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LOCAL EXISTENCE AND BLOW-UP OF SOLUTIONS FOR COUPLED VISCOELASTIC WAVE EQUATIONS WITH DEGENERATE DAMPING TERMS

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Abstract. In this paper, we investigate a nonlinear system of viscoelastic equations with degenerate damping and source terms in a bounded domain. Under appropriate assumptions on the parameters, degenerate damping terms and the relaxation functions ϖ_i, $(i = 1, 2)$, we prove local existence and uniqueness of the solution by using the Faedo–Galerkin method with a new scenario. Then, we prove the blow-up of weak solutions to problem (1.1). This improves earlier results in the literature [6, 23, 25].

1. Introduction and preliminaries

In this article, we consider the following coupled system of hyperbolic equations with degenerate damping terms and strong nonlinear sources

$$
(1.1)\qquad \begin{cases} u_{tt} - \Delta u + \int_0^t \varpi_1(t-s)\Delta u(s)ds \\ \qquad\qquad +(|u|^k + |v|^l)|u_t|^{p-1}u_t = f_1(u,v), \quad (x,t) \in \Omega \times (0,T), \\ v_{tt} - \Delta v + \int_0^t \varpi_2(t-s)\Delta v(s)ds \\ \qquad\qquad +(|v|^\theta + |u|^\varrho)|v_t|^{q-1}v_t = f_2(u,v), \quad (x,t) \in \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in \Omega, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \qquad x \in \Omega, \end{cases}
$$

where $\Omega \subset R^n$ $(n = 1, 2, 3)$ is a bounded domain with smooth boundary $\partial\Omega$, and $f_i(.,.)\colon R^2 \longrightarrow R$, $\varpi_i(.)\colon R^+ \to R^+$ $(i = 1, 2)$ are given functions to be specified later.

Problems of this type of evolution equations with damping are numerous in the context of material science and physics. For instance, in the context of fluid flows, viscosity effects often arise as damping terms in evolution equations. The situation

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is, however, different when the damping term is degenerate, leading to the degeneracy of the monotonicity argument (problems of this type often arise in physics when the friction is modulated by the strains). In addition, in classical mechanics, when it comes to the physicalv problems of vibrating membranes, strings or shells in elastic media, damping terms reflect the internal energy that is dissipated by the motion (see [21]).

In recent years, numerous mathematicians have paid attention to the dynamic systems of wave equations with degenerate damping terms, viscoelastic equations and hyperbolic systems.

Problem (1.1) was studied by Wu [25], who obtained a general decay of solutions. Also, there are many more works considering the case of $k = l = \theta = \varrho = 0$ for problem (1.1).

Without viscoelastic terms ($\varpi_i \equiv 0$, $i = 1, 2$), system (1.1) becomes the following famous problem

$$\begin{cases} u_{tt} - \Delta u + (|u|^k + |v|^l)|u_t|^{p-1}u_t = f_1(u, v), \\ v_{tt} - \Delta v + (|v|^\theta + |u|^\varrho)|v_t|^{q-1}v_t = f_2(u, v). \end{cases} \tag{1.2}$$

Rammaha and Sakuntasathien [23] investigated the well-posedness of solutions to problem (1.2). Later, in [3, 27], authors studied the same problem treated in [23], and considered the exponential growth and blow-up properties.

Other hyperbolic equations with degenerate damping terms are investigated by numerous authors, see [4, 8, 12, 16–18, 26, 28].

When $k = l = \theta = \varrho = 0$, system (1.1) becomes the following problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t \varpi_1(t-s)\Delta u(s)ds + |u_t|^{p-1}u_t = f_1(u, v), \\ v_{tt} - \Delta v + \int_0^t \varpi_2(t-s)\Delta v(s)ds + |v_t|^{q-1}v_t = f_2(u, v). \end{cases} \tag{1.3}$$

There are many more works related to problem (1.3) in the literature. For studies related to the existence, blow-up and decay of the solutions, we refer the interested readers to [5, 6, 14, 19, 20, 24]. Messaoudi and Tatar [13] studied uniform decay for problem (1.3) without damping terms and in [11] Liu improved the results previously obtained in [13] to weaker conditions on the relaxation functions and for more general forms of nonlinearities.

As far as the present authors are concerned, the model in the current study is both quite difficult and interesting and the analysis more subtle due to the degenerate damping term and relaxation functions ϖ_i, ($i = 1, 2$).

This paper consists of four sections, including the introduction and preliminaries. In Section 1, firstly, we introduce our problem and give some related problems. Next, we state the assumptions and definition of weak solution to problem (1.1), and then we give theorems about the main results. In Sections 2 to 4 are devoted to the proofs of the main results.

Now, we will present some assumptions, notations, and known results to state the main results. We denote $|u|_{s,\Omega} = |.|_{H^s(\Omega)}$, $|.|_p = |u|_{L^p(\Omega)}$ and $\langle u, v\rangle = \langle u, v\rangle_{L^2(\Omega)}$. Also, the following Sobolev embedding will be used often, and sometimes without mention

$$\begin{cases} H_0^1(\Omega) \hookrightarrow L^q(\Omega), & \text{for } 1 \leqslant q \leqslant 6,\ n=3, \\ H_0^1(\Omega) \hookrightarrow L^q(\Omega), & \text{for } 1 \leqslant q < \infty,\ n=1,2. \end{cases} \tag{1.4}$$

We need the following assumptions to state and prove our results.

(A1) For nonlinearity in damping, we suppose that

(1) The exponents $p, q > 0$. In addition

$$\begin{cases} r \geqslant 3 & \text{if } n=1,2, \\ r = 3 & \text{if } n=3. \end{cases}$$

(2) The parameters $k, l, \theta, \varrho \geqslant 1$, and, if $n=3$, we require

$$\max\{k,l\} \leqslant 3(1-p) \quad \text{and} \quad \max\{\theta,\varrho\} \leqslant 3(1-q).$$

(3) The initial conditions $u_0, v_0 \in H_0^1(\Omega)$, $u_1, v_1 \in L^2(\Omega)$.

(4) There exist c_0, c_1 such that

$$c_0(|u|^{r+1} + |v|^{r+1}) \leqslant F(u,v) \leqslant c_1(|u|^{r+1} + |v|^{r+1}), \tag{1.5}$$

where $uf_1(u,v) + vf_2(u,v) = (r+1)F(u,v)$ for all $(u,v) \in \mathbb{R}^2$.

We pick up the functions $f_1(u,v)$ and $f_2(u,v)$ as follows

$$\begin{aligned} f_1(u,v) &= (r+1)[a|u+v|^{r-1}(u+v) + b|u|^{\frac{r-3}{2}}u|v|^{\frac{r+1}{2}}], \\ f_2(u,v) &= (r+1)[a|u+v|^{r-1}(u+v) + b|v|^{\frac{r-3}{2}}v|u|^{\frac{r+1}{2}}], \end{aligned} \tag{1.6}$$

where $a, b > 0$ are constants and

$$F(u,v) = a|u+v|^{r+1} + 2b|uv|^{\frac{r+1}{2}}.$$

(A2) The relaxation functions ϖ_i $(i=1,2)$ are of class C^1 and satisfy, for $s \geqslant 0$,

$$\varpi_i(s) \geqslant 0, \quad \varpi_i'(s) \leqslant 0, \quad \int_0^\infty \varpi_i(s)ds < 1. \tag{1.7}$$

(A3) Assume that $\varpi_i(s)$ satisfy

$$\max\left\{\int_0^\infty \varpi_1(s)ds, \int_0^\infty \varpi_2(s)ds\right\} < \frac{r-1}{r+1}, \quad i=1,2. \tag{1.8}$$

Throughout this article, we use the following notations

$$\begin{aligned} 0 < \rho_i = 1 - \int_0^\infty \varpi_i(\tau)d\tau < 1, \quad (i=1,2), \\ (\varpi_i \diamond \vartheta)(t) = \int_0^t \varpi_i(t-\tau)|\vartheta(t) - \vartheta(\tau)|_2^2 d\tau, \quad (i=1,2), \\ \rho = \min\{\rho_1, \rho_2\}. \end{aligned} \tag{1.9}$$

Firstly, we present the definition of a weak solution to problem (1.1).

Definition 1.1. Under the stated assumptions, a pair of functions (u,v) is said to be a weak solution to (1.1) on the interval $[0,T]$ if

(1) $u, v \in C_w([0,T], H_0^1(\Omega))$,
(2) $u_t, v_t \in C_w([0,T], L^2(\Omega))$,
(3) $(u(0), v(0)) = (u^0, v^0) \in H_0^1(\Omega) \times H_0^1(\Omega)$,
(4) $(u_t(0), v_t(0)) = (u^1, v^1) \in L^2(\Omega) \times L^2(\Omega)$,
(5) (u, v) satisfies

$$\begin{cases} \langle u'(t), \varphi\rangle - \langle u^1, \varphi\rangle + \int_0^t \langle \nabla u(s), \nabla\varphi\rangle ds \\ \qquad - \int_0^t \langle \int_0^s \varpi_1(s-\tau)\nabla u(\tau)d\tau, \nabla\varphi\rangle ds \\ \qquad + \int_0^t \langle (|u|^k + |v|^l)|u'(s)|^{p-1}u'(s), \varphi\rangle ds \\ \quad = \int_0^t \langle f_1(u(s), v(s)), \varphi\rangle ds, \\ \langle v'(t), \vartheta\rangle - \langle v^1, \vartheta\rangle + \int_0^t \langle \nabla v(s), \nabla\vartheta\rangle ds \\ \qquad - \int_0^t \langle \int_0^s \varpi_2(s-\tau)\nabla v(\tau)d\tau, \nabla\vartheta\rangle ds \\ \qquad + \int_0^t \langle (|v|^\theta + |u|^\varrho)|v'(s)|^{q-1}v'(s), \vartheta\rangle ds \\ \quad = \int_0^t \langle f_2(u(s), v(s)), \vartheta\rangle ds, \end{cases} \tag{1.10}$$

for all test functions $\varphi, \vartheta \in H_0^1(\Omega)$ and for almost all $t \in [0,T]$.

The next theorem deals with the existence and uniqueness of a local weak solution to (1.1) in the sense of Definition 1.1.

Theorem 1.1 (Local Weak Solution). *Assume* (A1)–(A3) *hold. Then, there exists a unique local weak solution* (u, v) *to* (1.1) *defined on* $[0, T_0]$ *for some* $T_0 > 0$. *In addition, the said solution satisfies the energy identity*

$$\begin{aligned} E(t) &+ \frac{1}{2}\int_0^t [\varpi_1(s)|\nabla u(s)|^2 + \varpi_2(s)|\nabla v(s)|^2] ds \\ &- \frac{1}{2}\int_0^t [(\varpi_1' \diamond \nabla u)(s) + (\varpi_2' \diamond \nabla v)(s)] ds \\ &+ \int_0^t \int_\Omega (|u(s)|^k + |v(s)|^l)|u'(s)|^{p+1} dx\, ds \\ &+ \int_0^t \int_\Omega (|v(s)|^\theta + |u(s)|^\varrho)|v'(s)|^{q+1} dx\, ds = E(0), \end{aligned} \tag{1.11}$$

where

$$\begin{aligned} E(t) = &\frac{1}{2}(|u_t|^2 + |v_t|^2) \\ &+ \frac{1}{2}\left[(1 - \int_0^t \varpi_1(s)ds)|\nabla u(t)|^2 + (1 - \int_0^t \varpi_2(s)ds)|\nabla v(t)|^2\right] \\ &+ \frac{1}{2}[(\varpi_1 \diamond \nabla u)(t) + (\varpi_2 \diamond \nabla v)(t)] - \int_\Omega F(u(t), v(t))dx. \end{aligned} \tag{1.12}$$

The following result is about a finite time blow-up property of the weak solutions to (1.1).

Theorem 1.2 (Blow-up of Solutions). *Assume that* (A1)–(A3) *and further that*

$$r > \max\{k + p, l + p, \theta + q, \varrho + q\}$$

and $E(0) < 0$, *where* $E(0)$ *is the initial energy given by*

$$E(0) = \frac{1}{2}(|u^1|^2 + |v^1|^2 + |\nabla u^0|^2 + |\nabla v^0|^2) - \int_\Omega F(u^0, v^0)dx.$$

Then, the weak solution (u, v) *in Theorem* 1.1 *blows up in finite time.*

We state the elementary inequalities which will be used throughout this paper as follows

$$(|x|^{p-1}x - |y|^{p-1}y)(x - y) \geqslant C|x - y|^{p+1}, \tag{1.13}$$

with some constant $C > 0$, all $p > 0$, and all $x, y \in R$, In addition,

$$||x|^l - |y|^l| \leqslant C|x - y|(|x|^{l-1} + |y|^{l-1}), \tag{1.14}$$

with some constant $C > 0$, all $l \geqslant 1$, and all $x, y \in \mathbb{R}$. Also,

$$||x|^m x - |y|^m y| \leqslant C|x - y|(|x|^m + |y|^m), \tag{1.15}$$

where some constant $C > 0$, all $m \geqslant 0$, and all $x, y \in \mathbb{R}$. We obtain the following useful inequalities by using (1.14), (1.15), f_1, f_2 and F

$$\begin{aligned} |f_1(u,v) - f_1(\tilde{u},\tilde{v})| \leqslant{}& C_0(|u - \tilde{u}| + |v - \tilde{v}|)(|u|^{r-1} + |v|^{r-1} + |\tilde{u}|^{r-1} + |\tilde{v}|^{r-1}) \\ &+ C_1\Big[|v - \tilde{v}||u|^{\frac{r-1}{2}}\big(|v|^{\frac{r-1}{2}} + |\tilde{v}|^{\frac{r-1}{2}}\big) \\ &\qquad + |u - \tilde{u}||\tilde{v}|^{\frac{r+1}{2}}\big(|u|^{\frac{r-3}{2}} + |\tilde{u}|^{\frac{r-3}{2}}\big)\Big], \end{aligned} \tag{1.16}$$

$$\begin{aligned} |f_2(u,v) - f_2(\tilde{u},\tilde{v})| \leqslant{}& C_0(|u - \tilde{u}| + |v - \tilde{v}|)(|u|^{r-1} + |v|^{r-1} + |\tilde{u}|^{r-1} + |\tilde{v}|^{r-1}) \\ &+ C_1\Big[|u - \tilde{u}||v|^{\frac{r-1}{2}}\big(|u|^{\frac{r-1}{2}} + |\tilde{u}|^{\frac{r-1}{2}}\big) \\ &\qquad + |v - \tilde{v}||\tilde{u}|^{\frac{r+1}{2}}\big(|v|^{\frac{r-3}{2}} + |\tilde{v}|^{\frac{r-3}{2}}\big)\Big], \end{aligned} \tag{1.17}$$

$$\begin{aligned} |F(u,v) - F(\tilde{u},\tilde{v})| \leqslant{}& C_0(|u - \tilde{u}| + |v - \tilde{v}|)(|u|^r + |v|^r + |\tilde{u}|^r + |\tilde{v}|^r) \\ &+ C_1(|u - \tilde{u}||v| + |\tilde{u}||v - \tilde{v}|)\big(|u|^{\frac{r-1}{2}}|v|^{\frac{r-1}{2}} + |\tilde{u}|^{\frac{r-1}{2}}|\tilde{v}|^{\frac{r-1}{2}}\big), \end{aligned} \tag{1.18}$$

where $\forall u, v, \tilde{u}, \tilde{v} \in \mathbb{R}$, and C_0 and C_1 are positive constants.

2. Local solution and uniqueness

In this section, we prove Theorem 1.1, following the next steps.

Step 1: Approximate solution. We use the standard Galerkin approximation to prove our result. Let $A = -\Delta$ with zero Dirichlet boundary condition and its domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. It is well known that A is positive, self-adjoint and it is also the inverse of a compact operator. Besides, A has an infinite sequence of positive eigenvalues $\{\lambda_j : j = 1, 2, \dots\}$ and a corresponding sequence of eigenfunctions $\{e_j : j = 1, 2, \dots\}$ that forms an orthonormal basis for $L^2(\Omega)$. Also, the sequence $\{e_j : j = 1, 2, \dots\}$ is an orthogonal basis for $H_0^1(\Omega)$.

Let V_N = the linear span of $\{e_N\}$ and P_N be the orthogonal projection of $L^2(\Omega)$ onto V_N. In order to establish the existence of a local weak solution to the system (1.1) we will use a standard Galerkin approximation scheme based on the eigenfunctions $\{e_j\}_{j=1}^{\infty}$ of the operator $A = -\Delta$. More precisely, let

$$u_N(t) = \sum_{j=1}^{N} u_{N,j}(t) e_j$$

and

$$v_N(t) = \sum_{j=1}^{N} v_{N,j}(t) e_j$$

be the approximate solutions to (1.1) in V_N, then, $u_N(t), v_N(t)$ satisfy the following system of ODEs

$$\begin{cases} \langle u_N''(t), e_j\rangle + \langle \nabla u_N(t), \nabla e_j\rangle - \langle \int_0^t \varpi_1(t-s)\nabla u_N(s)ds, \nabla e_j\rangle \\ \qquad +\langle (|u_N(t)|^k + |v_N(t)|^l)|u_N'(t)|^{p-1}u_N'(t), e_j\rangle \\ \qquad = \langle f_1(u_N(t), v_N(t)), e_j\rangle, \\ \langle v_N''(t), e_j\rangle + \langle \nabla v_N(t), \nabla e_j\rangle - \langle \int_0^t \varpi_2(t-s)\nabla v_N(s)ds, \nabla e_j\rangle \\ \qquad +\langle (|v_N(t)|^\theta + |u_N(t)|^\varrho)|v_N'(t)|^{q-1}v_N'(t), e_j\rangle \\ \qquad = \langle f_2(u_N(t), v_N(t)), e_j\rangle, \\ u_N(0) = P_N u^0, \quad v_N(0) = P_N v^0, \quad u_N'(0) = P_N u^1, \quad v_N'(0) = P_N v^1, \end{cases} \tag{2.1}$$

for $j = 1, 2, \ldots, N$. More particularly, $(2.1)_3$ is equal to

$$u_{N,j}(0) = u_j^0, \quad v_{N,j}(0) = v_j^0, \quad u_{N,j}'(0) = u_j^1, \quad v_{N,j}'(0) = v_j^1 \tag{2.2}$$

for

$$u_j^0 = \langle u^0, e_j\rangle$$

$$v_j^0 = \langle v^0, e_j\rangle$$

$$u_j^1 = \langle u^1, e_j\rangle$$

and

$$v_j^1 = \langle v^1, e_j\rangle,$$

for $j = 1, 2, \ldots, N$.

It is clear that (2.1) is an initial value problem for a second order $2N \times 2N$ system of ODEs with continuous nonlinearities in the unknown functions $u_{N,j}$, $v_{N,j}$ and their derivatives. Thus, it follows from the Picard's iteration method that the system (2.1) has a solution $u_{N,j}, v_{N,j} \in C^2[0, T_N]$ for some $T_N > 0$.

Now, we shall get the priori estimate to show that T_N can be replaced by some $T > 0$, for all $N \geqslant 1$.

LEMMA 2.1. *Let $T > 0$ be a constant such that the sequences of approximate solutions $\{u_N\}$ and $\{v_N\}$ satisfy the following*

(1) *$\{u_N\}$, $\{v_N\}$ are uniformly bounded sequences in $L^\infty(0, T; H_0^1(\Omega))$.*
(2) *$\{u_N'\}$, $\{v_N'\}$ are uniformly bounded sequences in $L^\infty(0, T; L^2(\Omega))$.*

(3) *The sequences*

$$\left\{\int_0^t\int_\Omega(|u_N(s)|^k+|v_N(s)|^l)|u_N'(s)|^{p+1}dx\,ds\right\}$$

and

$$\left\{\int_0^t\int_\Omega(|v_N(s)|^\theta+|u_N(s)|^\varrho)|v_N'(s)|^{q+1}dx\,ds\right\}$$

are uniformly bounded in $L^\infty(0,T)$.

Proof. Multiply $(2.1)_1$ by $u_{N,j}'(t)$, $(2.1)_1$ by $v_{N,j}'(t)$, and summing with respect to j from 1 to N, we get

$$(2.3)\quad \begin{cases} \frac{1}{2}\frac{d}{dt}(|u_N'(t)|^2+(1-\int_0^t\varpi_1(s)ds)|\nabla u_N(t)|^2+(\varpi_1\diamond\nabla u_N)(t)) \\ \qquad +\frac{1}{2}\varpi_1(t)|\nabla u_N(t)|^2-\frac{1}{2}(\varpi_1'\diamond\nabla u_N)(t) \\ \qquad +\int_\Omega(|u_N(t)|^k+|v_N(t)|^l)|u_N'(t)|^{p+1}dx \\ \quad =\int_\Omega f_1(u_N(t),v_N(t))u_N'(t)dx, \\ \frac{1}{2}\frac{d}{dt}(|v_N'(t)|^2+(1-\int_0^t\varpi_2(s)ds)|\nabla v_N(t)|^2+(\varpi_2\diamond\nabla v_N)(t)) \\ \qquad +\frac{1}{2}\varpi_2(t)|\nabla v_N(t)|^2-\frac{1}{2}(\varpi_2'\diamond\nabla v_N)(t) \\ \qquad +\int_\Omega(|v_N(t)|^\theta+|u_N(t)|^\varrho)|v_N'(t)|^{q+1}dx \\ \quad =\int_\Omega f_2(u_N(t),v_N(t))v_N'(t)dx. \end{cases}$$

Summing and integrating (2.3) from 0 to $t\leqslant T_N$, we obtain

$$\begin{aligned}(2.4)\quad &\frac{1}{2}\left[|u_N'(t)|^2+|v_N'(t)|^2+\left(1-\int_0^t\varpi_1(s)ds\right)|\nabla u_N(t)|^2\right.\\ &\quad\left.+\left(1-\int_0^t\varpi_2(s)ds\right)|\nabla v_N(t)|^2\right]\\ &\quad+\frac{1}{2}[(\varpi_1\diamond\nabla u_N)(t)+(\varpi_2\diamond\nabla v_N)(t)]\\ &\quad+\frac{1}{2}\int_0^t[\varpi_1(s)|\nabla u_N(s)|^2+\varpi_2(s)|\nabla v_N(s)|^2]ds\\ &\quad-\frac{1}{2}\int_0^t[(\varpi_1'\diamond\nabla u_N)(s)+(\varpi_2'\diamond\nabla v_N)(s)]ds\\ &\quad+\int_0^t\int_\Omega(|u_N(s)|^k+|v_N(s)|^l)|u_N'(s)|^{p+1}dx\,ds\\ &\quad+\int_0^t\int_\Omega(|v_N(s)|^\theta+|u_N(s)|^\varrho)|v_N'(s)|^{q+1}dx\,ds\\ &=\frac{1}{2}[|u_N'(0)|^2+|v_N'(0)|^2+|\nabla u_N(0)|^2+|\nabla v_N(0)|^2]\\ &\quad+\int_0^t\int_\Omega[f_1(u_N(s),v_N(s))u_N'(s)+f_2(u_N(s),v_N(s))v_N'(s)]dx\,ds\end{aligned}$$

$$\leqslant C_0 + \int_0^t \int_\Omega [f_1(u_N(s), v_N(s))u_N'(s) + f_2(u_N(s), v_N(s))v_N'(s)]dx\, ds,$$

where positive constant $C_0 = C(|u^0|_{1,\Omega}, |v^0|_{1,\Omega}, |u^1|_{0,\Omega}, |v^1|_{0,\Omega})$ and we have used in (2.4) the fact that

$$\begin{aligned} &u_N(0) \to u^0, \quad v_N(0) \to v^0 \quad \text{strongly in } H_0^1(\Omega) \\ &u_N'(0) \to u^1, \quad v_N'(0) \to v^1 \quad \text{strongly in } L^2(\Omega). \end{aligned} \tag{2.5}$$

To estimate the last term in (2.4) applying (1.6) and using Hölder and Young's inequalities and Sobolev embedding in (1.4), we have

$$\begin{aligned} \left| \int_\Omega f_1(u_N, v_N)u_N' dx \right| &\leqslant C \int_\Omega \left(|u_N + v_N|^r |u_N'| + |v_N|^{\frac{r+1}{2}} |u_N|^{\frac{r-1}{2}} |u_N'| \right) dx \\ &\leqslant C \Big[(|u_N|_{2r}^r + |v_N|_{2r}^r)|u_N'| + |u_N|_{3(r-1)}^{\frac{r-1}{2}} |v_N|_{\frac{3(r+1)}{2}}^{\frac{r+1}{2}} |u_N'| \Big] \\ &\leqslant C \Big[|u_N|_{2r}^{2r} + |v_N|_{2r}^{2r} + |u_N|_{3(r-1)}^{r-1} |v_N|_{\frac{3(r+1)}{2}}^{r+1} + |u_N'|^2 \Big] \\ &\leqslant C[|\nabla u_N|^{2r} + |\nabla v_N|^{2r} + |\nabla u_N|^{r-1} |\nabla v_N|^{r+1} + |u_N'|^2], \end{aligned} \tag{2.6}$$

where we have used the fact that when $n = 3$ then $2r = 3(r-1) = \frac{3(r+1)}{2} = 6$.

In the same way, we obtain

$$\left| \int_\Omega f_2(u_N, v_N)v_N' dx \right| \leqslant C\big[|\nabla u_N|^{2r} + |\nabla v_N|^{2r} + |\nabla v_N|^{r-1}|\nabla u_N|^{r+1} + |v_N'|^2\big]. \tag{2.7}$$

Let $y_N(t) := |u_N'(t)|^2 + |v_N'(t)|^2 + |\nabla u_N(t)|^2 + |\nabla v_N(t)|^2$. Then, we infer from (2.4)–(2.7), using assumption (1.9), that

$$\begin{aligned} y_N(t) &+ (\varpi_1 \diamond \nabla u_N)(t) + (\varpi_2 \diamond \nabla v_N)(t) \\ &+ \frac{1}{\rho} \int_0^t [\varpi_1(s)|\nabla u_N(s)|^2 + \varpi_2(s)|\nabla v_N(s)|^2] ds \\ &- \frac{1}{\rho} \int_0^t [(\varpi_1' \diamond \nabla u_N)(s) + (\varpi_2' \diamond \nabla v_N)(s)] ds \\ &+ \frac{2}{\rho} \int_0^t \int_\Omega (|u_N(s)|^k + |v_N(s)|^l)|u_N'(s)|^{p+1} dx\, ds \\ &+ \frac{2}{\rho} \int_0^t \int_\Omega (|v_N(s)|^\theta + |u_N(s)|^\varrho)|v_N'(s)|^{q+1} dx\, ds \\ &\leqslant C_0 + C \int_0^t y_N(s)^r ds, \end{aligned} \tag{2.8}$$

where $C_0 = C(|u^0|_{1,\Omega}, |v^0|_{1,\Omega}, |u^1|_{0,\Omega}, |v^1|_{0,\Omega}) > 0$. Particularly, $y_N(t)$ satisfies the inequality

$$y_N(t) \leqslant C_0 + C \int_0^t y_N(s)^r ds. \tag{2.9}$$

Therefore, using a standard comparison Theorem (see, for instance, [10]), (2.9) yields $y_N(t) \leqslant z(t)$, where $z(t) = [C_0^{1-r} - C(r-1)t]^{-\frac{1}{r-1}}$ is the solution to the Volterra integral equation

$$z(t) = C_0 + C\int_0^t z(s)^r ds. \tag{2.10}$$

Although $z(t)$ blows up in finite time (since $r \geqslant 3$), nonetheless, there exists a time $0 < T < T_N$ such that

$$y_N(t) \leqslant z(t) \leqslant C_1 \quad \text{for all } t \in [0,T],$$

where C_1 is a positive constant independent of N. Therefore, for all $N \geqslant 1$, we obtain

$$y_N(t) \leqslant C_1 \quad \text{for all } t \in [0,T],$$

which established the first two parts of Lemma 2.1. The last part immediately follows from (2.8). □

Remark 2.1. The proof of Lemma 2.1 is not based on the assumption that the exponents of velocities satisfy $0 < p, q < 1$. Thus, the statements in Lemma 2.1 are true for all values $p, q > 0$. Now we easily conclude from Lemma 2.1 that there exist subsequences of (u_N, v_N) still denoted by (u_N, v_N) such that

$$\begin{cases} u_N \to u,\ v_N \to v & \text{weakly } * \text{ in } L^\infty(0,T,H_0^1(\Omega)),\\ u_N' \to u',\ v_N' \to v' & \text{weakly } * \text{ in } L^\infty(0,T,L^2(\Omega)) \end{cases} \tag{2.11}$$

as $N \to \infty$ for some $u, v \in L^\infty(0,T,H_0^1(\Omega))$.

Furthermore, we can get more information about the convergence of the approximate solutions. Indeed, we have the following strong convergence result.

Lemma 2.2. *The sequences of approximate solutions u_N and v_N satisfy the following*

$$\begin{cases} \{u_N\},\{v_N\} \text{ are Cauchy sequences in } L^\infty(0,T,H_0^1(\Omega)),\\ \{u_N'\},\{v_N'\} \text{ are Cauchy sequences in } L^\infty(0,T,L^2(\Omega)). \end{cases} \tag{2.12}$$

Proof. We take two families of approximate solutions (u_N, v_N) and (u_L, v_L). Without loss of generality, we get $N > L$. Set

$$u_{NL}(t) = u_N(t) - u_L(t) = \sum_{j=1}^N (u_{N,j}(t) - u_{L,j}(t))e_j$$

and

$$v_{NL}(t) = v_N(t) - v_L(t) = \sum_{j=1}^N (v_{N,j}(t) - v_{L,j}(t))e_j,$$

with

$$u_{L,j}(t) = v_{L,j}(t) \equiv 0, \quad \text{as } j > L.$$

Then, u_{NL} and v_{NL} verify

$$\left\{\begin{array}{l}\langle u''_{NL}, e_j\rangle + \langle \nabla u_{NL}, \nabla e_j\rangle - \langle \int_0^t \varpi_1(t-s)\nabla u_{NL}(s)ds, \nabla e_j\rangle \\ \qquad +\langle (|u_N|^k + |v_N|^l)|u'_N|^{p-1}u'_N - (|u_L|^k + |v_L|^l)|u'_L|^{p-1}u'_L, e_j\rangle \\ \qquad = \langle f_1(u_N, v_N) - f_1(u_L, v_L), e_j\rangle, \\ \langle v''_{NL}, e_j\rangle + \langle \nabla v_{NL}, \nabla e_j\rangle - \langle \int_0^t \varpi_2(t-s)\nabla v_{NL}(s)ds, \nabla e_j\rangle \\ \qquad +\langle (|v_N|^\theta + |u_N|^\varrho)|v'_N|^{q-1}v'_N - (|v_L|^\theta + |u_L|^\varrho)|v'_L|^{q-1}v'_L, e_j\rangle \\ \qquad = \langle f_2(u_N, v_N) - f_2(u_L, v_L), e_j\rangle, \\ u_{NL}(0) = P_N u^0 - P_L u^0, \quad v_{NL}(0) = P_N v^0 - P_L v^0, \\ u'_{NL}(0) = P_N u^1 - P_L u^1, \quad v'_{NL}(0) = P_N v^1 - P_L v^1,\end{array}\right. \tag{2.13}$$

for $j = 1, 2, \ldots, N$.

Multiply $(2.13)_1$ by $u'_{N,j}(t) - u'_{L,j}(t)$ and $(2.13)_1$ by $v'_{N,j}(t) - v'_{L,j}(t)$ and sum over j from 1 to N, to deduce

$$\left\{\begin{array}{l}\frac{1}{2}\frac{d}{dt}(|u'_{NL}(t)|^2 + (1 - \int_0^t \varpi_1(s)ds)|\nabla u_{NL}(t)|^2 \\ \qquad +(\varpi_1 \diamond \nabla u_{NL})(t)) + \frac{1}{2}\varpi_1(t)|\nabla u_{NL}(t)|^2 - \frac{1}{2}(\varpi'_1 \diamond \nabla u_{NL})(t) \\ \qquad +\langle (|u_N|^k + |v_N|^l)|u'_N|^{p-1}u'_N - (|u_L|^k + |v_L|^l)|u'_L|^{p-1}u'_L, u'_{NL}\rangle \\ \qquad = \langle f_1(u_N(t), v_N(t)) - f_1(u_L(t), v_L(t)), u'_{NL}(t)\rangle \\ \frac{1}{2}\frac{d}{dt}(|v'_{NL}(t)|^2 + (1 - \int_0^t \varpi_2(s)ds)|\nabla v_{NL}(t)|^2) \\ \qquad +(\varpi_2 \diamond \nabla v_{NL})(t) + \frac{1}{2}\varpi_2(t)|\nabla v_{NL}(t)|^2 - \frac{1}{2}(\varpi'_2 \diamond \nabla v_{NL})(t) \\ \qquad +\langle (|v_N|^\theta + |u_N|^\varrho)|v'_N|^{q-1}v'_N - (|v_L|^\theta + |u_L|^\varrho)|v'_L|^{q-1}v'_L, v'_{NL}\rangle \\ \qquad = \langle f_2(u_N(t), v_N(t)) - f_2(u_L(t), v_L(t)), v'_{NL}(t)\rangle.\end{array}\right. \tag{2.14}$$

We can use (1.16)–(1.17) to estimate the last terms in (2.14) as follows:

$$|\langle f_1(u_N(t), v_N(t)) - f_1(u_L(t), v_L(t)), u'_{NL}(t)\rangle| \leqslant G_1 + G_2 + G_3, \tag{2.15}$$

where

$$G_1 = C_0 \int_\Omega (|u_{NL}| + |v_{NL}|)(|u_N|^{r-1} + |v_N|^{r-1} + |u_L|^{r-1} + |v_L|^{r-1})|u'_{NL}|dx,$$
$$G_2 = C_1 \int_\Omega |v_{NL}||u_N|^{\frac{r-1}{2}}\left(|v_N|^{\frac{r-1}{2}} + |v_L|^{\frac{r-1}{2}}\right)|u'_{NL}|dx,$$
$$G_3 = C_1 \int_\Omega |u_{NL}||v_L|^{\frac{r+1}{2}}\left(|u_N|^{\frac{r-3}{2}} + |u_L|^{\frac{r-3}{2}}\right)|u'_{NL}|dx.$$

All terms in G_1 are estimated in a similar way, by applying Hölder' inequality, Sobolev embedding Theorem,Young's inequality and the boundedness statement in Lemma 2.1. Particularly, for a typical term in G_1, we get

$$\begin{aligned}\int_\Omega (|u_{NL}| + |v_{NL}|)|u_N|^{r-1}|u'_{NL}|dx &\leqslant (|u_{NL}|_6 + |v_{NL}|_6)|u_N|^{r-1}_{3(r-1)}|u'_{NL}|_2 \\ &\leqslant C(|\nabla u_{NL}| + |\nabla v_{NL}|)|\nabla u_N|^{r-1}|u'_{NL}| \\ &\leqslant C(|\nabla u_{NL}| + |\nabla v_{NL}|)|u'_{NL}|,\end{aligned} \tag{2.16}$$

where $3(r-1) = 6$ when $n = 3$.

Therefore, we get

$$G_1 \leqslant C(|\nabla(u_N - u_L)|^2 + |\nabla(v_N - v_L)|^2 + |u'_{NL}|^2). \tag{2.17}$$

Similarly, we estimate terms in G_2 as follows

$$\begin{aligned}\int_\Omega |v_{NL}||u_N|^{\frac{r-1}{2}}|v_N|^{\frac{r-1}{2}}|u'_{NL}|dx &\leqslant |v_{NL}|_6 |u_N|^{\frac{r-1}{2}}_{3(r-1)}|v_N|^{\frac{r-1}{2}}_{3(r-1)}|u'_{NL}| \\ &\leqslant C|\nabla v_{NL}||\nabla u_N|^{\frac{r-1}{2}}|\nabla v_N|^{\frac{r-1}{2}}|u'_{NL}| \\ &\leqslant C|\nabla v_{NL}||u'_{NL}|.\end{aligned} \tag{2.18}$$

Now, by using

$$G_2 \leqslant C(|\nabla v_{NL}|^2 + |u'_{NL}|^2) \tag{2.19}$$

and noting that $r = 3$ as $n = 3$, a typical term in G_3 is estimated as follows.

$$\int_\Omega |u_{NL}||v_N|^{\frac{r+1}{2}}|v_L|^{\frac{r-3}{2}}|u'_{NL}|dx \leqslant |u_{NL}|_6 |v_L|^{\frac{r+1}{2}}_{\frac{3(r+1)}{2}}|u'_{NL}| \leqslant C|\nabla u_{NL}||u'_{NL}|. \tag{2.20}$$

Likewise, if $n = 1, 2$ the last estimate in G_3 can be easily obtained. We have

$$G_3 \leqslant C(|\nabla u_{NL}|^2 + |u'_{NL}|^2). \tag{2.21}$$

By (2.15), we obtain

$$\begin{aligned}&|\langle f_1(u_N(t), v_N(t)) - f_1(u_L(t), v_L(t)), u'_{NL}(t)\rangle| \\ &\qquad\leqslant C(|\nabla u_{NL}|^2 + |\nabla v_{NL}|^2 + |u'_{NL}|^2).\end{aligned} \tag{2.22}$$

Likewise, by recalling (1.17), we obtain

$$\begin{aligned}&|\langle f_2(u_N(t), v_N(t)) - f_2(u_L(t), v_L(t)), v'_{NL}(t)\rangle| \\ &\qquad\leqslant C(|\nabla u_{NL}|^2 + |\nabla v_{NL}|^2 + |v'_{NL}|^2).\end{aligned} \tag{2.23}$$

Then, combining (2.14) and (2.22)–(2.23) it follows that

$$\begin{cases}\frac{1}{2}\frac{d}{dt}(|u'_{NL}(t)|^2 + (1 - \int_0^t \varpi_1(s)ds)|\nabla u_{NL}(t)|^2 + (\varpi_1 \diamond \nabla u_{NL})(t)) \\ \quad + \frac{1}{2}\int_0^t \varpi_1(s)|\nabla u_{NL}(s)|^2 ds - \frac{1}{2}(\varpi'_1 \diamond \nabla u_{NL})(t) \\ \quad + \int_\Omega [(|u_N|^k + |v_N|^l)|u'_N|^{p-1}u'_N - (|u_L|^k + |v_L|^l)|u'_L|^{p-1}u'_L]u'_{NL}dx \\ \leqslant C(|\nabla u_{NL}|^2 + |\nabla v_{NL}|^2 + |u'_{NL}|^2) \\ \frac{1}{2}\frac{d}{dt}(|v'_{NL}(t)|^2 + (1 - \int_0^t \varpi_2(s)ds)|\nabla v_{NL}(t)|^2 + (\varpi_2 \diamond \nabla v_{NL})(t)) \\ \quad + \frac{1}{2}\int_0^t \varpi_2(s)|\nabla v_{NL}(s)|^2 ds - \frac{1}{2}(\varpi'_2 \diamond \nabla v_{NL})(t) \\ \quad + \int_\Omega [(|v_N|^\theta + |u_N|^\varrho)|v'_N|^{q-1}v'_N - (|v_L|^\theta + |u_L|^\varrho)|v'_L|^{q-1}v'_L]v'_{NL}dx \\ \leqslant C(|\nabla u_{NL}|^2 + |\nabla v_{NL}|^2 + |v'_{NL}|^2).\end{cases} \tag{2.24}$$

Then, adding the term

$$\int_\Omega (|u_L(t)|^k + |v_L(t)|^l)|u'_N(t)|^{p-1}u'_N(t)u'_{NL}(t)dx$$

to both sides of $(2.24)_1$, we have

$$\begin{aligned}(2.25)\quad & \frac{1}{2}\frac{d}{dt}\left[|u'_{NL}(t)|^2+\left(1-\int_0^t \varpi_1(s)ds\right)|\nabla u_{NL}(t)|^2+(\varpi_1\diamond\nabla u_{NL})(t)\right]\\ &\quad +\frac{1}{2}\int_0^t \varpi_1(s)|\nabla u_{NL}(s)|^2ds-\frac{1}{2}(\varpi_1'\diamond\nabla u_{NL})(t)\\ &+\int_\Omega(|u_L(t)|^k+|v_L(t)|^l)(|u'_N(t)|^{p-1}u'_N(t)-|u'_L(t)|^{p-1}u'_L(t))u'_{NL}(t)dx\\ &\leqslant C(|\nabla u_{NL}|^2+|\nabla v_{NL}|^2+|u'_{NL}|^2)+H_2(t),\end{aligned}$$

where

$$(2.26)\quad H_2(t)=-\int_\Omega[|u_N(t)|^k-|u_L(t)|^k+|v_N(t)|^l-|v_L(t)|^l]|u'_N(t)|^{p-1}u'_N(t)u'_{NL}(t)dx.$$

Let

$$(2.27)\quad H_1(t)=\int_\Omega(|u_L(t)|^k+|v_L(t)|^l)(|u'_N(t)|^{p-1}u'_N(t)-|u'_L(t)|^{p-1}u'_L(t))u'_{NL}(t)dx.$$

Now, by using the monotonicity inequality (1.13), we obtain

$$(2.28)\quad H_1(t)\geqslant C\int_\Omega(|u_L(t)|^k+|v_L(t)|^l)|u'_{NL}(t)|^{p+1}dx.$$

Further, we have

$$(2.29)\quad |H_2(t)|\leqslant\int_\Omega[||u_N(t)|^k-|u_L(t)|^k|+||v_N(t)|^l-|v_L(t)|^l|]|u'_N(t)|^p|u'_{NL}(t)|dx.$$

We estimate now, both terms in (2.29) in a similar way. To this end, we pick $\delta=\frac{6}{6-3p-k}$, and we note that $\left\{\frac{6}{k-1},6,\frac{2}{p},\delta\right\}$ are Hölder conjugate indices. Also, our assumption that $k\leqslant 3(1-p)$ when $n=3$ and the fact that $0<p<1$ imply that $1<\delta\leqslant 2$. Thus, by applying (1.14), Hölder's inequality, the Sobolev embedding Theorem and Lemma 2.1, we obtain

$$\begin{aligned}(2.30)\quad & \int_\Omega||u_N(t)|^k-|u_L(t)|^k||u'_N(t)|^p|u'_{NL}(t)|dx\\ &\leqslant C\int_\Omega|u_N(t)-u_L(t)|(|u_N(t)|^{k-1}+|u_L(t)|^{k-1})|u'_N(t)|^p|u'_{NL}(t)|dx\\ &\leqslant C(|u_N(t)|_6^{k-1}+|u_L(t)|_6^{k-1})|u_{NL}(t)|_6|u'_N(t)|_2^{\frac{p}{2}}|u'_{NL}(t)|_\delta\\ &\leqslant C|\nabla u_{NL}(t)||u'_{NL}(t)|.\end{aligned}$$

The second term in (2.29) is estimated in a similar way. By choosing $\delta=\frac{6}{6-3p-l}$, we have

$$(2.31)\quad \int_\Omega||v_N(t)|^l-|v_L(t)|^l||u'_N(t)|^p|u'_{NL}(t)|dx\leqslant C|\nabla v_{NL}(t)||u'_{NL}(t)|.$$

Thus, combining (2.29)–(2.31), we get

$$(2.32)\quad |H_2(t)|\leqslant C_1[|\nabla u_{NL}(t)|^2+|\nabla v_{NL}(t)|^2+|u'_{NL}(t)|^2].$$

It follows from (2.25), (2.28) and (2.32) that

$$
\begin{aligned}
(2.33)\quad & \frac{1}{2}\frac{d}{dt}\left(|u'_{NL}(t)|^2+\left(1-\int_0^t \varpi_1(s)ds\right)|\nabla u_{NL}(t)|^2+(\varpi_1\diamond\nabla u_{NL})(t)\right)\\
&\quad +\frac{1}{2}\int_0^t \varpi_1(s)|\nabla u_{NL}(s)|^2ds-\frac{1}{2}(\varpi_1'\diamond\nabla u_{NL})(t)\\
&\quad +C\int_\Omega (|u_L(t)|^k+|v_L(t)|^l)|u'_{NL}(t)|^{p+1}dx\\
&\quad\leqslant C_2(|\nabla u_{NL}|^2+|\nabla v_{NL}|^2+|u'_{NL}|^2+|v'_{NL}|^2).
\end{aligned}
$$

Likewise, for $(2.24)_2$, we apply the same steps as above and get

$$
\begin{aligned}
(2.34)\quad & \frac{1}{2}\frac{d}{dt}\left(|v'_{NL}(t)|^2+\left(1-\int_0^t \varpi_2(s)ds\right)|\nabla v_{NL}(t)|^2+(\varpi_2\diamond\nabla v_{NL})(t)\right)\\
&\quad +\frac{1}{2}\int_0^t \varpi_2(s)|\nabla v_{NL}(s)|^2ds-\frac{1}{2}(\varpi_2'\diamond\nabla v_{NL})(t)\\
&\quad +C\int_\Omega (|v_L(t)|^\theta+|u_L(t)|^\varrho)|v'_{NL}(t)|^{q+1}dx\\
&\quad\leqslant C_3(|\nabla u_{NL}|^2+|\nabla v_{NL}|^2+|u'_{NL}|^2+|v'_{NL}|^2).
\end{aligned}
$$

Set

$$Y_{NL}(t):=|\nabla u_{NL}(t)|^2+|\nabla v_{NL}(t)|^2+|u'_{NL}(t)|^2+|v'_{NL}(t)|^2.$$

Then from (2.33)–(2.34) we obtain

$$
\begin{aligned}
(2.35)\quad & Y_{NL}(t)+(\varpi_1\diamond\nabla u_{NL})(t)+(\varpi_2\diamond\nabla v_{NL})(t)\\
&\quad +\frac{1}{\rho}\int_0^t[\varpi_1(s)|\nabla u_{NL}(s)|^2+\varpi_2(s)|\nabla v_{NL}(s)|^2]ds\\
&\quad -\frac{1}{\rho}\int_0^t[(\varpi_1'\diamond\nabla u_{NL})(s)+(\varpi_2'\diamond\nabla v_{NL})(s)]ds\\
&\quad +C\int_0^t\int_\Omega(|u_L(t)|^k+|v_L(t)|^l)|u'_{NL}(t)|^{p+1}dx\,ds\\
&\quad +C\int_0^t\int_\Omega(|v_L(t)|^\theta+|u_L(t)|^\varrho)|v'_{NL}(t)|^{q+1}dx\,ds\\
&\leqslant\frac{1}{\rho}Y_{NL}(0)+C\int_0^t Y_{NL}(s)ds,
\end{aligned}
$$

where $Y_{NL}(0):=|\nabla u_{NL}(0)|^2+|\nabla v_{NL}(0)|^2+|u'_{NL}(0)|^2+|v'_{NL}(0)|^2$.

Then, it follows from Gronwall's inequality that

$$(2.36)\qquad Y_{NL}(t)\leqslant\frac{1}{\rho}Y_{NL}(0)e^{CT}\to 0\quad\text{as } L,N\to\infty,$$

for all $t\in[0,T]$. Then we can obtain (2.12) by (2.35) and (2.36). By strong convergence in (2.5), namely $Y_{NL}(0)\to 0$ as $L,N\to\infty$. Therefore, $Y_{NL}(t)\to 0$ as $L,N\to\infty$ and the proof of the Lemma is complete. □

REMARK 2.2. Assume that (u_N, v_N) and (u_L, v_L) are two couples of approximate solutions as presented in the proof of Lemma 2.2. Then, it follows from (2.35) that

$$\lim_{N,L\to\infty}\int_\Omega (|u_L(t)|^k+|v_L(t)|^l)|u'_{NL}(t)|^{p+1}dx=0$$
$$\lim_{N,L\to\infty}\int_\Omega (|v_L(t)|^\theta+|u_L(t)|^\varrho)|v'_{NL}(t)|^{q+1}dx=0.$$

Furthermore, there exists a pair of functions (u, v) such that

$$\begin{cases} u_N\to u \quad \text{and} \quad v_N\to v \quad \text{strongly in } L^\infty(0,T:H_0^1(\Omega)),\\ u'_N\to u' \quad \text{and} \quad v'_N\to v' \quad \text{strongly in } L^\infty(0,T:L^2(\Omega)).\end{cases} \tag{2.37}$$

As a consequence of Lemma 2.2, we have two corollaries. The first one addresses the source term and the second addresses the damping terms.

COROLLARY 2.1. *The sequences of approximate solutions $\{u_N\}$ and $\{v_N\}$ satisfy as $N\to\infty$*

$$f_1(u_N,v_N)\longrightarrow f_1(u,v),\ f_2(u_N,v_N)\longrightarrow f_2(u,v) \text{ strongly in } L^\infty(0,T,L^2(\Omega)). \tag{2.38}$$

PROOF. The proof is similar to that of [1]. We give it for the convenience of the reader. By using (1.16), we get

$$|f_1(u_N(t),v_N(t))-f_1(u(t),v(t))|^2\leqslant K_1+K_2+K_3+K_4, \tag{2.39}$$

where

$$K_1=C_0\int_\Omega |u_N-u|^2(|u_N|^{2(r-1)}+|v_N|^{2(r-1)}+|u|^{2(r-1)}+|v|^{2(r-1)})dx$$
$$K_2=C_0\int_\Omega |v_N-v|^2(|u_N|^{2(r-1)}+|v_N|^{2(r-1)}+|u|^{2(r-1)}+|v|^{2(r-1)})dx$$
$$K_3=C_1\int_\Omega |v_N-v|^2|u_N|^{r-1}(|v_N|^{r-1}+|v|^{r-1})dx$$
$$K_4=C_1\int_\Omega |u_N-u|^2|v|^{r+1}(|u_N|^{r-3}+|u|^{r-3})dx.$$

Now, a typical term in K_1 and K_2 is estimated by using Hölder's inequality, Sobolev Embedding theorem and the bounds furnished by Lemma 2.1 and we get

$$\begin{aligned}\int_\Omega |u_N-u|^2|u_N|^{2(r-1)}dx &\leqslant |u_N-u|_6^2|u_N|_{3(r-1)}^{2(r-1)}\\ &\leqslant C|\nabla(u_N-u)|^2|\nabla u_N|^{2(r-1)}\\ &\leqslant C|\nabla(u_N-u)|^2,\end{aligned} \tag{2.40}$$

where if $n=3$, then $3(r-1)=6$. Thus, for all $t\in[0,T]$, we have

$$K_1+K_2\leqslant C(|\nabla(u_N(t)-u(t))|^2+|\nabla(v_N(t)-v(t))|^2). \tag{2.41}$$

Likewise, a typical term in K_3 is as follows,

$$\int_\Omega |v_N-v|^2|u_N|^{r-1}|v_N|^{r-1}dx\leqslant |v_N-v|_6^2|u_N|_{3(r-1)}^{r-1}|v_N|_{3(r-1)}^{r-1} \tag{2.42}$$

$$\leqslant C|\nabla(v_N(t)-v(t))|^2.$$

In addition, a typical term in K_4 is estimated in a similar way and we obtain

$$K_3+K_4 \leqslant C(|\nabla(u_N(t)-u(t))|^2+|\nabla(v_N(t)-v(t))|^2). \tag{2.43}$$

Thus, the strong convergence furnished by Lemma 2.2 combined with (2.39), (2.41) and (2.43) proves the first convergence in (2.38). □

Corollary 2.2. *There exist subsequences of approximate solutions, which we still denote by $\{u_N\}$ and $\{v_N\}$, which satisfy the following:*

$$\begin{cases} |u_N|^k|u_N'|^{p-1}u_N' \longrightarrow |u|^k|u'|^{p-1}u' & \text{weakly in } L^2(W_T)\\ |v_N|^l|u_N'|^{p-1}u_N' \longrightarrow |v|^l|u'|^{p-1}u' & \text{weakly in } L^2(W_T)\\ |v_N|^\theta|v_N'|^{q-1}v_N' \longrightarrow |v|^\theta|v'|^{q-1}v' & \text{weakly in } L^2(W_T)\\ |u_N|^\varrho|v_N'|^{q-1}v_N' \longrightarrow |v|^\varrho|v'|^{q-1}v' & \text{weakly in } L^2(W_T), \end{cases} \tag{2.44}$$

where $W_T=\Omega\times(0,T)$.

Proof. By the strong convergence in Lemma 2.2 there exist subsequences of the approximate solutions, which we still denote by $\{u_N\}$ and $\{v_N\}$, and which satisfy

$$\begin{cases} u_N\to u \quad \text{and} \quad v_N\to v \quad \text{a.e. in } W_T,\\ u_N'\to u' \quad \text{and} \quad v_N'\to v' \quad \text{a.e. in } W_T. \end{cases} \tag{2.45}$$

Particularly, we have

$$|u_N|^k|u_N'|^{p-1}u_N' \longrightarrow |u|^k|u'|^{p-1}u' \quad \text{a.e. in } W_T. \tag{2.46}$$

By combining (2.46) with the fact that

$$||u_N|^k|u_N'|^{p-1}u_N'| \leqslant |u_N|^k_{\frac{2k}{1-p}}|u_N'|^p \leqslant |\nabla u_N|^k|u_N'|^p \leqslant C_T, \tag{2.47}$$

for all $N\geqslant 1$, the first convergence in (2.44) follows from a standard result in the analysis. The other statements in (2.44) are proven similarly. □

Step 2: Limiting process. We need to show several things for completing the proof of the existence statement in Theorem 1.1. Firstly, integrating (2.1) over $(0,t)$, we obtain

$$\begin{cases} \langle u_N'(t),e_j\rangle-\langle u_N'(0),e_j\rangle+\int_0^t\langle\nabla u_N(s),\nabla e_j\rangle ds\\ \qquad -\int_0^t\langle\int_0^s\varpi_1(s-\tau)\nabla u_N(\tau)d\tau,\nabla e_j\rangle ds\\ \qquad +\int_0^t\langle(|u_N|^k+|v_N|^l)|u_N'|^{p-1}u_N',e_j\rangle ds\\ \qquad =\int_0^t\langle f_1(u_N(s),v_N(s)),e_j\rangle ds,\\ \langle v_N'(t),e_j\rangle-\langle v_N'(0),e_j\rangle+\int_0^t\langle\nabla v_N(s),\nabla e_j\rangle ds\\ \qquad -\int_0^t\langle\int_0^s\varpi_2(s-\tau)\nabla v_N(\tau)d\tau,\nabla e_j\rangle ds\\ \qquad +\int_0^t\langle(|v_N|^\theta+|u_N|^\varrho)|v_N'|^{q-1}v_N',e_j\rangle ds\\ \qquad =\int_0^t\langle f_2(u_N(s),v_N(s)),e_j\rangle ds, \end{cases} \tag{2.48}$$

for all $t\in[0,T]$.

By using the strong convergence in Lemma 2.2, Corollary 2.1 and (2.5) and the weak convergence in Corollary 2.2, we can pass to the limit in (2.48). Actually, there exists a couple of functions (u,v) with $u,v \in L^{\infty}(0,T;H_0^1(\Omega))$, $u_t, v_t \in L^{\infty}(0,T;L^2(\Omega))$ and (u,v), which satisfies all the requirements of Definition 1.1. Therefore, the local existence of weak solutions has been established.

Moreover, we can obtain the following result concerning the second order derivatives.

Lemma 2.3. *Let $u,v \in L^{\infty}(0,T;H_0^1(\Omega))$, $u_t, v_t \in L^{\infty}(0,T;L^2(\Omega))$ and (u,v) satisfy (1.10). Then,*

$$\Delta u, \Delta v, u'', v'', \int_0^t \varpi_1(t-s)\nabla u(s)ds, \int_0^t \varpi_2(t-s)\nabla v(s)ds \in L^{\infty}(0,T;H_0^{-1}(\Omega)),$$

where $H_0^{-1}(\Omega)$ is dual space of $H_0^1(\Omega)$.

Proof. The proof can be easily established by combining [6, Lemma 3.3] and [23, Lemma 2.7], thus we omit it. □

Step 3: Proof of energy identity in Theorem 1.1. We prove that the weak solution to (1.1) satisfies (1.11). We have shown that the approximate solutions (u_N, v_N) satisfy

$$\begin{aligned}
&\frac{1}{2}\left[|u_N'(t)|^2+|v_N'(t)|^2+\left(1-\int_0^t \varpi_1(s)ds\right)|\nabla u_N(t)|^2+\left(1-\int_0^t \varpi_2(s)ds\right)|\nabla v_N(t)|^2\right]\\
&+\frac{1}{2}[(\varpi_1 \diamond \nabla u_N)(t)+(\varpi_2 \diamond \nabla v_N)(t)]+\frac{1}{2}\int_0^t [\varpi_1(s)|\nabla u_N(s)|^2+\varpi_2(s)|\nabla v_N(s)|^2]ds\\
&-\frac{1}{2}\int_0^t [(\varpi_1' \diamond \nabla u_N)(s)+(\varpi_2' \diamond \nabla v_N)(s)]ds+\int_0^t\int_\Omega (|u|^k+|v|^l)|u_t|^{p+1}dx\,ds\\
&+\int_0^t\int_\Omega (|v|^\theta+|u|^\varrho)|v_t|^{q+1}dx\,ds\\
&=\frac{1}{2}[|u_N'(0)|^2+|v_N'(0)|^2+|\nabla u_N(0)|^2+|\nabla v_N(0)|^2]\\
&+\int_0^t\int_\Omega [f_1(u_N(s),v_N(s))u_N'(s)+f_2(u_N(s),v_N(s))v_N'(s)]dx\,ds.
\end{aligned}$$

It follows from (1.6) that

$$\begin{aligned}
(2.49)\qquad &\int_0^t\int_\Omega [f_1(u_N(s),v_N(s))u_N'(s)+f_2(u_N(s),v_N(s))v_N'(s)]dx\,ds\\
&=\int_0^t\int_\Omega \frac{d}{ds}F(u_N(s),v_N(s))dx\,ds\\
&=\int_\Omega F(u(t),v(t))dx-\int_\Omega F(u^0,v^0)dx.
\end{aligned}$$

Define

$$E_N(t) = \frac{1}{2}\bigg[|u'_N(t)|^2 + |v'_N(t)|^2 + \Big(1 - \int_0^t \varpi_1(s)ds\Big)|\nabla u_N(t)|^2 + \Big(1 - \int_0^t \varpi_2(s)ds\Big)|\nabla v_N(t)|^2\bigg] + \frac{1}{2}[(\varpi_1 \diamond \nabla u_N)(t) + (\varpi_2 \diamond \nabla v_N)(t)] - \int_\Omega F(u(t), v(t))dx,$$

and then we have

$$\begin{aligned} E_N(t) &+ \frac{1}{2}\int_0^t [\varpi_1(s)|\nabla u_N(s)|^2 + \varpi_2(s)|\nabla v_N(s)|^2]ds \\ &- \frac{1}{2}\int_0^t [(\varpi_1' \diamond \nabla u_N)(s) + (\varpi_2' \diamond \nabla v_N)(s)]ds \\ &+ \int_0^t \int_\Omega (|u|^k + |v|^l)|u_t|^{p+1}dx\,ds \\ &+ \int_0^t \int_\Omega (|v|^\theta + |u|^\varrho)|v_t|^{q+1}dx\,ds = E_N(0). \end{aligned} \tag{2.50}$$

We need to pass to the limit in (2.50), to the convergence of the nonlinear term

$$\int_\Omega F(u_N(t), v_N(t))dx \to \int_\Omega F(u(t), v(t))dx \quad \text{as } N \to \infty. \tag{2.51}$$

By using (1.18), we get

$$\begin{aligned} &\left|\int_\Omega (F(u_N(t), v_N(t)) - F(u(t), v(t)))dx\right| \\ &\leqslant C_0 \int_\Omega (|u_N - u| + |v_N - v|)(|u_N|^r + |v_N|^r + |u|^r + |v|^r)dx \\ &+ C_1 \int_\Omega (|u_N||v_N - v| + |v||u_N - u|)\big(|u_N|^{\frac{r-1}{2}}|v_N|^{\frac{r-1}{2}} + |u|^{\frac{r-1}{2}}|v|^{\frac{r-1}{2}}\big)dx \\ &= I_1 + I_2. \end{aligned} \tag{2.52}$$

In order to estimate I_1, I_2, we use Hölder's inequality, Sobolev embedding Theorem and Lemma 2.1. I_1 can be estimated as

$$\begin{aligned} \int_\Omega |u_N - u||u_N|^r dx &\leqslant |u_N - u||u_N|_{2r}^r \\ &\leqslant |u_N - u||\nabla u_N|^r \\ &\leqslant C|u_N - u|. \end{aligned}$$

Thus,

$$I_1 \leqslant C(|u_N - u| + C|v_N - v|) \quad \text{for all } t \in [0, T]. \tag{2.53}$$

Similarly, I_2 can be estimated as

$$\int_\Omega |u_N||v_N - v||u_N|^{\frac{r-1}{2}}|v_N|^{\frac{r-1}{2}}dx \leqslant |u_N|_6|v_N - v||u_N|_{3(r-1)}^{\frac{r-1}{2}}|v_N|_{3(r-1)}^{\frac{r-1}{2}}$$

$$\begin{aligned} &\leqslant C|\nabla u_N||v_N - v||\nabla u_N|^{\frac{r-1}{2}}|\nabla v_N|^{\frac{r-1}{2}} \\ &\leqslant C|v_N - v|. \end{aligned}$$

Therefore, we get

$$I_2 \leqslant C(|u_N - u| + C|v_N - v|). \tag{2.54}$$

It follows from (2.52)–(2.54) that

$$\left| \int_\Omega (F(u_N(t), v_N(t)) - F(u(t), v(t)))dx \right| \leqslant C(|u_N - u| + C|v_N - v|). \tag{2.55}$$

Then, (2.51) follows from (2.37) and 2.55. Similarly,

$$\int_\Omega F(u_N(0), v_N(0))dx \to \int_\Omega F(u^0, v^0)dx. \tag{2.56}$$

Noting the strong convergence of $\{u_N\}$, $\{v_N\}$ in $L^\infty(0, T; H_0^1(\Omega))$, and using Lebesgue's dominated convergence theorem, we can see that as $N \to \infty$

$$\begin{cases} (\varpi_1 \diamond \nabla u_N)(t) \to (\varpi_1 \diamond \nabla u)(t), \\ (\varpi_2 \diamond \nabla v_N)(t) \to (\varpi_2 \diamond \nabla v)(t), \\ \int_0^t \varpi_1(s)|\nabla u_N(s)|^2 ds \to \int_0^t \varpi_1(s)|\nabla u(s)|^2 ds, \\ \int_0^t \varpi_2(s)|\nabla v_N(s)|^2 ds \to \int_0^t \varpi_2(s)|\nabla v(s)|^2 ds, \\ \int_0^t (\varpi_1' \diamond \nabla u_N)(s) ds \to \int_0^t (\varpi_1' \diamond \nabla u)(s) ds, \\ \int_0^t (\varpi_2' \diamond \nabla v_N)(s) ds \to \int_0^t (\varpi_2' \diamond \nabla v)(s) ds. \end{cases} \tag{2.57}$$

Then, by using Lemma 2.2, (2.51), (2.56) and (2.57), we can pass to the limit in (2.50) and obtain

$$\begin{aligned} E(t) &+ \frac{1}{2} \int_0^t [\varpi_1(s)|\nabla u(s)|^2 + \varpi_2(s)|\nabla v(s)|^2] ds \\ &- \frac{1}{2} \int_0^t [(\varpi_1' \diamond \nabla u)(s) + (\varpi_2' \diamond \nabla v)(s)] ds + \int_0^t \int_\Omega (|u|^k + |v|^l)|u_t|^{p+1} dx\, ds \\ &+ \int_0^t \int_\Omega (|v|^\theta + |u|^\varrho)|v_t|^{q+1} dx\, ds \\ &= E(0), \end{aligned}$$

where

$$\begin{aligned} E(t) = \frac{1}{2} \bigg[&|u'(t)|^2 + |v'(t)|^2 + \left(1 - \int_0^t \varpi_1(s) ds\right) |\nabla u(t)|^2 \\ &+ \left(1 - \int_0^t \varpi_2(s) ds\right) |\nabla v(t)|^2 \bigg] \\ &+ \frac{1}{2} [(\varpi_1 \diamond \nabla u)(t) + (\varpi_2 \diamond \nabla v)(t)] - \int_\Omega F(u(t), v(t)) dx. \end{aligned}$$

Then, an energy identity is obtained.

3. Uniqueness

Our goal in this section is to prove the uniqueness statement in Theorem 1.1.

Lemma 3.1. *Assume (u,v) and $(\tilde u,\tilde v)$ are two weak solutions to the initial boundary value problem (1.1) on $[0,T]$ in the sense of Definition 1.1. Then, $(u,v)=(\tilde u,\tilde v)$.*

Proof. Let $y=u-\tilde u$ and $z=v-\tilde v$. Then, y and z satisfy

$$\begin{cases} y_{tt}-\Delta y+\int_0^t \varpi_1(t-s)\Delta y(s)ds+(|u|^k+|v|^l)|u_t|^{p-1}u_t \\ \qquad -(|\tilde u|^k+|\tilde v|^l)|\tilde u_t|^{p-1}\tilde u_t = f_1(u,v)-f_1(\tilde u,\tilde v), \quad \text{in } \Omega\times(0,T), \\ z_{tt}-\Delta z+\int_0^t \varpi_2(t-s)\Delta z(s)ds+(|v|^\theta+|u|^\varrho)|v_t|^{q-1}v_t \\ \qquad -(|\tilde v|^\theta+|\tilde u|^\varrho)|\tilde v_t|^{q-1}\tilde v_t = f_2(u,v)-f_2(\tilde u,\tilde v), \quad \text{in } \Omega\times(0,T), \\ y(x,0)=0, y_t(x,0)=0, \quad z(x,0)=0, \quad z_t(x,0)=0, \qquad \text{in } \Omega \\ y=z=0, \quad \text{on } \partial\Omega\times(0,T). \end{cases} \tag{3.1}$$

Since y_t, $z_t\in C_w([0,T],L^2(\Omega))$, we cannot directly test the equations in (3.1) with y_t and z_t and apply standard energy estimates to obtain the desired uniqueness result. In order to overcome this difficulty, we use the difference quotients $D_h y$ and $D_h z$ (as in [1]) and their well-known properties (see [9]).

For any function $y\in C_w([0,T],L^2(\Omega))$ and $h>0$, set

$$D_h y(s)=\frac{1}{2h}(y_e(s+h)-y_e(s-h)) \quad \text{for } s\in(0,T),$$

where $y_e(s)$ denotes the extension of $y(s)$ to $\mathbb{R}$ given by $y_e(s)=y(s)$ for $s\in(0,T)$; $y_e(s)=y(t)$ for $s\geqslant T$; and $y_e(s)=y(0)$ for $s\leqslant 0$.

$$\begin{cases} \lim_{h\to0}\int_0^t\langle y_t(s),D_h y_t(s)\rangle ds=\frac12[|y_t(t)|^2-|y_t(0)|^2]=\frac12|y_t(t)|^2 \\ \lim_{h\to0}\int_0^t\langle z_t(s),D_h z_t(s)\rangle ds=\frac12[|z_t(t)|^2-|z_t(0)|^2]=\frac12|z_t(t)|^2 \\ \lim_{h\to0}\int_0^t\langle \nabla y(s),D_h \nabla y(s)\rangle ds=\frac12[|\nabla y(t)|^2-|\nabla y(0)|^2]=\frac12|\nabla y(t)|^2 \\ \lim_{h\to0}\int_0^t\langle \nabla z(s),D_h \nabla z(s)\rangle ds=\frac12[|\nabla z(t)|^2-|\nabla z(0)|^2]=\frac12|\nabla z(t)|^2 \\ \lim_{h\to0}\int_0^t\langle \int_0^s\varpi_1(s-\tau)\nabla y(\tau),D_h\nabla y(s)\rangle ds=-\frac12\int_0^t\varpi_1(s)ds|\nabla y(t)|^2 \\ \qquad +\frac12(\varpi_1\diamond\nabla y)(t)+\frac12\int_0^t\varpi_1(s)|\nabla y(s)|^2ds-\frac12\int_0^t(\varpi_1'\diamond\nabla y)(s)ds \\ \lim_{h\to0}\int_0^t\langle \int_0^s\varpi_2(s-\tau)\nabla z(\tau),D_h\nabla z(s)\rangle ds=-\frac12\int_0^t\varpi_2(s)ds|\nabla z(t)|^2 \\ \qquad +\frac12(\varpi_2\diamond\nabla z)(t)+\frac12\int_0^t\varpi_2(s)|\nabla z(s)|^2ds-\frac12\int_0^t(\varpi_2'\diamond\nabla z)(s)ds. \end{cases} \tag{3.2}$$

In addition,

$$\begin{cases} \lim_{h\to0}D_h y=y_t, \ \lim_{h\to0}D_h z=z_t \quad \text{strongly in } L^2(0,t;L^2(\Omega)), \\ \lim_{h\to0}D_h y(0)=\frac12 y_t(0), \ \lim_{h\to0}D_h z(0)=\frac12 z_t(0) \quad \text{weakly in } L^2(\Omega), \\ \lim_{h\to0}D_h y(t)=\frac12 y_t(t), \ \lim_{h\to0}D_h z(t)=\frac12 z_t(t) \quad \text{weakly in } L^2(\Omega), \end{cases} \tag{3.3}$$

and for every $s\in(0,t)$, we have

$$\lim_{h\to0}D_h y(s)=y_t(s) \quad \text{and} \quad \lim_{h\to0}D_h z(s)=z_t(s) \qquad \text{weakly in } L^2(\Omega).$$

By Hölder's inequality, Sobolev embedding Theorem and the assumption that $r = 3$ when $n = 3$, one has

$$\begin{aligned}\int_\Omega |f_1(u,v)|^2 dx &\leqslant C\int_\Omega [|u|^{2r} + |v|^{2r} + |u|^{r-1}|v|^{r+1}]dx \\ &\leqslant C\int_\Omega \left[|u|_{2r}^{2r} + |v|_{2r}^{2r} + |u|_{3(r-1)}^{r-1}|v|_{\frac{3(r+1)}{2}}^{r+1}\right] \\ &\leqslant C[|\nabla u|^{2r} + |\nabla v|^{2r} + |\nabla u|^{r-1}|\nabla v|^{r+1}] \\ &\leqslant C(|\nabla u|^{2r} + |\nabla v|^{2r}) \leqslant C.\end{aligned}$$

Hence, $f_1(u,v)$ is in $L^2(\Omega)$ and, similarly, we can show $f_2(u,v) \in L^2(\Omega)$.

Noting that $(|u|^k + |v|^l)|u_t|^{p-1}u_t$, $(|v|^\theta + |u|^\varrho)|v_t|^{q-1}v_t \in L^2(\Omega\times(0,t))$ and by (3.1), we get

$$\begin{cases} y_{tt}(s) - \Delta y(s) + \int_0^s \varpi_1(s-\tau)\Delta y(\tau)d\tau \in L^2(\Omega\times(0,t)), \\ z_{tt}(s) - \Delta z(s) + \int_0^s \varpi_2(s-\tau)\Delta z(\tau)d\tau \in L^2(\Omega\times(0,t)). \end{cases} \tag{3.4}$$

Multiplying the equations in (3.1) by $D_h y$ and $D_h z$, and then integrating over $\Omega\times(0,t)$, we obtain,

$$\begin{cases} \int_0^t\int_\Omega (y_{tt}(s) - \Delta y(s) + \int_0^s \varpi_1(s-\tau)\Delta y(\tau)d\tau)D_h y(s)dx\,ds \\ \qquad - \int_0^t\int_\Omega [f_1(u(s),v(s)) - f_1(\tilde u(s),\tilde v(s))]D_h y(s)dx\,ds \\ \quad = -\int_0^t\int_\Omega [(|u|^k + |v|^l)|u_t|^{p-1}u_t - (|\tilde u|^k + |\tilde v|^l)|\tilde u_t|^{p-1}\tilde u_t]D_h y(s)dx\,ds, \\ \int_0^t\int_\Omega (z_{tt}(s) - \Delta z(s) + \int_0^s \varpi_2(s-\tau)\Delta z(\tau)d\tau)D_h z(s)dx\,ds \\ \qquad - \int_0^t\int_\Omega [f_2(u(s),v(s)) - f_2(\tilde u(s),\tilde v(s))]D_h z(s)dx\,ds \\ \quad = -\int_0^t\int_\Omega [(|v|^\theta + |u|^\varrho)|v_t|^{q-1}v_t - (|\tilde v|^\theta + |\tilde u|^\varrho)|\tilde v_t|^{q-1}\tilde v_t]D_h z(s)dx\,ds. \end{cases} \tag{3.5}$$

Now, we try to pass to the limit in (3.5) as $h \to 0$ is straightforward. It follows from Lemma 2.3 that

$$\begin{aligned} &\int_0^t\int_\Omega \left(y_{tt}(s) - \Delta y(s) + \int_0^s \varpi_1(s-\tau)\Delta y(\tau)d\tau\right)D_h y(s)dx\,ds \\ &= \int_0^t \langle y_{tt}(s), D_h y(s)\rangle ds - \int_0^t \langle \Delta y(s), D_h y(s)\rangle ds \\ &\quad + \int_0^t \left\langle \int_0^s \varpi_1(s-\tau)\Delta y(\tau)d\tau, D_h y(s)\right\rangle ds \\ &= \langle y_t(t), D_h y(t)\rangle - \langle y_t(0), D_h y(0)\rangle - \int_0^t \langle y_t(s), \partial_t D_h y(s)\rangle ds \\ &\quad + \int_0^t \langle \nabla y(s), \nabla D_h y(s)\rangle ds - \int_0^t \left\langle \int_0^s \varpi_1(s-\tau)\nabla y(\tau)d\tau, \nabla D_h y(s)\right\rangle ds \\ &= \langle y_t(t), D_h y(t)\rangle - \int_0^t \langle y_t(s), D_h y_t(s)\rangle ds + \int_0^t \langle \nabla y(s), \nabla D_h y(s)\rangle ds \\ &\quad - \int_0^t \left\langle \int_0^s \varpi_1(s-\tau)\nabla y(\tau)d\tau, \nabla D_h y(s)\right\rangle ds. \end{aligned} \tag{3.6}$$

By using (3.3) and (3.4), we can pass to the limit in the above equality and obtain

$$\begin{aligned}(3.7)\qquad &\lim_{h\to 0}\int_0^t\int_\Omega\left(y_{tt}(s)-\Delta y(s)+\int_0^s\varpi_1(s-\tau)\Delta y(\tau)d\tau\right)D_hy(s)dx\,ds\\ &=\int_0^t\int_\Omega\left(y_{tt}(s)-\Delta y(s)+\int_0^s\varpi_1(s-\tau)\Delta y(\tau)d\tau\right)y_t(s)dx\,ds\\ &=\frac{1}{2}|y_t(t)|^2+\frac{1}{2}\left(1-\int_0^t\varpi_1(s)ds\right)|\nabla y(t)|^2+\frac{1}{2}(\varpi_1\diamond\nabla y)(t)\\ &\quad+\frac{1}{2}\int_0^t\varpi_1(s)ds|\nabla y(t)|^2-\frac{1}{2}\int_0^t(\varpi_1'\diamond\nabla y)(s)ds.\end{aligned}$$

Likewise, we have

$$\begin{aligned}(3.8)\qquad &\int_0^t\int_\Omega\left(z_{tt}(s)-\Delta z(s)+\int_0^s\varpi_2(s-\tau)\Delta z(\tau)d\tau\right)D_hz(s)dx\,ds\\ &=\frac{1}{2}|z_t(t)|^2+\frac{1}{2}\left(1-\int_0^t\varpi_2(s)ds\right)|\nabla z(t)|^2+\frac{1}{2}(\varpi_2\diamond\nabla z)(t)\\ &\quad+\frac{1}{2}\int_0^t\varpi_2(s)ds|\nabla z(t)|^2-\frac{1}{2}\int_0^t(\varpi_2'\diamond\nabla z)(s)ds.\end{aligned}$$

Combining (3.3), (3.7) and (3.8), we can pass to the limit in (3.5) to obtain

$$(3.9)\qquad \begin{cases}\frac{1}{2}[|y_t(t)|^2+(1-\int_0^t\varpi_1(s)ds)|\nabla y(t)|^2+(\varpi_1\diamond\nabla y)(t)+\int_0^t\varpi_1(s)|\nabla y(s)|^2ds]\\ \quad-\frac{1}{2}\int_0^t(\varpi_1'\diamond\nabla y)(s)ds+\int_0^t\int_\Omega[(|u(s)|^k+|v(s)|^l)|u_t(s)|^{p-1}u_t(s)\\ \quad-(|\tilde u(s)|^k+|\tilde v(s)|^l)|\tilde u_t(s)|^{p-1}\tilde u_t(s)]y_t(s)dx\,ds\\ \quad=\int_0^t\int_\Omega[f_1(u(s),v(s))-f_1(\tilde u(s),\tilde v(s))]y_t(s)dx\,ds,\\ \frac{1}{2}[|z_t(t)|^2+(1-\int_0^t\varpi_2(s)ds)|\nabla z(t)|^2+(\varpi_2\diamond\nabla z)(t)+\int_0^t\varpi_2(s)|\nabla z(s)|^2ds]\\ \quad-\frac{1}{2}\int_0^t(\varpi_2'\diamond\nabla z)(s)ds+\int_0^t\int_\Omega[(|v(s)|^\theta+|u(s)|^\varrho)|v_t(s)|^{q-1}v_t(s)\\ \quad-(|\tilde v(s)|^\theta+|\tilde u(s)|^\varrho)|\tilde v_t(s)|^{q-1}\tilde v_t(s)]z_t(s)dx\,ds\\ \quad=\int_0^t\int_\Omega[f_2(u(s),v(s))-f_2(\tilde u(s),\tilde v(s))]z_t(s)dx\,ds,\end{cases}$$

where we have used the fact that $y(x,0)=y_t(x,0)=z(x,0)=z_t(x,0)=0$.

By adding respectively the terms

$$\int_0^t\int_\Omega(|\tilde u|^k+|\tilde v|^l)|u_t|^{p-1}u_ty_tdx\,ds,\qquad \int_0^t\int_\Omega(|\tilde v|^\theta+|\tilde u|^\varrho)|v_t|^{q-1}v_tz_tdx\,ds$$

to both sides of (3.9), one finds

$$(3.10)\qquad \begin{cases}\frac{1}{2}[|y_t(t)|^2+(1-\int_0^t\varpi_1(s)ds)|\nabla y(t)|^2+(\varpi_1\diamond\nabla y)(t)]\\ \quad+\frac{1}{2}\int_0^t\varpi_1(s)|\nabla y(s)|^2ds-\frac{1}{2}\int_0^t(\varpi_1'\diamond\nabla y)(s)ds+\int_0^tJ_1(s)ds\\ \quad=\int_0^t\int_\Omega[f_1(u(s),v(s))-f_1(\tilde u(s),\tilde v(s))]y_t(s)dx\,ds+\int_0^tJ_2(s)ds,\\ \frac{1}{2}[|z_t(t)|^2+(1-\int_0^t\varpi_2(s)ds)|\nabla z(t)|^2+(\varpi_2\diamond\nabla z)(t)]\\ \quad+\frac{1}{2}\int_0^t\varpi_2(s)|\nabla z(s)|^2ds-\frac{1}{2}\int_0^t(\varpi_2'\diamond\nabla z)(s)ds+\int_0^tK_1(s)ds\\ \quad=\int_0^t\int_\Omega[f_2(u(s),v(s))-f_2(\tilde u(s),\tilde v(s))]z_t(s)dx\,ds+\int_0^tK_2(s)ds,\end{cases}$$

where

$$
\begin{aligned}
J_1(s) &= \int_\Omega (|\tilde{u}|^k + |\tilde{v}|^l)[|u_t|^{p-1}u_t - |\tilde{u}_t|^{p-1}\tilde{u}_t]y_t dx,\\
J_2(s) &= -\int_\Omega (|u|^k - |\tilde{u}|^k + |v|^l - |\tilde{v}|^l)|u_t|^{p-1}u_t y_t dx,\\
K_1(s) &= \int_\Omega (|\tilde{v}|^\theta + |\tilde{u}|^\varrho)[|v_t|^{q-1}v_t - |\tilde{v}_t|^{q-1}\tilde{v}_t]z_t dx,\\
K_2(s) &= -\int_\Omega (|v|^\theta - |\tilde{v}|^\theta + |u|^\varrho - |\tilde{u}|^\varrho)|v_t|^{q-1}v_t z_t dx.
\end{aligned}
\tag{3.11}
$$

By recalling the monotonicity problem (1.13), we have

$$
\begin{aligned}
J_1(s) &\geqslant c_0 \int_\Omega (|\tilde{u}|^k + |\tilde{v}|^l)|y_t|^{p+1} dx,\\
K_1(s) &\geqslant c_0 \int_\Omega (|\tilde{v}|^\theta + |\tilde{u}|^\varrho)|z_t|^{q+1} dx.
\end{aligned}
\tag{3.12}
$$

Also, in order to estimate the right hand terms of (3.10), we use (2.22) and (2.23) as follows

$$
\begin{aligned}
\left| \int_\Omega [f_1(u(s),v(s)) - f_1(\tilde{u}(s),\tilde{v}(s))]y_t(s)dx \right| &\leqslant C(|\nabla y(s)|^2 + |\nabla z(s)|^2 + |y_t(s)|^2)\\
\left| \int_\Omega [f_2(u(s),v(s)) - f_1(\tilde{u}(s),\tilde{v}(s))]z_t(s)dx \right| &\leqslant C(|\nabla y(s)|^2 + |\nabla z(s)|^2 + |z_t(s)|^2),
\end{aligned}
$$

and

$$
\begin{aligned}
|J_2(s)| &\leqslant C(|\nabla y(s)|^2 + |\nabla z(s)|^2 + |y_t(s)|^2)\\
|K_2(s)| &\leqslant C(|\nabla y(s)|^2 + |\nabla z(s)|^2 + |z_t(s)|^2).
\end{aligned}
\tag{3.13}
$$

Now, owing to (1.13), (1.9), let

$$
Y(t) := |y_t(t)|^2 + |z_t(t)|^2 + |\nabla y(t)|^2 + |\nabla z(t)|^2.
$$

Then, it follows from (3.10)–(3.13) that

$$
\begin{aligned}
& Y(t) + (\varpi_1 \diamond \nabla y)(t) + (\varpi_2 \diamond \nabla z)(t)\\
&\quad + \frac{1}{\rho}\int_0^t [\varpi_1(s)|\nabla y(s)|^2 + \varpi_2(s)|\nabla z(s)|^2] ds\\
&\quad - \frac{1}{\rho}\int_0^t [(\varpi_1' \diamond \nabla y)(s) + (\varpi_2' \diamond \nabla z)(s)] ds\\
&\quad + c_0 \int_0^t \int_\Omega [(|\tilde{u}|^k + |\tilde{v}|^l)|y_t|^{p+1} + (|\tilde{v}|^\theta + |\tilde{u}|^\varrho)|z_t|^{q+1}] dx\, ds\\
&\leqslant C\int_0^t Y(s) ds.
\end{aligned}
\tag{3.14}
$$

By Gronwall's inequality, $Y(t) = 0$ for all $t \in [0,T]$. Hence, $(u,v) = (\tilde{u},\tilde{v})$, which completes the proof. □

4. Blow-up solutions

In this section, we will give the proof of Theorem 1.2.

PROOF. Let (u, v) be the weak solution to problem (1.1) provided by Definition 1.1. Along the proof we assume that $r > \max\{k+p, l+p, \theta+q, \varrho+q\}$ and $E(0) < 0$.
For $t \in [0, T)$, set

$$M(t) = |u(t)|^2 + |v(t)|^2, G(t) = \int_\Omega F(u(t), v(t))dx, \tag{4.1}$$

and

$$\begin{aligned} H(t) = -E(t) = -\frac{1}{2}\bigg(|u'(t)|^2 + |v'(t)|^2 + \bigg(1 - \int_0^t \varpi_1(s)ds\bigg)|\nabla u(t)|^2 \\ + \bigg(1 - \int_0^t \varpi_2(s)ds\bigg)|\nabla v(t)|^2\bigg) - \frac{1}{2}[(\varpi_1 \diamond \nabla u)(t) + (\varpi_2 \diamond \nabla v)(t)] + G(t). \end{aligned} \tag{4.2}$$

The basic ideas of this proof have been used in [1, 2, 6, 7, 15, 22, 23].

Then, we have $H(0) > 0$ and the energy identity (1.11) implies that

$$\begin{aligned} H'(t) = \frac{1}{2}[\varpi_1(t)|\nabla u(t)|^2 + \varpi_2(t)|\nabla v(t)|^2] \\ - \frac{1}{2}[(\varpi_1' \diamond \nabla u)(t) + (\varpi_2' \diamond \nabla v)(t)] \\ + \int_\Omega (|u(t)|^k + |v(t)|^l)|u'(t)|^{p+1}dx \\ + \int_\Omega (|v(t)|^\theta + |u(t)|^\varrho)|v'(t)|^{q+1}dx \geqslant 0. \end{aligned} \tag{4.3}$$

Thus, from (1.5), we have

$$0 < H(0) \leqslant H(t) \leqslant G(t) \leqslant c_1(|u(t)|_{r+1}^{r+1} + |v(t)|_{r+1}^{r+1}), \tag{4.4}$$

for $t \in [0, T)$. Also, recalling (1.5), we obtain

$$c_0(|u(t)|_{r+1}^{r+1} + |v(t)|_{r+1}^{r+1}) \leqslant G(t), \quad 0 \leqslant t < T. \tag{4.5}$$

Let

$$\begin{aligned} 0 < \lambda < \min\bigg\{\frac{r-(k+p)}{(p+1)(r+1)}, \frac{r-(l+p)}{(p+1)(r+1)}, \\ \frac{r-(\varrho+q)}{(q+1)(r+1)}, \frac{r-(\theta+q)}{(q+1)(r+1)}, \frac{r-1}{2(r+1)}\bigg\}. \end{aligned} \tag{4.6}$$

Particularly, it follows easily from (4.6) that $0 < \lambda < \frac{1}{2}$. We introduce the following notation for simplification and we take the constants as follows

$$\begin{aligned} \gamma_1 = \frac{r+1}{8}H(0)^{\frac{r-(k+p)}{(p+1)(r+1)}}, \quad \gamma_2 = \frac{r+1}{8}H(0)^{\frac{r-(l+p)}{(p+1)(r+1)}}, \\ \gamma_3 = \frac{r+1}{8}H(0)^{\frac{r-(\varrho+q)}{(q+1)(r+1)}}, \quad \gamma_4 = \frac{r+1}{8}H(0)^{\frac{r-(\theta+q)}{(q+1)(r+1)}}, \end{aligned} \tag{4.7}$$

and

$$(4.8)\quad \begin{aligned} \sigma_1 &= c_0^{-\frac{k+p+1}{(p+1)(r+1)}}|\Omega|^{\frac{r-(k+p)}{(p+1)(r+1)}}, \quad \sigma_2 = c_0^{-\frac{l+p+1}{(p+1)(r+1)}}|\Omega|^{\frac{r-(l+p)}{(p+1)(r+1)}},\\ \sigma_3 &= c_0^{-\frac{\varrho+q+1}{(q+1)(r+1)}}|\Omega|^{\frac{r-(\varrho+q)}{(q+1)(r+1)}}, \quad \sigma_4 = c_0^{-\frac{\theta+q+1}{(q+1)(r+1)}}|\Omega|^{\frac{r-(\theta+q)}{(q+1)(r+1)}}, \end{aligned}$$

where $|\Omega|$ indicates the Lebesgue measure of Ω and $c_0 > 0$ is the constant that arises in (4.5). Since $0 < \lambda < \frac{1}{2}$, we can pick $0 < \varepsilon \leqslant 1$ small enough such that

$$(4.9)\quad \begin{aligned} A := 1 - \lambda - 2\varepsilon\Big[&\sigma_1^{\frac{p+1}{p}}\gamma_1^{-\frac{1}{p}}H(0)^{\frac{k+p-r}{(p+1)(r+1)}+\lambda} + \sigma_2^{\frac{p+1}{p}}\gamma_2^{-\frac{1}{p}}H(0)^{\frac{l+p-r}{(p+1)(r+1)}+\lambda}\\ &+ \sigma_3^{\frac{q+1}{q}}\gamma_3^{-\frac{1}{q}}H(0)^{\frac{\varrho+q-r}{(q+1)(r+1)}+\lambda} + \sigma_4^{\frac{q+1}{q}}\gamma_4^{-\frac{1}{q}}H(0)^{\frac{\theta+q-r}{(q+1)(r+1)}+\lambda}\Big] \geqslant 0. \end{aligned}$$

Later, we may need to adjust ε again.

We indicate that the smallness condition on ε in (4.9) is such that

$$(4.10)\quad \begin{aligned} 1 - \lambda \geqslant C\varepsilon\Big[&H(0)^{\frac{k+p-r}{(p+1)(r+1)}+\lambda} + H(0)^{\frac{l+p-r}{(p+1)(r+1)}+\lambda}\\ &+ H(0)^{\frac{\varrho+q-r}{(q+1)(r+1)}+\lambda} + H(0)^{\frac{\theta+q-r}{(q+1)(r+1)}+\lambda}\Big] \geqslant C\varepsilon H(0)^{-\eta}, \end{aligned}$$

where $\eta > 0$ is given by

$$(4.11)\quad \eta = \begin{cases} -\lambda + \max\left\{\frac{r-(k+p)}{p(r+1)}, \frac{r-(l+p)}{p(r+1)}, \frac{r-(\varrho+q)}{q(r+1)}, \frac{r-(\theta+q)}{q(r+1)}\right\}: & \text{if } H(0) \geqslant 1,\\ -\lambda + \min\left\{\frac{r-(k+p)}{p(r+1)}, \frac{r-(l+p)}{p(r+1)}, \frac{r-(\varrho+q)}{q(r+1)}, \frac{r-(\theta+q)}{q(r+1)}\right\}: & \text{if } 0 < H(0) < 1. \end{cases}$$

Thus, is such (4.9) will be valid provided we pick $\varepsilon = cH(0)^{\eta}$ for a sufficiently small generic constant $c > 0$. Therefore, ε is chosen as

$$(4.12)\quad \varepsilon = \min\{1, cH(0)^{\eta}\}$$

where our interest lies if $H(0) > 0$ is sufficiently small.

From the definition of $M(t)$, we get

$$(4.13)\quad M'(t) = 2\int_{\Omega}[u'(t)u(t) + v'(t)v(t)]dx.$$

Also, from (1.10), we have

$$(4.14)\quad \begin{cases} \langle u''(t), \varphi\rangle = \frac{d}{dt}\langle u'(t), \varphi\rangle = \frac{d}{dt}\langle u'(t), \varphi\rangle\\ \quad = -\langle\nabla u(t), \nabla\varphi\rangle + \langle\int_0^t \varpi_1(t-s)\nabla u(s)ds, \nabla\varphi\rangle\\ \qquad -\langle(|u(t)|^k + |v(t)|^l)|u'(t)|^{p-1}u'(t), \varphi\rangle + \langle f_1(u(t), v(t)), \varphi\rangle,\\ \langle v''(t), \vartheta\rangle = \frac{d}{dt}\langle v'(t), \vartheta\rangle = \frac{d}{dt}\langle u'(t), \vartheta\rangle\\ \quad = -\langle\nabla v(t), \nabla\vartheta\rangle + \langle\int_0^t \varpi_2(t-s)\nabla v(s)ds, \nabla\vartheta\rangle\\ \qquad -\langle(|v(t)|^\theta + |v(t)|^\varrho)|v'(t)|^{q-1}v'(t), \vartheta\rangle + \langle f_2(u(t), v(t)), \vartheta\rangle, \end{cases}$$

for all $\varphi, \vartheta \in H_0^1(\Omega)$. In (4.14), $\langle .,.\rangle$ denotes the standard duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Now, by using (4.14), we have

$$(4.15)\quad \frac{d}{dt}\int_{\Omega} u'(t)u(t)dx = |u'(t)|^2 + \langle u''(t), u(t)\rangle$$

$$= |u'(t)|^2 - |\nabla u(t)|^2 + \int_0^t \varpi_1(t-s)\int_\Omega \nabla u(s)\nabla u(t)dx\,ds - \int_\Omega (|u(t)|^k + |v(t)|^l)|u'(t)|^{p-1}u'(t)u(t)dx + \int_\Omega f_1(u(t),v(t))u(t)dx.$$

Likewise,

$$\frac{d}{dt}\int_\Omega v'(t)v(t)dx = |v'(t)|^2 + \langle v''(t), v(t)\rangle = |v'(t)|^2 - |\nabla v(t)|^2 + \int_0^t \varpi_2(t-s)\int_\Omega \nabla v(s)\nabla v(t)dx\,ds - \int_\Omega (|v(t)|^\theta + |v(t)|^\varrho)|v'(t)|^{q-1}v'(t)v(t)dx + \int_\Omega f_2(u(t),v(t))v(t)dx. \tag{4.16}$$

Thus, by using (4.13), (4.15)–(4.16) and from (1.6), we have

$$M''(t) = 2(|u'(t)|^2 + |v'(t)|^2) - 2(|\nabla u(t)|^2 + |\nabla v(t)|^2) + 2(r+1)\int_\Omega F(u(t),v(t))dx + 2\Big(\int_0^t \varpi_1(t-s)\int_\Omega \nabla u(s)\nabla u(t)dx\,ds + \int_0^t \varpi_2(t-s)\int_\Omega \nabla v(s)\nabla v(t)dx\,ds\Big) - 2\int_\Omega (|u(t)|^k + |v(t)|^l)|u'(t)|^{p-1}u'(t)u(t)dx - 2\int_\Omega (|v(t)|^\theta + |v(t)|^\varrho)|v'(t)|^{q-1}v'(t)v(t)dx. \tag{4.17}$$

As in [1, 2, 7, 15], we let

$$Y(t) = H(t)^{1-\lambda} + \varepsilon M'(t), \tag{4.18}$$

where $0 < \varepsilon \leqslant 1$ is chosen as in (4.9), then (4.17)–(4.18) yield

$$Y'(t) = (1-\lambda)H(t)^{-\lambda}H'(t) + 2\varepsilon(|u'(t)|^2 + |v'(t)|^2) - 2\varepsilon(|\nabla u(t)|^2 + |\nabla v(t)|^2) + 2\varepsilon\Big(\int_0^t \varpi_1(t-s)\int_\Omega \nabla u(s)\nabla u(t)dx\,ds + \int_0^t \varpi_2(t-s)\int_\Omega \nabla v(s)\nabla v(t)dx\,ds\Big) + 2\varepsilon(r+1)G(t) - 2\epsilon\int_\Omega (|u(t)|^k + |v(t)|^l)|u'(t)|^{p-1}u'(t)u(t)dx - 2\varepsilon\int_\Omega (|v(t)|^\theta + |v(t)|^\varrho)|v'(t)|^{q-1}v'(t)v(t)dx. \tag{4.19}$$

By using Young's inequality, we get

$$\begin{aligned}(4.20)\quad &\int_0^t \varpi_1(t-s)\int_\Omega \nabla u(s)\nabla u(t)dx\,ds\\ &= \int_0^t \varpi_1(t-s)\int_\Omega \nabla u(t)(\nabla u(s)-\nabla u(t))dx\,ds + \int_0^t \varpi_1(s)ds|\nabla u(t)|^2\\ &\geqslant -\int_0^t \varpi_1(s)ds|\nabla u(t)|^2 - \frac{1}{4}(\varpi_1 \diamond \nabla u)(t) + \int_0^t \varpi_1(s)ds|\nabla u(t)|^2\\ &= -\frac{1}{4}(\varpi_1 \diamond \nabla u)(t).\end{aligned}$$

Likewise,

$$(4.21)\qquad \int_0^t \varpi_2(t-s)\int_\Omega \nabla v(s)\nabla v(t)dx\,ds \geqslant -\frac{1}{4}(\varpi_2 \diamond \nabla v)(t).$$

Then, it follows from (4.19)–(4.21) that

$$\begin{aligned}(4.22)\qquad Y'(t) \geqslant\ & (1-\lambda)H(t)^{-\lambda}H'(t) + 2\varepsilon(|u'(t)|^2+|v'(t)|^2)\\ &- 2\varepsilon(|\nabla u(t)|^2+|\nabla v(t)|^2)\\ &- \varepsilon\frac{1}{2}(\varpi_1\diamond\nabla u)(t) - \varepsilon\frac{1}{2}(\varpi_2\diamond\nabla v)(t) + 2\varepsilon(r+1)G(t)\\ &- 2\varepsilon\int_\Omega (|u(t)|^k+|v(t)|^l)|u'(t)|^{p-1}u'(t)u(t)dx\\ &- 2\varepsilon\int_\Omega (|v(t)|^\theta+|v(t)|^\varrho)|v'(t)|^{q-1}v'(t)v(t)dx.\end{aligned}$$

Now, from the definition of $H(t)$

$$\begin{aligned}(4.23)\quad -(|\nabla u(t)|^2+|\nabla v(t)|^2) \geqslant\ & \frac{2}{\rho}H(t) + \frac{1}{\rho}(|u'(t)|^2+|v'(t)|^2)\\ &+\frac{1}{\rho}((\varpi_1\diamond\nabla u)(t)+(\varpi_2\diamond\nabla v)(t)) - \frac{2}{\rho}G(t).\end{aligned}$$

Combining (4.22) and (4.23), we get

$$\begin{aligned}(4.24)\quad Y'(t) \geqslant\ & (1-\lambda)H(t)^{-\lambda}H'(t) + \varepsilon\frac{4}{\rho}H(t) + 2\varepsilon\Big(1+\frac{1}{\rho}\Big)(|u'(t)|^2+|v'(t)|^2)\\ &+ 2\varepsilon\Big(\frac{1}{\rho}-\frac{1}{4}\Big)[(\varpi_1\diamond\nabla u)(t)+(\varpi_2\diamond\nabla v)(t)] + 2\varepsilon\Big(r+1-\frac{1}{\rho}\Big)G(t)\\ &- 2\varepsilon\int_\Omega (|u(t)|^k+|v(t)|^l)|u'(t)|^{p-1}u'(t)u(t)dx\\ &- 2\varepsilon\int_\Omega (|v(t)|^\theta+|u(t)|^\varrho)|v'(t)|^{q-1}v'(t)v(t)dx.\end{aligned}$$

We estimate the last two terms in (4.24) as follows

$$(4.25)\quad |\int_\Omega (|u(t)|^k+|v(t)|^l)|u'(t)|^{p-1}u'(t)u(t)dx|$$

$$\leqslant \int_\Omega |u(t)|^{k+1}|u'(t)|^p dx + \int_\Omega |v(t)|^l |u(t)||u'(t)|^p dx = L_1 + L_2.$$

By recalling Hölder's inequality and using the assumption that $r > k+p$, we obtain

$$\begin{aligned} L_1 &= \int_\Omega |u(t)|^{k+1-\frac{kp}{p+1}} |u'(t)|^{\frac{kp}{p+1}} |u'(t)|^p dx \\ &\leqslant \left(\int_\Omega |u(t)|^k |u'(t)|^{p+1} dx \right)^{\frac{p}{p+1}} \left(\int_\Omega |u(t)|^{k+p+1} dx \right)^{\frac{1}{p+1}} \\ &\leqslant H'(t)^{\frac{p}{p+1}} |u(t)|_{k+p+1}^{\frac{k+p+1}{p+1}} \\ &\leqslant H'(t)^{\frac{p}{p+1}} |u(t)|_{r+1}^{\frac{k+p+1}{p+1}} |\Omega|^{\frac{r-(k+p)}{(r+1)(p+1)}}. \end{aligned} \tag{4.26}$$

By noting that $0 < H(0) \leqslant H(t) \leqslant G(t)$, $k + p - r < 0$, and (4.5) and then using Young's inequality, we obtain

$$\begin{aligned} L_1 &\leqslant |\Omega|^{\frac{r-(k+p)}{(r+1)(p+1)}} c_0^{-\frac{k+p+1}{(r+1)(p+1)}} G(t)^{\frac{k+p+1}{(r+1)(p+1)}} H'(t)^{\frac{p}{p+1}} \\ &= \sigma_1 G(t)^{\frac{k+p-r}{(r+1)(p+1)}} G(t)^{\frac{1}{p+1}} H'(t)^{\frac{p}{p+1}} \\ &\leqslant H(t)^{\frac{k+p-r}{(r+1)(p+1)}} \left(\sigma_1 G(t)^{\frac{1}{p+1}} H'(t)^{\frac{p}{p+1}} \right) \\ &\leqslant H(t)^{\frac{k+p-r}{(r+1)(p+1)}} \left[\gamma_1 G(t) + \sigma_1^{\frac{p+1}{p}} \gamma_1^{-\frac{1}{p}} H'(t) \right] \\ &\leqslant \gamma_1 H(0)^{\frac{k+p-r}{(r+1)(p+1)}} G(t) + \sigma_1^{\frac{p+1}{p}} \gamma_1^{-\frac{1}{p}} H(t)^{\frac{k+p-r}{(r+1)(p+1)}} H'(t). \end{aligned} \tag{4.27}$$

By recalling $\frac{k+p-r}{(r+1)(p+1)} + \lambda < 0$, implied by the definition of λ, we obtain

$$\begin{aligned} L_1 &\leqslant \gamma_1 H(0)^{\frac{k+p-r}{(r+1)(p+1)}} G(t) + \sigma_1^{\frac{p+1}{p}} \gamma_1^{-\frac{1}{p}} H(t)^{\frac{k+p-r}{(r+1)(p+1)}+\lambda} H^{-\lambda}(t) H'(t) \\ &\leqslant \gamma_1 H(0)^{\frac{k+p-r}{(r+1)(p+1)}} G(t) + \sigma_1^{\frac{p+1}{p}} \gamma_1^{-\frac{1}{p}} H(0)^{\frac{k+p-r}{(r+1)(p+1)}+\lambda} H^{-\lambda}(t) H'(t). \end{aligned} \tag{4.28}$$

In order to estimate L_2, we use Hölder's inequality and (4.5) as follows

$$\begin{aligned} L_2 &= \int_\Omega |v(t)|^l |u'(t)|^p |u(t)| dx \\ &= \int_\Omega \left\{ |v(t)|^{\frac{lp}{p+1}} |u'(t)|^p \right\} \left\{ |v(t)|^{l-\frac{lp}{p+1}} |u(t)| \right\} dx \\ &\leqslant \left(\int_\Omega |v(t)|^l |u'(t)|^{p+1} dx \right)^{\frac{p}{p+1}} \left(\int_\Omega |v(t)|^l |u(t)|^{p+1} dx \right)^{\frac{1}{p+1}} \\ &\leqslant H'(t)^{\frac{p}{p+1}} |v(t)|_{r+1}^{\frac{l}{p+1}} |u(t)|_{r+1} |\Omega|^{\frac{r-(l+p)}{(p+1)(r+1)}} \\ &\leqslant \sigma_2 H'(t)^{\frac{p}{p+1}} G(t)^{\frac{l+p+1}{(r+1)(p+1)}}. \end{aligned} \tag{4.29}$$

Now, similarly as in (4.27) (replacing k with l), we obtain

$$L_2 \leqslant \gamma_2 H(0)^{\frac{l+p-r}{(r+1)(p+1)}} G(t) + \sigma_2^{\frac{p+1}{p}} \gamma_2^{-\frac{1}{p}} H(0)^{\frac{l+p-r}{(r+1)(p+1)}+\lambda} H^{-\lambda}(t) H'(t). \tag{4.30}$$

In order to estimate the last term in (4.24) in a similar way:

$$|\int_\Omega (|v(t)|^\theta + |u(t)|^\varrho)|v'(t)|^{q-1}v'(t)v(t)dx| \leqslant \int_\Omega |v(t)|^{\theta+1}|v'(t)|^q dx + \int_\Omega |u(t)|^\varrho |v'(t)|^q |v(t)| dx = K_1 + K_2, \tag{4.31}$$

where K_1, K_2 are estimated as we have done for L_1 and L_2, respectively, replacing k by θ, p by q, l by ϱ, u by v, and v by u.

Indeed, we have

$$K_2 \leqslant \gamma_3 H(0)^{\frac{\varrho+q-r}{(r+1)(q+1)}} G(t) + \sigma_3^{\frac{q+1}{q}} \gamma_3^{-\frac{1}{q}} H(0)^{\frac{\varrho+q-r}{(r+1)(q+1)}+\lambda} H^{-\lambda}(t)H'(t), \tag{4.32}$$

and

$$K_1 \leqslant \gamma_4 H(0)^{\frac{\theta+q-r}{(r+1)(q+1)}} G(t) + \sigma_4^{\frac{q+1}{q}} \gamma_4^{-\frac{1}{q}} H(0)^{\frac{\theta+q-r}{(r+1)(q+1)}+\lambda} H^{-\lambda}(t)H'(t). \tag{4.33}$$

Combining (4.24)–(4.33) and the definition of A in (4.9), we obtain

$$\begin{aligned}
Y'(t) &\geqslant (1-\lambda)H(t)^{-\lambda}H'(t) + \varepsilon\frac{4}{\rho}H(t) + 2\varepsilon\Big(1+\frac{1}{\rho}\Big)(|u'(t)|^2 + |v'(t)|^2) \\
&\quad + 2\varepsilon\Big(\frac{1}{\rho}-\frac{1}{4}\Big)[(\varpi_1 \diamond \nabla u)(t) + (\varpi_2 \diamond \nabla v)(t)] + 2\varepsilon\Big(r+1-\frac{1}{\rho}\Big)G(t) \\
&\quad - 2\varepsilon[L_1 + L_2 + K_2 + K_1] \\
&\geqslant AH^{-\lambda}(t)H'(t) + \varepsilon\frac{4}{\rho}H(t) + 2\varepsilon\Big(1+\frac{1}{\rho}\Big)(|u'(t)|^2 + |v'(t)|^2) \\
&\quad + 2\varepsilon\Big(\frac{1}{\rho}-\frac{1}{4}\Big)[(\varpi_1 \diamond \nabla u)(t) + (\varpi_2 \diamond \nabla v)(t)] \\
&\quad + 2\varepsilon G(t)\Big(r+1-\frac{1}{\rho} - \big\{\gamma_1 H(0)^{\frac{k+p-r}{(p+1)(r+1)}} + \gamma_2 H(0)^{\frac{l+p-r}{(p+1)(r+1)}} \\
&\quad + \gamma_3 H(0)^{\frac{\varrho+q-r}{(q+1)(r+1)}} + \gamma_4 H(0)^{\frac{\theta+q-r}{(q+1)(r+1)}}\big\}\Big).
\end{aligned} \tag{4.34}$$

By definitions of $\gamma_1, \dots, \gamma_4$, we obtain

$$\gamma_1 H(0)^{\frac{k+p-r}{(p+1)(r+1)}} + \gamma_2 H(0)^{\frac{l+p-r}{(p+1)(r+1)}} + \gamma_3 H(0)^{\frac{\varrho+q-r}{(q+1)(r+1)}} + \gamma_4 H(0)^{\frac{\theta+q-r}{(q+1)(r+1)}} = \frac{r+1}{2}, \tag{4.35}$$

and from (1.8) and (1.9), we easily notice that

$$\frac{r+1}{2} - \frac{1}{\rho} > 0, \quad \frac{1}{\rho} - \frac{1}{4} > 0,$$

and since $A \geqslant 0$, it follows from (4.34) that

$$\begin{aligned}
Y'(t) &\geqslant \varepsilon\frac{4}{\rho}H(t) + 2\varepsilon\Big(1+\frac{1}{\rho}\Big)(|u'(t)|^2 + |v'(t)|^2) + 2\varepsilon\Big(\frac{r+1}{2}-\frac{1}{\rho}\Big)G(t) \\
&\geqslant C\varepsilon\big[H(t) + |u'(t)|^2 + |v'(t)|^2 + |u(t)|_{r+1}^{r+1} + |v(t)|_{r+1}^{r+1}\big],
\end{aligned} \tag{4.36}$$

where $C > 0$ is a positive constant. Obviously, (4.36) indicates that $Y(t)$ is increasing on $(0, T]$ with

$$Y(t) = H(t)^{1-\lambda} + \varepsilon M'(t) \geqslant H(0)^{1-\lambda} + \varepsilon M'(0). \tag{4.37}$$

If $M'(0) \geqslant 0$, then no further restriction on ε is needed. However, if $M'(0) < 0$, then we need a further condition $0 < \varepsilon \leqslant -\frac{H(0)^{1-\lambda}}{2M'(0)}$ / In both cases, one has

$$Y(t) \geqslant \frac{1}{2}H(0)^{1-\lambda} > 0 \quad \text{for } t \in (0,T]. \tag{4.38}$$

Now, we prove that

$$Y'(t) \geqslant \varepsilon^{1+\rho} C Y(t)^{\xi} \quad \text{for } t \in (0,T], \tag{4.39}$$

where

$$\xi := \frac{1}{1-\lambda}, \rho := \frac{1}{\eta}\Big(1 - \frac{2}{(1-2\lambda)(r+1)}\Big),$$

and C is a positive constant. It is important to note that the definition of λ implies that $1 < \xi < 2$ and $\rho > 0$.

Case 1: $M'(t) \leqslant 0$ for some $t \in (0,T]$, then for such values of t we have

$$Y(t)^{\xi} = [H(t)^{1-\lambda} + \varepsilon M'(t)]^{\xi} \leqslant H(t), \tag{4.40}$$

and in this case (4.36) and (4.40) yield

$$Y'(t) \geqslant C\epsilon H(t) \geqslant C\varepsilon^{1+\rho}H(t) \geqslant C\varepsilon^{1+\rho}Y(t)^{\xi}. \tag{4.41}$$

Hence, (4.39) holds for all $t \in (0,T]$ for which $M'(t) \leqslant 0$. However, if $t \in (0,T]$ is such that $M'(t) > 0$, then (4.39) is still valid, but with some more work. First we note that

$$Y(t)^{\xi} \leqslant 2^{\xi-1}[H(t) + M'(t)^{\xi}]. \tag{4.42}$$

In order to estimate $M'(t)^{\xi}$ we use Hölder's and Young's inequalities and, since $1 < \xi < 2$, we obtain

$$\begin{aligned} M'(t)^{\xi} &= 2^{\xi}\left[\int_{\Omega}(u(t)u'(t) + v(t)v'(t))dx\right]^{\xi} \\ &\leqslant 2^{\xi}(|u(t)||u'(t)| + |v(t)||v'(t)|)^{\xi} \\ &\leqslant C(|u(t)|_{r+1}|u'(t)| + |v(t)|_{r+1}|v'(t)|)^{\xi} \\ &\leqslant C(|u(t)|_{r+1}^{\xi}|u'(t)|^{\xi} + |v(t)|_{r+1}^{\xi}|v'(t)|^{\xi}) \\ &\leqslant C\big(|u(t)|_{r+1}^{\frac{2\xi}{2-\xi}} + |u'(t)|^2 + |v(t)|_{r+1}^{\frac{2\xi}{2-\xi}} + |v'(t)|^2\big). \end{aligned} \tag{4.43}$$

Using the definition of λ, namely, $\lambda < \frac{r-1}{2(r+1)}$, it is easy to see that

$$\frac{2\xi}{(2-\xi)(r+1)} - 1 = \frac{2}{(1-2\lambda)(r+1)} - 1 < 0.$$

Thus, by noting (4.4) and using (4.10) and the definition of ρ, we obtain

$$\begin{aligned} |u(t)|_{r+1}^{\frac{2\xi}{2-\xi}} &= (|u(t)|_{r+1}^{r+1})^{\frac{2\xi}{(2-\xi)(r+1)}} \\ &\leqslant CG(t)G(t)^{\frac{2\xi}{(2-\xi)(r+1)}-1} \\ &\leqslant CG(t)H(0)^{\frac{2\xi}{(2-\xi)(r+1)}-1} \\ &\leqslant C\varepsilon^{-\rho}G(t). \end{aligned} \tag{4.44}$$

Likewise,

$$|v(t)|_{r+1}^{\frac{2\xi}{2-\xi}} \leqslant C\varepsilon^{-\rho}G(t). \tag{4.45}$$

Combining (4.43)–(4.45), we have

$$M'(t)^{\xi} \leqslant C\varepsilon^{-\rho}(|u'(t)|^2 + |v'(t)|^2 + G(t)). \tag{4.46}$$

Finally, (4.36), (4.42)–(4.43) and (4.46) allow us to conclude that

$$\begin{aligned} Y'(t) &\geqslant C\varepsilon[H(t) + |u'(t)|^2 + |v'(t)|^2 + G(t)] \\ &\geqslant C\varepsilon^{1+\rho}Y(t)^{\xi}, \end{aligned} \tag{4.47}$$

for all values of $t \in (0, T]$ for which $M'(t) > 0$. Therefore, (4.39) is valid and thus $Y(t)$ blows up in finite time T, where

$$T < C\varepsilon^{-1-\rho}Y(0)^{-\lambda\backslash(1-\lambda)}. \tag{4.48}$$

Also, (4.38) and (4.48) yield the following upper bound for the life span of the solution

$$T < C\varepsilon^{-1-\rho}[H(0)^{1-\lambda} + \varepsilon M'(0)]^{-\lambda\backslash(1-\lambda)} \leqslant C\varepsilon^{-1-\rho}H(0)^{-\lambda}. \tag{4.49}$$

Finally, by recalling (4.12) for Case 2 ($H(0) > 0$ is sufficiently small), we have

$$T < CH(0)^{-[\lambda+\eta(1+\rho)]}. \tag{4.50}$$

This completes the proof. □

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ЛОКАЛНО ПОСТОЈАЊЕ И НАРУШАВАЊЕ (BLOW-UP) РЕШЕЊА ЗА ВЕЗАНЕ ВИСКОЕЛАСТИЧНЕ ТАЛАСЕ СА ДЕГЕНЕРИСАНИМ ПРИГУШЕЊИМА

Резиме. У раду се проучава нелинеарни систем вискоеластичних једначина са дегенерисаним пригушењима у ограниченом домену. Под одговарајућим претпоставкама на параметре, дегенерисана пригушења и функције релаксације ϖ_i, $(i = 1, 2)$, коришћењем методе Фаедо-Галеркина, доказује се локално постојање и јединственост решења. Даље, показује се нарушавање (blow-up) слабих решења проблема (1.1), што побољшава раније резултате у литератури [6, 23, 25].

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