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HEREDITARINESS AND NON-LOCALITY IN WAVE PROPAGATION MODELING

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Abstract. The classical wave equation is generalized within the framework of fractional calculus in order to account for the memory and non-local effects that might be material features. Both effects are included in the constitutive equation, while the equation of motion of the deformable body and strain are left unchanged. Memory effects in viscoelastic materials are modeled through the distributed-order fractional constitutive equation that generalizes all linear models having differentiation orders up to order one. The microlocal approach in analyzing singularity propagation is utilized in the case of viscoelastic material described by the fractional Zener model, as well as in the case of two non-local models: non-local Hookean and fractional Eringen.

1. Introduction

This review article aims to present results previously published in [13, 14, 18] related to the fractional wave equations including hereditary and non-local effects.

The classical wave equation for a one-dimensional infinite deformable body positioned along the x-axis, for time $t>0$, considered as a system of: the equation of motion

$$\frac{\partial}{\partial x}\sigma(x,t)=\rho\frac{\partial^2}{\partial t^2}u(x,t), \tag{1.1}$$

where σ, u, and ρ are stress, displacement, and material density; strain ε for small local deformations

$$\varepsilon(x,t)=\frac{\partial}{\partial x}u(x,t); \tag{1.2}$$

and the Hooke law

$$\sigma(x,t)=E\varepsilon(x,t),$$

representing the connection between stress and strain local in both time and space, where E is the Young modulus of elasticity, is generalized by considering memory

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effects through the distributed-order fractional constitutive model

$$\int_0^1 \phi_\sigma(\alpha)\, {}_0\mathrm{D}_t^\alpha \sigma(x,t)\mathrm{d}\alpha = E\int_0^1 \phi_\varepsilon(\alpha)\, {}_0\mathrm{D}_t^\alpha \varepsilon(x,t)\mathrm{d}\alpha, \tag{1.3}$$

where ϕ_σ and ϕ_ε are constitutive functions or distributions, while the non-locality effects are taken into account either through the non-local fractional Hooke law

$$\sigma(x,t) = \frac{E}{\ell^{1-\alpha}}\mathcal{E}_x^\alpha u(x,t) = \frac{E}{\ell^{1-\alpha}}\frac{1}{2\Gamma(1-\alpha)}\frac{1}{|x|^\alpha} *_x \varepsilon(x,t), \tag{1.4}$$

or through the fractional Eringen constitutive equation

$$\sigma(x,t) - \ell^\alpha \mathrm{D}^\alpha \sigma(x,t) = E\varepsilon(x,t), \quad \alpha \in (1,3), \tag{1.5}$$

where ℓ is the non-locality parameter.

Constitutive equations (1.3), (1.4), and (1.5) contain different types of fractional derivatives: the Riemann–Liouville fractional derivative of order $\alpha \in (0,1)$, defined by

$${}_0\mathrm{D}_t^\alpha y(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{t^{-\alpha}}{\Gamma(1-\alpha)} *_t y(t)\Big), \quad t>0,$$

see [15], with $*_t$ denoting the convolution in time: $f(t) *_t g(t) = \int_0^t f(\eta)g(t-\eta)\mathrm{d}\eta$, $t>0$, the Riesz symmetrized fractional derivative

$$\mathcal{E}_x^\alpha w(x) = \frac{1}{2\Gamma(1-\alpha)}\frac{1}{|x|^\alpha} *_x \frac{\mathrm{d}}{\mathrm{d}x}w(x), \quad \alpha \in (0,1),$$

with $*_x$ denoting the convolution in space: $f(t) *_x g(t) = \int_{\mathbb{R}} f(\eta)g(x-\eta)\mathrm{d}\eta$, $x \in \mathbb{R}$, and two Riesz-type symmetrized fractional derivatives

$$\mathrm{D}^\alpha w(x) = \frac{1}{2\Gamma(2-\alpha)}\frac{1}{|x|^{\alpha-1}} *_x \frac{\mathrm{d}^2}{\mathrm{d}x^2}w(x), \quad \alpha \in (1,2),$$
$$\mathrm{D}^\alpha w(x) = \frac{1}{2\Gamma(3-\alpha)}\frac{\operatorname{sgn} x}{|x|^{\alpha-2}} *_x \frac{\mathrm{d}^3}{\mathrm{d}x^3}w(x), \quad \alpha \in [2,3),$$

taken so that in both cases if $\alpha \to 2$, then $\mathrm{D}^\alpha \to \frac{\partial^2}{\partial x^2}$.

The distributed-order constitutive stress-strain relation (1.3) generalizes integer and fractional order constitutive models of linear viscoelasticity

$$\sum_{i=1}^n a_i\, {}_0\mathrm{D}_t^{\alpha_i}\sigma(x,t) = E\sum_{j=1}^m b_i\, {}_0\mathrm{D}_t^{\beta_j}\varepsilon(x,t), \tag{1.6}$$

having differentiation orders up to the first order, if the constitutive distributions ϕ_σ and ϕ_ε are chosen as

$$\phi_\sigma(\alpha) := \sum_{i=1}^n a_i\delta(\alpha-\alpha_i), \quad \phi_\varepsilon(\alpha) := \sum_{j=1}^m b_i\delta(\alpha-\beta_j),$$

with model parameters: $a_i, b_j > 0$ as generalized time constants, and $\alpha_i, \beta_j \in (0,1)$ as orders of fractional differentiation, where $i = 1,\dots,n$, $j = 1,\dots,m$. Thermodynamical consistency of linear fractional constitutive equation (1.6) is examined in [3], where it is shown that there are four cases of (1.6) when the restrictions on model parameters guarantee its thermodynamical consistency. The power type distributed-order model of the viscoelastic body

$$\int_0^1 a^\alpha\, {}_0\mathrm{D}_t^\alpha \sigma(x,t)\mathrm{d}\alpha = E\int_0^1 b^\alpha\, {}_0\mathrm{D}_t^\alpha \varepsilon(x,t)\mathrm{d}\alpha, \tag{1.7}$$

considered in [1] and revisited in [3], is obtained if the constitutive functions ϕ_σ and ϕ_ε are chosen as

$$\phi_\sigma(\alpha) := a^\alpha, \quad \phi_\varepsilon(\alpha) := b^\alpha,$$

with the time constants $a, b > 0$ satisfying $a \leqslant b$ guaranteeing the model's dimensional homogeneity and thermodynamical consistency.

Generalizations of the classical wave equation are considered on an infinite domain in [16, 17], where the constitutive equation, representing a single class of thermodynamically consistent linear fractional constitutive equation (1.6), is chosen to be either the fractional Zener model, or its generalization having an arbitrary number of fractional derivatives of the same orders acting on both stress and strain. Wave propagation speed is obtained from the support property of the fundamental solution in [16,17], while in [13] tools of microlocal analysis were employed in order to examine the singularity propagation properties in the case of the fractional Zener wave equation. The fractional wave equation, with power type distributed order model (1.7) as the constitutive equation, is considered on a finite domain in [5, 6]. The fractional Zener wave equation on the bounded domain is analyzed in [20], while the modified Zener and modified Maxwell wave equations are considered for bounded and semi-bounded domains in [19, 21]. Several other problems involving generalizations of the classical wave equations are reviewed in [4, 22].

The non-local wave equation, using the non-local fractional Hooke law (1.4), is formulated in [7] in the context of fractionalization of the strain measure. Further, it was used in [2], along with the time-fractional Zener model in order to account for both the non-locality and memory effects. The wave equation that uses stress and strain gradient variant of the Eringen constitutive equation is considered in [8] for harmonic wave propagation in order to obtain a dispersion equation, used for finding the optimal values of the model parameters by comparison with the Born–Kármán model of lattice dynamics. By the same method, the optimal value of the order of the fractional differentiation α, appearing in (1.5), is obtained in [9] as well. A non-local wave equation with the fractional Eringen constitutive equation (1.5) is formulated in [14] and analyzed using the tools of microlocal analysis.

2. Distributed-order fractional wave equation

The distributed-order fractional wave equation, modeling disturbance propagation in the hereditary medium, consists of the equation of motion (1.1), strain (1.2), and distributed-order constitutive equation (1.3) and it is analyzed in [18].

2.1. General results. Consider the Cauchy problem on the real line $\mathbb{R}$, with $t > 0$, for the dimensionless system of equations

$$\frac{\partial}{\partial x}\sigma(x,t) = \frac{\partial^2}{\partial t^2}u(x,t), \tag{2.1}$$

$$\int_0^1 \phi_\sigma(\alpha)\, {}_0\mathrm{D}_t^\alpha \sigma(x,t)\mathrm{d}\alpha = \int_0^1 \phi_\varepsilon(\alpha)\, {}_0\mathrm{D}_t^\alpha \varepsilon(x,t)\mathrm{d}\alpha, \tag{2.2}$$

$$\varepsilon(x,t) = \frac{\partial}{\partial x} u(x,t), \tag{2.3}$$

subject to initial and boundary conditions:

$$u(x,0) = u_0(x), \quad \frac{\partial}{\partial t} u(x,0) = v_0(x), \quad \sigma(x,0) = 0, \quad \varepsilon(x,0) = 0,$$
$$\lim_{x\to\pm\infty} u(x,t) = 0, \quad \lim_{x\to\pm\infty} \sigma(x,t) = 0$$

obtained from the initial system of equations (1.1), (1.2), and (1.3) after introducing dimensionless quantities.

The first result provides equivalence of system of equations (2.1)–(2.3) and equation

$$\frac{\partial^2}{\partial t^2} u(x,t) - L(t)\frac{\partial^2}{\partial x^2} u(x,t) = 0, \quad x\in\mathbb{R},\ t>0, \tag{2.4}$$

with L being a linear operator of convolution type acting on $\mathcal{S}'(\mathbb{R})$, given below. Equation (2.4) will be called the wave equation for distributed order type viscoelastic media. Assume that the constitutive relation (2.2) is determined by compactly supported distributions $\phi_\sigma, \phi_\varepsilon \in \mathcal{E}'(\mathbb{R})$ with support in $[0,1]$, while the initial conditions u_0 and v_0 are assumed to be elements of $\mathcal{S}'(\mathbb{R})$.

THEOREM 2.1. *Let $\phi_\sigma, \phi_\varepsilon \in \mathcal{E}'(\mathbb{R})$ with support in $[0,1]$. Set*

$$\Phi_\sigma(s) = \langle \phi_\sigma(\alpha), s^\alpha\rangle \quad \text{and} \quad \Phi_\varepsilon(s) = \langle \phi_\varepsilon(\alpha), s^\alpha\rangle, \quad \operatorname{Re} s > 0.$$

Suppose that the following assumption holds.

(A1): $\mathcal{L}^{-1}\big(\frac{\Phi_\varepsilon(s)}{\Phi_\sigma(s)}\big)$ *exists as an element of* $\mathcal{S}'_+(\mathbb{R})$.

Then system of equations (2.1)–(2.3) and equation (2.4) with $L(t) := \mathcal{L}^{-1}\big(\frac{\Phi_\varepsilon(s)}{\Phi_\sigma(s)}\big)*_t$ are equivalent.

The generalized Cauchy problem for the operator P takes the following form $Pu(x,t) = u_0(x)\delta'(t) + v_0(x)\delta(t)$, or equivalently

$$\frac{\partial^2}{\partial t^2} u(x,t) = \mathcal{L}^{-1}\Big(\frac{\Phi_\varepsilon(s)}{\Phi_\sigma(s)}\Big) *_t \frac{\partial^2}{\partial x^2} u(x,t) + u_0(x)\delta'(t) + v_0(x)\delta(t). \tag{2.5}$$

The following theorem provides conditions that guarantee existence and uniqueness of a solution to the generalized Cauchy problem (2.5).

THEOREM 2.2. *Let $u_0, v_0 \in \mathcal{S}'(\mathbb{R})$. Suppose that assumption* (A1) *holds. Further, assume the following.*

(A2): $s^2 \frac{\Phi_\sigma(s)}{\Phi_\varepsilon(s)} \in \mathbb{C}\setminus(-\infty,0]$, *for all* $s\in\mathbb{C}$ *with* $\operatorname{Re} s>0$.

(A3): $\mathcal{L}^{-1}\big(\frac{\Phi_\sigma(s)}{\Phi_\varepsilon(s)}\big)$ *exists as an element of* $\mathcal{S}'_+$.

Then there exists a unique solution $u\in\mathcal{S}'(\mathbb{R}\times\mathbb{R}_+)$ to (2.5) given by

$$u(x,t) = S(x,t) *_{x,t} (u_0(x)\delta'(t) + v_0(x)\delta(t)),$$

where $S\in\mathcal{S}'(\mathbb{R}\times\mathbb{R}_+)$ is a fundamental solution for the operator P.

In Theorem 2.2 the existence of the inverse Laplace transform of the fundamental solution is proved and solution u as the convolution of fundamental solution S with initial conditions is obtained. Closer inspection of the proof of the theorem indicates that in order to calculate S explicitly one needs to impose additional assumptions. This is considered in the following statement.

THEOREM 2.3. *Suppose that assumptions* (A1), (A2) *and* (A3) *of Theorems* 2.1 *and* 2.2 *hold. Suppose in addition the following.*

(A4): *Multiform function*

$$\tilde{K}(x,s) := \frac{1}{2}\sqrt{\frac{\Phi_\sigma(s)}{\Phi_\varepsilon(s)}}e^{-|x|s\sqrt{\frac{\Phi_\sigma(s)}{\Phi_\varepsilon(s)}}}, \quad x \in \mathbb{R},\ \operatorname{Re} s > 0,$$

has only two branch points $s = 0$ and $s = \infty$.

(A5): $\lim_{R\to\infty}\left|\sqrt{\frac{\Phi_\sigma(Re^{i\varphi})}{\Phi_\varepsilon(Re^{i\varphi})}}\right| = k$,*for* $\varphi \in \left(\frac{\pi}{2},\pi\right)\cup\left(-\pi,-\frac{\pi}{2}\right)$ *and* $k \geqslant 0$.

(A6): $\lim_{\eta\to 0}\left|\eta\sqrt{\frac{\Phi_\sigma(\eta e^{i\varphi})}{\Phi_\varepsilon(\eta e^{i\varphi})}}\right| = 0$, *for* $\varphi \in (-\pi,\pi)$.

Then the solution u reads

$$u(x,t) = K(x,t) *_{x,t} (u_0(x)\delta(t) + v_0(x)H(t)),$$

where the fundamental solution K can be calculated as

$$K(x,t) = \frac{1}{4\pi i}\int_0^\infty\left(\sqrt{\frac{\Phi_\sigma(q\mathrm{e}^{-i\pi})}{\Phi_\varepsilon(q\mathrm{e}^{-i\pi})}}\mathrm{e}^{|x|q\sqrt{\frac{\Phi_\sigma(q\mathrm{e}^{-i\pi})}{\Phi_\varepsilon(q\mathrm{e}^{-i\pi})}}} - \sqrt{\frac{\Phi_\sigma(q\mathrm{e}^{i\pi})}{\Phi_\varepsilon(q\mathrm{e}^{i\pi})}}\mathrm{e}^{|x|q\sqrt{\frac{\Phi_\sigma(q\mathrm{e}^{i\pi})}{\Phi_\varepsilon(q\mathrm{e}^{i\pi})}}}\right)\mathrm{e}^{-qt}\mathrm{d}q, \tag{2.6}$$

$K \in \mathcal{S}'(\mathbb{R}\times\mathbb{R}_+)$, *and has the support in the cone* $|x| < ct$, *with* $c = 1/k$. *If* $|x| > ct$, *then* $K = 0$.

2.2. Cases of the linear fractional and power type distributed-order model. Thermodynamical restrictions for the linear fractional model of a viscoelastic body (1.6) are studied in [3], where four admissible model classes are obtained.

CASE 1.

$$\phi_\sigma(\alpha) := \sum_{i=1}^{n} a_i\delta(\alpha-\alpha_i), \quad \phi_\varepsilon(\alpha) := \sum_{i=1}^{n} b_i\delta(\alpha-\alpha_i), \tag{2.7}$$

with $0 \leqslant \alpha_1 < \ldots < \alpha_n < 1$, and $\frac{a_1}{b_1} \geqslant \frac{a_2}{b_2} \geqslant \ldots \geqslant \frac{a_n}{b_n} \geqslant 0$;

CASE 2.

$$\phi_\sigma(\alpha) := \sum_{i=1}^{n} a_i\delta(\alpha-\alpha_i), \quad \phi_\varepsilon(\alpha) := \sum_{i=1}^{n} b_i\delta(\alpha-\alpha_i) + \sum_{i=n+1}^{m} b_i\delta(\alpha-\beta_i), \tag{2.8}$$

with $0 \leqslant \alpha_1 < \ldots < \alpha_n < \beta_{n+1} < \ldots < \beta_m < 1$, and $\frac{a_1}{b_1} \geqslant \frac{a_2}{b_2} \geqslant \ldots \geqslant \frac{a_n}{b_n} \geqslant 0$;

CASE 3.

$$\phi_\sigma(\alpha) := \sum_{i=1}^{m} a_i\delta(\alpha-\alpha_i) + \sum_{i=m+1}^{n} a_i\delta(\alpha-\alpha_i), \quad \phi_\varepsilon(\alpha) := \sum_{i=m+1}^{n} b_i\delta(\alpha-\alpha_i), \tag{2.9}$$

with $0 \leqslant \alpha_1 < \ldots < \alpha_m < \alpha_{m+1} < \ldots < \alpha_n < 1$, and $\frac{a_{m+1}}{b_{m+1}} \geqslant \frac{a_{m+2}}{b_{m+2}} \geqslant \ldots \geqslant \frac{a_n}{b_n} \geqslant 0$;

CASE 4.

$$\phi_\sigma(\alpha) := \sum_{i=1}^{n} a_i\delta(\alpha-\alpha_i), \quad \phi_\varepsilon(\alpha) := \sum_{j=1}^{m} b_j\delta(\alpha-\beta_j), \tag{2.10}$$

with $\alpha_i \neq \beta_j$, for all $i \neq j$, and $0 \leqslant \alpha_1 < \ldots < \alpha_n < \beta_1 < \ldots < \beta_m < 1$.

In all four cases all coefficients a_i and b_i are supposed to be nonnegative.

Assumptions (A1)–(A6) are satisfied in the case of any of the four admissible linear fractional models described above, implying the existence and uniqueness of the solution to the corresponding hereditary wave equation, as well as the explicit form of its fundamental solution.

THEOREM 2.4. *Let $u_0, v_0 \in \mathcal{S}'(\mathbb{R})$. Let the constitutive distributions ϕ_σ and ϕ_ε in the stress-strain relation* (2.2) *be determined by any of the cases* (2.7), (2.8), (2.9) *or* (2.10). *Then there exists a unique solution $u \in \mathcal{S}'(\mathbb{R}\times\mathbb{R}_+)$ to the generalized Cauchy problem* (2.5) *given by* (2.6).

The power type distributed-order model of the viscoelastic body, as a genuine distributed-order model, in dimensionless form becomes

$$\int_0^1 \tau^\alpha\, {}_0\mathrm{D}_t^\alpha \sigma(x,t)\mathrm{d}\alpha = \int_0^1 {}_0\mathrm{D}_t^\alpha \varepsilon(x,t)\mathrm{d}\alpha,$$

yielding the corresponding generalized Cauchy problem in the form

$$\frac{\partial^2}{\partial t^2}u(x,t) = \mathcal{L}^{-1}\Big(\frac{s-1}{\tau s-1}\frac{\ln(\tau s)}{\ln s}\Big) *_t \frac{\partial^2}{\partial x^2}u(x,t) + u_0(x)\delta'(t) + v_0(x)\delta(t). \tag{2.11}$$

Again, assumptions (A1)–(A6) are satisfied in the case of the power type distributed-order, implying the existence and uniqueness of the solution to the corresponding hereditary wave equation, as well as the explicit form of its fundamental solution.

THEOREM 2.5. *Suppose $\phi_\sigma(\alpha) = \tau^\alpha$, $\phi_\varepsilon(\alpha) = 1$, with $0 < \tau < 1$, and $u_0, v_0 \in \mathcal{S}'(\mathbb{R})$. Then there exists a unique solution $u \in \mathcal{S}'(\mathbb{R}\times\mathbb{R}_+)$ to* (2.11), *supported in the cone $|x| < ct$, and given by* (2.6), *with $c = \frac{1}{\sqrt{\tau}}$ being the wave propagation speed. Outside the cone, i.e., for $|x| > ct$, $u = 0$.*

The wave propagation speed is closely related to the material properties in creep and stress relaxation through assumption (A5). Namely, the Laplace transform of the constitutive equation (2.2), solved with respect to stress is

$$\tilde{\sigma}(s) = \frac{\Phi_\varepsilon(s)}{\Phi_\sigma(s)}\tilde{\varepsilon}(s).$$

The creep compliance, i.e., strain in the creep experiment (stress is assumed to be the Heaviside function), is

$$\tilde{J}(s) = \frac{1}{s}\frac{\Phi_\sigma(s)}{\Phi_\varepsilon(s)}.$$

Then, the glass compliance is

$$J_g := \lim_{t\to 0} J(t) = \lim_{s\to\infty}(s\tilde{J}(s)) = \lim_{s\to\infty}\frac{\Phi_\sigma(s)}{\Phi_\varepsilon(s)} = k^2 = \frac{1}{c^2},$$

with k from assumption (A5) and $J(t) = \mathcal{L}^{-1}(\tilde{J}(s))(t)$. The relaxation modulus, i.e., stress in the stress relaxation experiment (strain is assumed to be the Heaviside function), is connected to the creep compliance by

$$s\tilde{G}(s) = \frac{1}{s\tilde{J}(s)} = \frac{\Phi_\varepsilon(s)}{\Phi_\sigma(s)},$$

so that the glass modulus is

$$G_g := \lim_{t\to 0} G(t) = \lim_{s\to\infty}(s\tilde{G}(s)) = \frac{1}{J_g} = \lim_{s\to\infty}\frac{\Phi_\varepsilon(s)}{\Phi_\sigma(s)},$$

where $G(t) = \mathcal{L}^{-1}(\tilde{G}(s))(t)$. Hence, the wave speed in the distributed order fractional viscoelastic media is obtained as

$$c = \sqrt{G_g} = \frac{1}{\sqrt{J_g}}$$

if the glass modulus (compliance) is finite (non-zero), i.e., the wave speed is determined by the finite initial value of the stress (strain) in the stress relaxation (creep) experiment. If the glass modulus (compliance) is infinite (zero), then we only conclude that the fundamental solution takes the form (2.6) for all $x \in \mathbb{R}$, and $t > 0$, without a straightforward indication about the wave speed through the solution support properties.

3. Microlocal approach in analyzing time- and space-fractional wave equations

Tools of microlocal analysis are employed in order to investigate the propagation of singularities introduced by the initial conditions in case of the hereditary type wave equation, represented by the Zener wave equation, as well as for the non-local type of wave equations, represented by the non-local Hookean and Eringen wave equations. These three wave equations are studied in [13, 14].

3.1. Time-fractional Zener wave equation. The time-fractional Zener wave equation, obtained from the system of equations (2.1)–(2.3) for the choice of constitutive distributions

$$\phi_\sigma(\gamma) := 1 + a\delta(\gamma - \alpha), \quad \phi_\varepsilon(\gamma) := 1 + b\delta(\gamma - \alpha),$$

yielding constitutive equation (2.2) in the form

$$(1 + a\,{}_0\mathrm{D}_t^\alpha)\sigma(x,t) = (1 + b\,{}_0\mathrm{D}_t^\alpha)\varepsilon(x,t),$$

is rewritten as

$$Zu(x,t) = \partial_t^2 u(x,t) - L_t^\alpha \partial_x^2 u(x,t) = u_0(x) \otimes \delta'(t) + v_0(x) \otimes \delta(t), \tag{3.1}$$

where the operator L_t^α, considered as a convolution operator in one variable, is linear and bounded $L^p(\mathbb{R}) \to L^p(\mathbb{R})$, $1 < p < \infty$, by the Hörmander's multiplier theorem, cf. [11, Corollary 8.11] or [12, Theorem 7.9.5], since l_α, defined by

$$l_\alpha(\tau) = \frac{1 + b\,\mathrm{e}^{\mathrm{i}\frac{\alpha\pi}{2}}(\tau - \mathrm{i}0)^\alpha}{1 + a\,\mathrm{e}^{\mathrm{i}\frac{\alpha\pi}{2}}(\tau - \mathrm{i}0)^\alpha} = \frac{1 + b\,i^\alpha \operatorname{sgn}(\tau)|\tau|^\alpha}{1 + a\,i^\alpha \operatorname{sgn}(\tau)|\tau|^\alpha},$$

is in $L^\infty(\mathbb{R}) \cap C^1(\mathbb{R} \smallsetminus \{0\})$ with the derivative bounded by a constant times $|\tau|^{-1}$. Unique solvability of (3.1) by distributions supported in a forward cone has been established in [16]. Here we show a kind of non-characteristic regularity of the solution u to problem (3.1).

The "Fourier symbol" of Z is $z(\xi,\tau) = -\tau^2 + l_\alpha(\tau)\xi^2$ to which we apply a conic cut-off to obtain a smooth symbol in both variables (ξ,τ).

Lemma 3.1. *Let $\Gamma \subseteq \mathbb{R}^2$ (representing the (ξ,τ)-plane) be the union of a closed disc around $(0,0)$ and a closed narrow cone containing the ξ-axis and being symmetric with respect to both axes. Let Γ' be a closed set of the same shape as Γ, but with a slightly larger disc and opening angle of the cone. Let $\tilde{b} \in S^0(\mathbb{R}^2 \times \mathbb{R}^2)$ such that $\tilde{b}(x,t,\xi,\tau)$ is real, constant with respect to (x,t), homogenous of degree 0 with respect to (ξ,τ) away from the disc contained in Γ', and such that $\tilde{b}(x,t,\xi,\tau) = 0$, if $(\xi,\tau) \in \Gamma$, $\tilde{b}(x,t,\xi,\tau) = 1$, if $(\xi,\tau) \notin \Gamma'$. Then $p := \tilde{b}z$ is a symbol belonging to the class $S^2(\mathbb{R}^2 \times \mathbb{R}^2)$.*

Theorem 3.1. *For the wave front set of u^+, the restriction of the solution u to (3.1) to forward time $t > 0$, we have the inclusion*

$$\mathrm{WF}(u^+) \subseteq \Big\{(x,t;\xi,\tau) \mid x \in \mathbb{R}, t > 0, \xi \neq 0, \tau^2 = \frac{b}{a}\xi^2 \text{ or } \tau = 0\Big\}.$$

3.2. Non-local Hookean wave equation. Consider the system of governing equations consisting of the equation of motion (2.1), non-local fractional Hooke law (1.4) as a constitutive equation, written as

$$\sigma(x,t) = \mathcal{E}_x^\alpha u(x,t) = \frac{1}{2\Gamma(1-\alpha)} \frac{1}{|x|^\alpha} *_x \varepsilon(x,t),$$

in dimensionless form, and strain (2.3). We consider the solution to the non-local Hookean wave equation rewritten as

$$Zu(x,t) = \partial_t^2 u(x,t) - \partial_x \mathcal{E}_x^\beta u(x,t) = u_0(x) \otimes \delta'(t) + v_0(x) \otimes \delta(t). \tag{3.2}$$

with $u_0, v_0 \in \mathcal{E}'(\mathbb{R})$, when $0 < \beta < 1$, which takes the form

$$u = u_0 *_x \underbrace{\mathcal{F}^{-1}_{\xi\to x}\big[\cos\big(b_\beta|\xi|^{\frac{1+\beta}{2}}t\big)H(t)\big]}_{E_0} + v_0 *_x \underbrace{\mathcal{F}^{-1}_{\xi\to x}\left[\frac{\sin\big(b_\beta|\xi|^{\frac{1+\beta}{2}}t\big)}{b_\beta|\xi|^{\frac{1+\beta}{2}}}H(t)\right]}_{E_1},$$

where $b_\beta := \sqrt{\sin\frac{\beta\pi}{2}}$, with $\mathrm{supp}(u) \subseteq \{t \geqslant 0\}$.

Lemma 3.2. *For $j = 0, 1$, let E_j^+ denote the restriction of E_j to the open half-space $\{t > 0\}$. Then the wave front sets are given by*

$$\mathrm{WF}(E_0^+) = \mathrm{WF}(E_1^+) = \{(0, t; \xi, 0) \mid t > 0, \xi \neq 0\} =: W_0.$$

Based on the results of Lemma 3.2 we will investigate the influence of the singularities in the initial data u_0 and v_0 on the wave front set of the solution u to (3.2).

Theorem 3.2. *Let $u_0, v_0 \in \mathcal{E}'(\mathbb{R})$ and denote by u^+ the restriction of the solution u to (3.2) to the half-space of future time $\mathbb{R}\times]0, \infty[$, then $\mathrm{WF}(u^+)$ is invariant under translations $(x, t) \mapsto (x, t + s)$ with $s > 0$ and*

$$\mathrm{WF}(u^+) \subseteq \{(x, t; \xi, 0) \mid t > 0, (x, \xi) \in \mathrm{WF}(u_0) \textit{ or } (x, \xi) \in \mathrm{WF}(v_0)\}.$$

Moreover, in case v_0 is smooth we have a more precise statement

$$\mathrm{WF}(u^+) = \{(x, t; \xi, 0) \mid t > 0, (x, \xi) \in \mathrm{WF}(u_0)\},$$

and similarly $\mathrm{WF}(u^+) = \{(x, t; \xi, 0) \mid t > 0, (x, \xi) \in \mathrm{WF}(v_0)\}$, if u_0 is smooth.

For the proof of the theorem we need a technical lemma on "symbol corrections".

Lemma 3.3. *Let $\sigma \in (0, 1)$ and $y(\xi, \tau) = -\tau + b_\beta|\xi|^\sigma$. Let $\Gamma \subseteq \mathbb{R}^2$ (representing the (ξ, τ)-plane) be the union of a closed disc around $(0, 0)$ and a closed narrow cone containing the τ-axis and being symmetric with respect to both axes. Let Γ' be a closed set of the same shape as Γ, but with slightly larger disc and opening angle of the cone. Let $\tilde{b} \in S^0(\mathbb{R}^2 \times \mathbb{R}^2)$ such that $\tilde{b}(x, t, \xi, \tau)$ is real, constant with respect to (x, t), homogenous of degree 0 with respect to (ξ, τ) away from the disc contained in Γ', and such that $\tilde{b}(x, t, \xi, \tau) = 0$, if $(\xi, \tau) \in \Gamma$, $\tilde{b}(x, t, \xi, \tau) = 1$, if $(\xi, \tau) \notin \Gamma'$. Then $y\tilde{b}$ is a symbol belonging to the class $S^1(\mathbb{R}^2 \times \mathbb{R}^2)$.*

Remark 3.1. The result on the wave front set of u^+ in the above theorem implies, in particular, smoothness of u^+ considered as a map from time into distributions on space (cf. [10, (23.65.5)]), i.e., $u^+ \in C^\infty(]0, \infty[, \mathcal{D}'(\mathbb{R}))$; in addition, we have $u^+(t) \in \mathcal{S}'(\mathbb{R})$ for every $t > 0$.

3.3. Eringen wave equation. Consider the system of governing equations consisting of the equation of motion (2.1), fractional Eringen model (1.5) as a constitutive equation, written as

$$\sigma(x, t) - \mathrm{D}^\alpha \sigma(x, t) = \varepsilon(x, t), \quad \alpha \in (1, 3),$$

in dimensionless form, and strain (2.3). The fractional Eringen wave equation takes the form

$$\partial_t^2 u(x, t) - L_x^\alpha \partial_x^2 u(x, t) = 0, \tag{3.3}$$

where

$$L_x^\alpha w(x, t) = \mathcal{F}^{-1}_{\xi\to x}\left(\frac{1}{\sqrt{1 + a_\alpha|\xi|^\alpha}}\right) *_x w(x, t), \quad \alpha \in (1, 3),$$

with constant $a_\alpha = -\cos\frac{\alpha\pi}{2}$.

Suppose we had a classical solution $u_{\rm cl}$ to (3.3), which is C^2 for $t>0$ and of class C^1 for $t \geqslant 0$, with initial data

$$u_{\rm cl}\,|_{t=0}= u_0 \in C^1(\mathbb{R}), \quad \partial_t u_{\rm cl}\,|_{t=0}= v_0 \in C(\mathbb{R}),$$

and we put $u_{\rm cl}(x,t)=0$ for $t<0$. Then we might define the distribution

$$u(x,t) = u_{\rm cl}(x,t)H(t), \quad x,t\in\mathbb{R},$$

where H denotes the Heaviside function, so that u has support in $t \geqslant 0$ and satisfies the fractional Eringen wave equation in the form

$$\partial_t^2 u(x,t) - L_x^\alpha \partial_x^2 u(x,t) = u_0(x)\otimes\delta'(t) + v_0(x)\otimes\delta(t), \quad (x,t)\in\mathbb{R}^2,$$

with $\operatorname{supp}(u)$ being contained in forward time $t>0$. If u_0 and v_0 are also temperate then by the Fourier transform one has

$$\partial_t^2\hat{u} + \frac{\xi^2}{1+a_\alpha|\xi|^\alpha}\hat{u} = \hat{v_0}\otimes\delta + \hat{u_0}\otimes\delta'.$$

Considered as an ordinary differential equation in t with parameter ξ, the latter is solved by

$$\hat{u}(t) = \left(\hat{u_0}(\xi)\cos\left(\frac{|\xi|t}{\sqrt{1+a_\alpha|\xi|^\alpha}}\right) + \hat{v_0}(\xi)\frac{\sin\left(\frac{|\xi|t}{\sqrt{1+a_\alpha|\xi|^\alpha}}\right)}{\frac{|\xi|}{\sqrt{1+a_\alpha|\xi|^\alpha}}}\right)H(t).$$

We recall the standard Sobolev spaces $H^s(\mathbb{R}) = \mathcal{F}^{-1}L_s^2(\mathbb{R})$, where $s\in\mathbb{R}$ and $L_s^2(\mathbb{R})$ is the set of L^2-functions w such that $\xi\mapsto(1+\xi^2)^{s/2}w(\xi)$ belongs to L^2 as well. Let us consider the operator P acting on elements $u\in L^1_{\rm loc}(\mathbb{R},H^s(\mathbb{R}))\cap\mathcal{S}'(\mathbb{R}^2)$ by

$$Pu := \partial_t^2 u - L_x^\alpha\partial_x^2 u = \partial_t^2 u - \mathcal{F}^{-1}_{\xi\to x}\left(\frac{1}{\sqrt{1+a_\alpha|\xi|^\alpha}}\right) *_x \partial_x^2 u.$$

THEOREM 3.3. *Let $s\in\mathbb{R}$, $u_0\in H^s(\mathbb{R})$, and $v_0\in H^{s+1-\alpha/2}(\mathbb{R})$. Then*

$$Pu = u_0\otimes\delta' + v_0\otimes\delta$$

has a unique solution $u\in L^1_{\rm loc}(\mathbb{R},H^s(\mathbb{R}))\cap\mathcal{S}'(\mathbb{R}^2)$ with $\operatorname{supp} u\subseteq\{(x,t)\in\mathbb{R}^2 \mid t\geqslant 0\}$ and $u\in C^\infty((0,\infty);\mathcal{S}'(\mathbb{R}))\cap C((0,\infty);H^s(\mathbb{R}))$, given by

$$u(t) = u_0 *_x E_0(t) + v_0 *_x E_1(t),$$

where

$$E_0(t) := \mathcal{F}^{-1}_{\xi\to x}\left[\cos\left(\frac{|\xi|t}{\sqrt{1+a_\alpha|\xi|^\alpha}}\right)H(t)\right],$$

$$E_1(t) := \mathcal{F}^{-1}_{\xi\to x}\left[\frac{\sin\left(\frac{|\xi|t}{\sqrt{1+a_\alpha|\xi|^\alpha}}\right)}{\left(\frac{|\xi|}{\sqrt{1+a_\alpha|\xi|^\alpha}}\right)}H(t)\right].$$

REMARK 3.2. Both E_0 and E_1 are weakly smooth with respect to t when $t \neq 0$, which implies the property $u \in C^\infty((0,\infty); \mathcal{S}'(\mathbb{R}))$ for the solution given in the theorem above. Note that, in addition, we have that $t \mapsto E_1(t)$ is continuous $\mathbb{R}t\mathcal{S}'(\mathbb{R})$ with $E_1(0) = 0$, whereas $\lim_{t\to 0+} E_0(t) = \delta \neq 0 = \lim_{t\to 0-} E_0(t)$. However, E_0 is weakly measurable with respect to $t \in \mathbb{R}$.

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ИСТОРИЈСКИ И НЕЛОКАЛНИ ЕФЕКТИ У МОДЕЛИМА ПРОСТИРАЊА ТАЛАСА

Резиме. Класична таласна једначина је уопштена у оквиру теорије фракционог рачуна узимањем у обзир меморијских и нелокалних својстава материјала. Оба својства су укључена кроз конститутивну једначину, док једначина кретања једнодимензионог континуума и деформација нису уопштаване. Меморијски ефекти вискоеластичних материјала су моделирани фракционим изводима расподељеног реда, тако да конститутивна релација уопштава све моделе линеарне вискоеластичности уколико редови извода не прелазе први извод. Простирање сингуларитета анализирано је коришћењем алата микролокалне анализе у случају фракционог Ценеровог модела вискоеластичног материјала, као и у случајевима нелокалних материјала моделираних нелокалним Хуковим законом, као и фракционим Ерингеновим моделом.

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