \nu^2\rho^2 = \nu^2\Big(\frac{a^2(1-e^2)^2}{(1+e\cos\upsilon)^2}\Big), \\ W_2 &= \frac{1}{R^3} = \frac{1}{\nu^3\rho^3} = \frac{1}{\nu^3}\Big(\frac{(1+e\cos\upsilon)^3}{a^3(1-e^2)^3}\Big), \\ W_3 &= \frac{\beta^2}{R^3} = \frac{1}{\nu^3}\cdot\frac{\beta^2}{\rho^3} = \frac{1}{\nu^3}\frac{(1+e\cos\upsilon)^3}{a^3(1-e^2)^3}\Big(a_{12}\frac{x}{R} + a_{22}\frac{y}{R} + a_{32}\frac{z}{R}\Big)^2, \\ W_4 &= \frac{\gamma^2}{R^3} = \frac{1}{\nu^3}\cdot\frac{\gamma^2}{\rho^3} = \frac{1}{\nu^3}\frac{(1+e\cos\upsilon)^3}{a^3(1-e^2)^3}\Big(a_{13}\frac{x}{R} + a_{23}\frac{y}{R} + a_{33}\frac{z}{R}\Big)^2. \end{aligned} \tag{5.9}$$

Note that equations (5.2)–(5.7) and (5.9) enable us to rewrite (5.8) and the perturbing functions (4.6) and (5.1) in terms of the Delaunay–Andoyer variables and to write out the right-hand sides of the equations of motion (4.5) and (4.7) in the explicit form. However, the corresponding expressions are quite cumbersome and we do not write them out here.

6. Evolution equations

In the absence of resonances, averaging the right-hand sides of equations (4.5) and (4.7) by l and l', we obtain the equations for secular perturbations of the Delaunay–Andoyer elements

$$\dot{L} = \frac{\partial\tilde{\tilde{F}}}{\partial l} = 0, \quad \dot{G} = \frac{\partial\tilde{\tilde{F}}}{\partial g}, \quad \dot{H} = \frac{\partial\tilde{\tilde{F}}}{\partial h}, \quad \dot{l} = -\frac{\partial\tilde{\tilde{F}}}{\partial L}, \quad \dot{g} = -\frac{\partial\tilde{\tilde{F}}}{\partial G}, \quad \dot{h} = -\frac{\partial\tilde{\tilde{F}}}{\partial H}, \tag{6.1}$$

(6.2)
$$\dot L' = \frac{\partial \tilde{\tilde F}'}{\partial l'} = 0, \quad \dot G' = \frac{\partial \tilde{\tilde F}'}{\partial g'}, \quad \dot H' = \frac{\partial \tilde{\tilde F}'}{\partial h'},$$
$$\dot l' = -\frac{\partial \tilde{\tilde F}'}{\partial L'}, \quad \dot g' = -\frac{\partial \tilde{\tilde F}'}{\partial G'}, \quad \dot h' = -\frac{\partial \tilde{\tilde F}'}{\partial H'}.$$

To find the secular parts of the perturbing functions $\tilde{\tilde F}$ and $\tilde{\tilde F}'$, we need to calculate integrals of the form

(6.3)
$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} F\,dl\,dl' = \frac{(1-e^2)^{3/2}}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{F}{(1+e\cos\upsilon)^2}d\upsilon\,dl'.$$

Note that the integration over the angle l in (6.3) can be conveniently replaced by the integration over the true anomaly υ.

Taking into account (3.8), (4.3), (4.4) and (5.3) and averaging expressions (5.9), we obtain

$$\begin{aligned}
\widetilde{\widetilde W}_1 &= \nu^2a^2\Big(1+\frac{3}{2}e^2\Big) = \frac{\nu^2L^4}{2M_0^2}\Big(5-\frac{3G^2}{L^2}\Big),\\
\widetilde{\widetilde W}_2 &= \frac{1}{\nu^3a^3(1-e^2)^{3/2}} = \frac{M_0^3}{\nu^3L^3G^3},\\
\widetilde{\widetilde W}_3 &= \frac{M_0^3}{4\nu^3L^3G^5}\Bigg((G^2-H^2)(\kappa_{31}^2+\kappa_{32}^2)\\
&\quad + G^2\big((\kappa_{11}\cos h+\kappa_{21}\sin h)^2+(\kappa_{12}\cos h+\kappa_{22}\sin h)^2\big)\\
&\quad + H^2\big((\kappa_{11}\sin h-\kappa_{21}\cos h)^2+(\kappa_{12}\sin h-\kappa_{22}\cos h)^2\big)\\
&\quad + 2GH\sqrt{1-\frac{H^2}{G^2}}\big((\kappa_{21}\kappa_{31}+\kappa_{22}\kappa_{32})\cos h\\
&\quad - (\kappa_{11}\kappa_{31}+\kappa_{12}\kappa_{32})\sin h\big)\Bigg),\\
\widetilde{\widetilde W}_4 &= \frac{M_0^3}{2\nu^3L^3G^5}\Bigg(a_{33}^2(G^2-H^2)\\
&\quad + 2GH\sqrt{1-\frac{H^2}{G^2}}(a_{23}a_{33}\cos h - a_{13}a_{33}\sin h)\\
&\quad + H^2(a_{23}\cos h - a_{13}\sin h)^2 + G^2(a_{13}\cos h + a_{23}\sin h)^2\Bigg).
\end{aligned}$$

As a result we obtain the secular parts of the perturbing functions $\tilde{\tilde F}$ and $\tilde{\tilde F}'$ in the form

$$\tilde{\tilde F} = \frac{M_0^2}{2\nu^2L^2} + \frac{f(m_1+m_2)}{2m_2}\big((A-B)(3\widetilde{\widetilde W}_3-\widetilde{\widetilde W}_2)+(A-C)(3\widetilde{\widetilde W}_4-\widetilde{\widetilde W}_2)\big) - \frac{1}{2}b\widetilde{\widetilde W}_1,$$

$$\tilde{\tilde F}' = -\frac{1}{2m\xi^2}\Big(\frac{G'^2}{A_0}+\frac{A_0-C_0}{A_0C_0}L'^2\Big) + \frac{1}{4}\Big(\frac{1}{A}-\frac{1}{B}\Big)(G'^2-L'^2) - \frac{1}{2}b\widetilde{\widetilde W}_1$$

$$+\frac{fm_1}{2}\big((A-B)(3\,\overset{\approx}{W}_3-\overset{\approx}{W}_2)+(A-C)(3\,\overset{\approx}{W}_4-\overset{\approx}{W}_2)\big).$$

On substituting the obtained expressions for $\tilde{\tilde{F}}$ and $\tilde{\tilde{F}}'$ in (6.1) and (6.2), we obtain the equations for secular perturbations of the Delaunay–Andoyer elements.

Note that all relevant cumbersome symbolic calculations are performed here with the computer algebra system Mathematica [20].

7. Conclusions

The equations for secular perturbations of the translational-rotational motion of a triaxial satellite in osculating Delaunay–Andoyer elements are obtained and can be used to analyse the dynamic evolution of the system of two non-stationary bodies attracting each other according to Newton's law of gravitation. Using the unperturbed motion is effective in the study of dynamics of bodies, the ellipsoid of inertia of which is close to the corresponding ellipsoids of rotation (spheroids).

Further development of this work involves the study of the obtained evolution equations of translational-rotational motion of a triaxial body of constant dynamic shape and variable size and mass using various numerical methods.

In cases where the ellipsoid of inertia of a triaxial body differs substantially from the corresponding ellipsoid of rotation (spheroids), it is convenient to use analogues of Poisson elements (action-angle variables) to describe the rotational motion. Based on the above Delaunay–Andoyer variables, one can introduce the Delaunay–Poisson variables.

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ЈЕДНАЧИНЕ ТРАНСЛАЦИОНАЛНО-РОТАЦИОНОГ КРЕТАЊА НЕСТАЦИОНАРНОГ ТРООСНОГ ТЕЛА У ЦЕНТРАЛНОМ ГРАВИТАЦИОНОМ ПОЉУ

Резиме. Испитује се транслационо-ротационо кретање троосног тела константног динамичког облика, а променљиве величине и масе у нестационарном Њутновом централном гравитационом пољу. Добијене су диференцијалне једначине кретања троосног тела у релативном координатном систему са почетком у средишту нестационарног сферног тела. Осе Декартовог координатног система причвршћене на нестационарно троосно тело подударају се с његовим главним осама инерције, и претпоставља се да ће њихова релативна оријентација током еволуције остати непромењена. Добијен је аналитички израз за функцију силе гравитационе интеракције троосног тела променљиве масе и величине са сферним телом променљиве величине и масе. Изведене су диференцијалне једначине транслационо-ротационог кретања нестационарног троосног тела у оскулаторним променљивама Јакобија које су проучаване методама теорије пертурбације. Функција поремећаја је развијена у ред по Делоне-Андоајеовим елеменатима до другог хармоника. Добијене су једначине еволуције транслационо-ротационог кретања нестационарног троосног тела у Делоне-Андоајеовим оскулаторним елементима.

Department of Mechanics
Al-Farabi Kazakh National University
Almaty
Kazakhstan
minglibayev@gmail.com

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Institute of Information Technology
Warsaw University of Life Sciences–SGGW
Warsaw
Poland
alexander_prokopenya@sggw.pl

Department of Mechanics
Al-Farabi Kazakh National University
Almaty
Kazakhstan
baisbayevaoral@gmail.com