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INVESTIGATION OF THE INTERACTION OF TWO PARALLEL SHIFTED CRACKS IN PLATE BENDING ADJUSTED FOR THEIR CLOSURE

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Abstract. The problem of interaction of two parallel shifted cracks in plate bending is considered. The cracks closure has been investigated in the classical two-dimensional statement, using the model of smooth contact along a line. The influence of the relative position of cracks and of the contact of their edges on the forces and moment intensity factors has been studied by the singular integral equations method.

The thin-walled elements of structures are widely used in modern technology and machine building. The presence of crack-like defects in plates or shells may significantly reduce the operating time of both separate elements and whole mechanisms. Under the action of the bending component of the load, the crack edges can interact due to the closing effect. In this case, the stress-strain state is redistributed at the defects tips and, as a result, the calculated parameters of the limiting equilibrium of the defective structure are changed. Hence, the investigation of the stress state in proximity to cracks in thin-walled elements adjusted for the reciprocal influence and closure of defects is currently a relevant problem of fracture mechanics.

The interaction of through-thickness defects without regard to the contact of their edges during plate bending was studied [1–4], and the classical theory and Reisner's theory were used. Most of the results in this direction have been collected and systematized by Murakami [5].

The main approaches to the modeling of cracks closure in plate bending within the classical theory and contact along the line model were described in works by Shatsky, Young, Khludnev, Zehnder [6–10]. Shatsky [11] was the first to investigate the cracks closure effect in the simultaneous extension and bending of a plate. As to the problem of reciprocal influence of contact cracks, the most studied cases are those of collinear defects in plates [12–15] and shells [16–20]. The periodical problems for parallel defects were obtained, taking into account crack closure in

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infinite [21] and half-infinite [22] plates. The interaction of contact cracks with slits in the bending of infinite plates was investigated in [23, 24].

Because of the need to take into account the asymmetric modes of deformation, the problem of the closure of the randomly oriented defects system has not been sufficiently investigated so far. Shatsky investigated only the cases of interaction of two non-shifted parallel cracks [25], and Dalyak solved the periodic problem for shifted parallel defects [26].

The aim of the paper is to investigate the impact of cracks closure with both symmetric and asymmetric deformation modes in the bending of an infinite plate with the system of two parallel shifted cracks.

1. Formulation and integral equations of the problem

Let us consider the infinite plate $(x, y, z) \in R^2 \times [-h, h]$ with the system of two cross-cutting cracks parallel to the axis x. The cracks' lengths are $2l$ and the centers of the cuts are on the straight line running through the origin and sloped to the abscissa axis at the angle β. Let us assume that d is the distance between the centers. We shall assume that the plate is on infiniteness under the influence of evenly distributed moment $m_y^\infty = m = const$, and its obverse surfaces are free from stresses (fig. 1). The interaction of defects is investigated, taking into consideration the contacts of their edges.

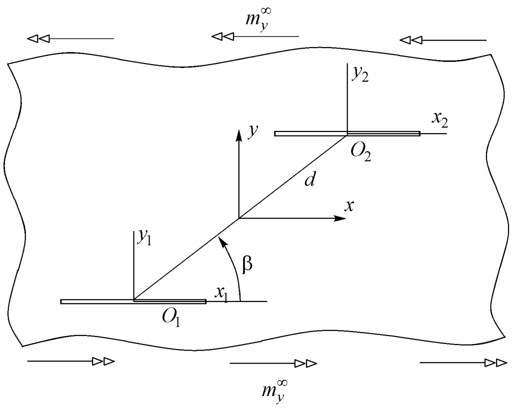


FIGURE 1. Two parallel shifted cracks

The stresses-strain state of the cracked plate is described by a pair of biharmonic equations of the plane stress state and technical bending theory:

$$\Delta\Delta\varphi = 0, \quad \Delta\Delta w = 0. \tag{1.1}$$

With this situation of cracks' location and external loading, the problem is symmetric to the origin of coordinates, hence it is sufficient to satisfy the boundary conditions of the smooth contact only on one of the cuts:

$$\begin{array}{lll} [u_y](x) = h|[\vartheta_y](x)|, & N_{xy}(x, 0) = 0, & M_y(x, 0) = hN_y(x, 0)\,\mathrm{sign}[\vartheta_y], \\ P_y(x, 0) = C, & N_y(x, 0) \leqslant 0, & x \in (-l,\ l). \end{array} \tag{1.2}$$

At infinity

$$
\begin{aligned}
&N_x = N_{xy} = N_y = 0, \qquad Q_y = Q_x = 0;\\
&M_x = M_{xy} = 0, \qquad M_y = m, \qquad (x, y) \to \infty.
\end{aligned} \tag{1.3}
$$

Here and hereafter, φ, w is Airy stress function, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is Laplasian operator, N_{ij}, Q_i are membrane and shear forces, M_{ij} are moments, $P_y = \int Q_y^* dx$, C is arbitrary constant, $Q^* = Q + \partial M_{xy}/\partial x$–generalized shear force according to Kirchhoff law, $[u_x], [u_y]$ are displacement discontinuities of the median surface, $[\vartheta_x], [\vartheta_y]$ are jumps of the angles of the normal rotation.

So, (1.1)–(1.3) are the boundary problem for the system of two parallel cracks with contacting edges.

By using the procedure of singular integral equations method [7], we reduced this formulated problem to the system of integral equations:

$$
\begin{aligned}
&hs\, f_1'(t) = \varkappa f_3'(t),\\
&\frac{1}{\pi}\int_{-1}^{1} \Big\{ f_1'(t)\, T_{21}(\tau,t) + f_2'(t)\Big(\frac{1}{\tau - t} + T_{22}(\tau,t)\Big)\Big\} d\tau = 0,\\
&\frac{1}{\pi}\int_{-1}^{1} \Big\{ f_3'(t)\Big(\frac{1+\varkappa}{\tau - t} + T_{33}(\tau,t) + \varkappa T_{11}(\tau,t)\Big)\Big\} d\tau\\
&\qquad + \frac{1}{\pi}\int_{-1}^{1} \{hs\, f_2'(t)\, T_{12}(\tau,t) + f_4'(yt)\, T_{34}(\tau,t)\} d\tau = -m,\\
&\frac{1}{\pi}\int_{-1}^{1} \Big\{ f_3'(t)\, T_{43}(\tau,t) + f_4'(t)\Big(\frac{1}{\tau - t} + T_{44}(\tau,t)\Big)\Big\} d\tau = C,
\end{aligned} \tag{1.4}
$$

with supplementary conditions: $\int_{-1}^{1} f_n'(\tau) d\tau = 0$, $\int_{-1}^{1} \tau\, f_4'(\tau) d\tau = 0$, $n = \overline{1, 4}$. Here $t = x/l$ is a non-dimensional coordinate, taken relative to the semi-length of the defect τ is integration variable;

$f_1'(t) = B[u_y]'(x)/4$, $\qquad f_2'(t) = B[u_x]'(x)/4$,

$f_3'(yt) - Da[\vartheta_y]'(x)/4$, $\qquad f_4'(t) = -Da[\vartheta_x]'(x)/4$,

$s = -\operatorname{sign} f_3(t)$

$$
T_{11}(\tau,t) = \lambda \frac{z_x}{r^2}\Big(1 + \frac{2z_y^2}{r^2}\Big), \qquad T_{12}(\tau,t) = T_{21}(\tau,t) = \lambda \frac{z_y}{r^2}\Big(1 - \frac{2z_y^2}{r^2}\Big),
$$

$$
T_{22}(\tau,t) = \lambda \frac{z_y}{r^2}\Big(1 - \frac{2z_y^2}{r^2}\Big), \qquad T_{33}(\tau,t) = \lambda \frac{z_x}{r^2}\Big(1 - \varkappa_0 \frac{2z_y^2}{r^2}\Big),
$$

$$
T_{34}(\tau,t) = T_{43}(\tau,t) = -\varkappa_0 \lambda \frac{z_y}{r^2}\Big(1 - \frac{2z_y^2}{r^2}\Big), \qquad T_{44}(\tau,t) = \lambda \frac{z_x}{r^2}\Big(1 + \varkappa_0 \frac{2z_y^2}{r^2}\Big);
$$

$r^2 = z_x^2 + z_y^2$, $\qquad z_x = \lambda(\tau - t)/2 + \cos\beta$,

$z_y = \sin\beta, \lambda = 2l/d$; $\qquad B = 2Eh$,

$D = 2Eh^3/(y3(1 - \nu^2))$, $\qquad a = (3+\nu)(1-\nu)$,

$\varkappa = 3(1+\nu)/(3+\nu)$, $\qquad \varkappa_0 = (1-\nu)/(3+\nu)$;

E and ν are Young's modulus and Poisson's ratio for the plate material.

2. Asymptotic problem solution

The asymptotic problem solution (1.1)–(1.3) for large distances between the defects was constructed by the small parameter method [7]. The analytical expressions of the required function look as follows:

$$f_1'(t) = -\varkappa \operatorname{sign} \frac{m}{h} f_3'(t), \tag{2.1}$$

$$\begin{aligned}
f_2'(t) = &-\varkappa \frac{|m|}{h}(1+\varkappa)\sqrt{1-t^2}\Big\{-\frac{1}{2}\lambda^2 a_{11}^{12} H_0(yt) + \frac{1}{2}\lambda^3 a_{21}^{12} H_1(t)\\
&+\frac{1}{4}\lambda^4\Big[2a_{31}^{12}H_2(t) - \frac{3}{2}a_{33}^{12}H_0(yt) + (a_{11}^{12}b_{11} + a_{11}^{12}a_{11}^{12})H_0(t)\Big]\\
&-\frac{1}{4}\lambda^5\Big[\Big(a_{21}^{12}b_{11} + a_{11}^{12}a_{21}^{22} - \frac{3}{2}a_{43}^{12}\Big)H_1(yt) - 2a_{41}^{12}H_3(t)\Big] + O(\lambda^6)\Big\},\\
f_3'(t) = &-\frac{m}{(1+\varkappa)\sqrt{1-t^2}}\Big\{-H_0(yt) + \frac{1}{2}\lambda^2 b_{11}H_0(t) + \frac{1}{2}\lambda^3 b_{21}H_1(t)\\
&+\frac{1}{4}\lambda^4\Big[2b_{31}H_2(t) - \Big(b_{11}^2 + \frac{\varkappa s^2}{1+\varkappa}(a_{11}^{12})^2 - \frac{3}{2}b_{33}\Big)H_0(t)\Big]\\
&-\frac{1}{4}\lambda^5\Big[\Big(b_{21}b_{11} + \frac{\varkappa s^2}{1+\varkappa}a_{11}^{12}a_{21}^{12} - \frac{3}{2}b_{43}\Big)H_1(t) - 2b_{41}H_3(t)\Big] + O(\lambda^6)\Big\},\\
f_4'(t) = &-\frac{m}{(1+\varkappa)\sqrt{1-t^2}}\Big\{\frac{1}{2}\lambda^3 a_{21}^{34}H_1(t) + \frac{1}{4}\lambda^4\Big(2H_2(t) - \frac{1}{2}H_0(t)\Big)a_{31}^{34}\\
&-\frac{1}{4}\lambda^5\Big[\Big(a_{21}^{34}b_{11} - \frac{3}{2}a_{43}^{34}\Big)H_1(t) - 2a_{41}^{12}H_3(t)\Big] + O(\lambda^6)\Big\},\\
C = &\frac{m}{1+\varkappa}\Big\{-\frac{1}{2}\lambda^2 a_{11}^{34} + \frac{1}{8}\lambda^4(2b_{11}a_{11}^{34} - a_{31}^{34} - 3a_{33}^{34}) + O(\lambda^6)\Big\}
\end{aligned}$$

Here

$$\begin{aligned}
a_{pq}^{11} &= (-1)^p(\tfrac{1}{2})^{p+2}\{-(p+1)C_p^q\cos(\pi p + \beta(p+3))\\
&\quad + (3C_p^q + pC_{p-1}^q + pC_{p-1}^{q-1})\cos(\pi p + \beta(p+1))\},\\
a_{pq}^{12} &= a_{pq}^{21} = (-1)^p(\tfrac{1}{2})^{p+2}\{(p+1)C_p^q\sin(\pi p + \beta(p+3))\\
&\quad - (C_p^q + pC_{p-1}^q + pC_{p-1}^{q-1})\sin(\pi p + \beta(p+1))\},\\
a_{pq}^{22} &= (-1)^p(\tfrac{1}{2})^{p+2}\{(p+1)C_p^q\cos(\pi p + \beta(p+3))\\
&\quad + (C_p^q - pC_{p-1}^q - pC_{p-1}^{q-1})\cos(\pi p + \beta(p+1))\},\\
a_{pq}^{33} &= (-1)^p(\tfrac{1}{2})^{p+2}\{\varkappa_0(p+1)C_p^q\cos(\pi p + \beta(p+3))\\
&\quad + ((2-\varkappa_0)C_p^q - \varkappa_0 pC_{p-1}^q - \varkappa_0 pC_{p-1}^{q-1})\cos(\pi p + \beta(p+1))\},\\
a_{pq}^{34} &= -\varkappa_0 a_{pq}^{12}\\
a_{pq}^{44} &= (-1)^p(\tfrac{1}{2})^{p+2}\{-\varkappa_0(yp+1)C_p^q\cos(\pi p + \beta(p+3))\\
&\quad + ((2+\varkappa_0)C_p^q + \varkappa_0 pC_{p-1}^q + \varkappa_0 pC_{p-1}^{q-1})\cos(\pi p + \beta(p+1))\},\\
b_{pq} &= \frac{a_{pq}^{33} + \varkappa a_{pq}^{11}}{1+\varkappa}, \quad C_p^n - \text{binomial coefficients;}
\end{aligned}$$

$$H_0(t) = -t, \quad H_{2m}(t) = tH_{2m-1}(t),$$
$$H_{2m-1}(t) = -t^{2m} + \sum_{\nu=0}^{m-1} \frac{(2m-2\nu-3)!!}{(2m-2\nu)!!} t^{2\nu}, \qquad (m = 1, 2, \ldots; \quad (-1)!! = 1).$$

With the known step functions we can determine the other characteristics of the stressed-deformed state of the plate. Namely, for the factors of the forces and moments intensity in the neighbourhood of the crack tips ($K_i^{\pm} = \mp\sqrt{l} \lim\limits_{t\to\pm1} \sqrt{1-t^2} f_i'(t)$, $i = \overline{1,4}$) we obtained the correlation:

$$\begin{aligned}
&K_1^{\pm} = \frac{\varkappa \operatorname{sign}(m)}{h} K_3^{\pm} = \frac{\varkappa}{h} |K_3^{\pm}|,\\
&K_2^{\pm} = \frac{|m|}{h} \frac{\varkappa}{1+\varkappa} \{ \tfrac{1}{8}\lambda^2 S_2 \pm \tfrac{3}{32}\lambda^3 S_3 + \tfrac{1}{64}\lambda^4 (2\cos 2\beta S_2 + \gamma_0 S_2 C_2 - 9S_4) \mp\\
&\qquad \mp \tfrac{1}{256}\lambda^5 (3(\cos 2\beta + \gamma_0 C_2) S_3 + (3\cos 5\beta + \cos 3\beta) S_2 - 6S_5) + O(\lambda^6)\},\\
&K_3^{\pm} = \frac{m}{1+\varkappa} \{ 1 - \tfrac{1}{8}\lambda^2 (\cos 2\beta + \gamma_0 C_2) \pm \tfrac{1}{32}\lambda^3 (y2\cos 3\beta + 3\gamma_0 C_3) S_3 +\\
&\qquad + \tfrac{1}{64}\lambda^4 ((\varkappa + \varkappa_0^2) S_2^2 - (\cos 2\beta + \gamma_0 C_2)^2 + \tfrac{9}{2}(\cos 4\beta + 2\gamma_0 C_4)) \pm\\
&\qquad \pm \tfrac{1}{256}\lambda^5 ((\varkappa + \varkappa_0^2) S_2 S_3 + (\cos 2\beta + \gamma_0 C_2)(2\cos 3\beta + 3\gamma_0 C_3) +\\
&\qquad + 6(2\cos 5\beta + 5\gamma_0 C_5)) + O(\lambda^6)\},\\
&K_4^{\pm} = \frac{|m|}{h} \frac{\varkappa}{1+\varkappa} \Big\{ \mp \frac{3}{32}\lambda^3 \varkappa_0 S_3 - \frac{3}{64}\lambda^4 \varkappa_0 S_4 \mp\\
&\qquad \mp \frac{3\varkappa_0}{256}\lambda^5 ((\cos 2\beta + \gamma_0 C_2) S_3 - 10 S_5) + O(\lambda^6) \Big\}.
\end{aligned} \tag{2.2}$$

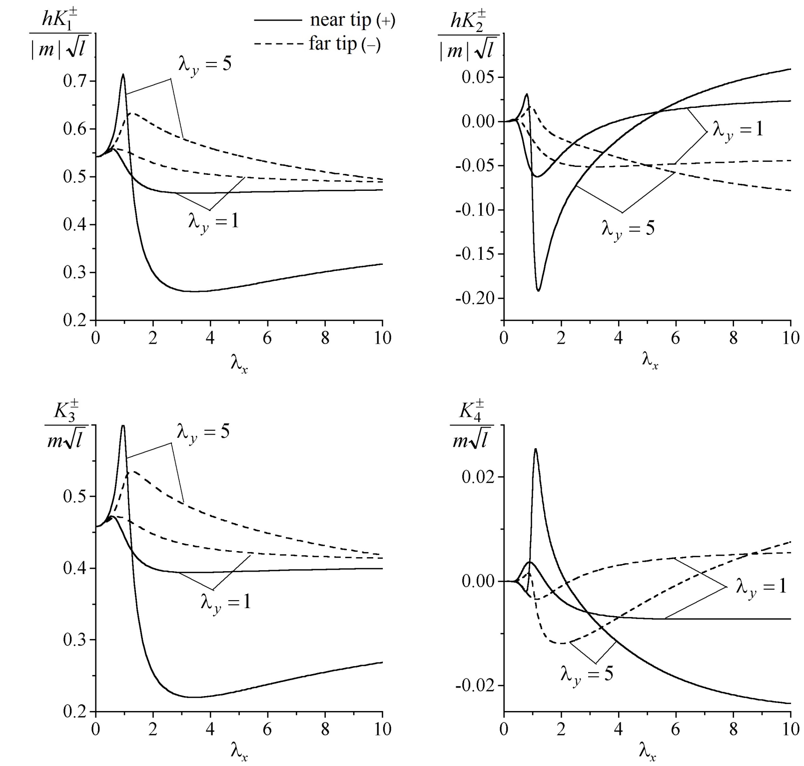
Here $S_n = \sin n\beta - \sin((n+2)\beta)$, $C_n = \cos n\beta - \cos((n+2)\beta)$, $\gamma_0 = (\varkappa - \varkappa_0)/(1+\varkappa)$.

It should be noted that by formal substitution $\varkappa = 0$ in formulae (2.1) and (2.2) we can reach a solution to the problem of the plate bending with cracks not adjusted for the contact of their edges [1,6]. In particular cases $\beta = 0$ and $\beta = \pi/2$ we obtain known solutions respectively for collinear [13] and parallel [11] contact cracks, and at $\lambda = 0$–the analytical solution to the closure of an isolated rectilinear crack problem [9,12,20].

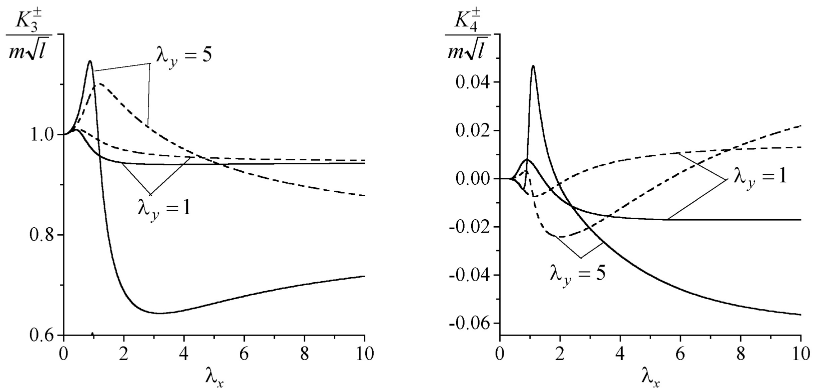
3. Analysis of results

The numerical solution to the system of singular integral equations (1.4) was constructed by the quadrature method [7] at $\nu = 0, 3$. We have examined the horizontal approaching of cracks: in this case the parameter $\lambda_x = \lambda/\cos\beta$ changes, and the value $\lambda_y = \lambda/\sin\beta$ remains fixed. Fig. 2 shows schematic dependencies of the membrane forces intensity and bending moments factors on λ_x at constant values $\lambda_y = 1;\ 5$; for the same parameters Fig. 3 shows the distribution of contact forces along the length of the crack.

We can see from analytical solutions (2.1), (2.2) and charts that the closure of the crack leads to a significant reduction in coefficients of the intensity of moments K_3, K_4 and to the emergence of nonzero coefficients of the intensity of forces K_1, K_2, yet the qualitative character of the dependencies of the intensity coefficients on λ_x, λ_y practically does not change.



a) solution with the contact of crack edges



b) classical solution without thecontact of crack edges

FIGURE 2. Dependencies of the membrane forces intensity and bending moments factors on the relative position of defects

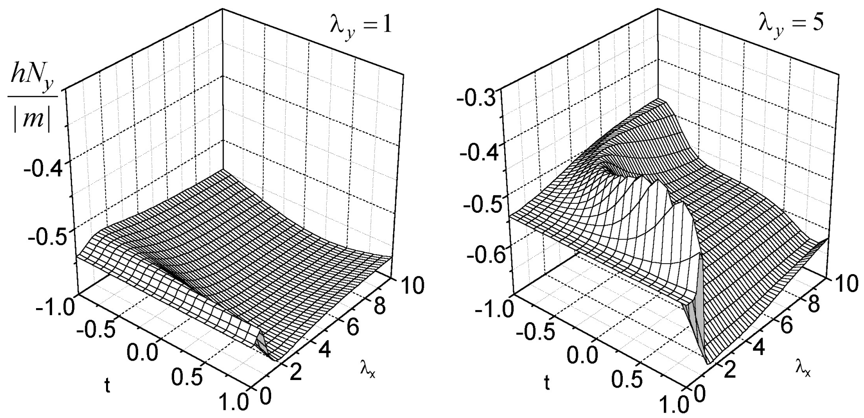


FIGURE 3. Distribution of contact forces along the length of the crack

4. Conclusions

The values of intensity factors are larger in external peaks for horizontally remote cracks, while the internal tips become more dangerous during significant nearing. The contact reaction between the edges of the crack recedes in close proximity to the peak of the neighbor defect.

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ИСТРАЖИВАЊЕ ИНТЕРАКЦИЈЕ ДВЕ ПАРАЛЕЛНО ПОМЕРЕНЕ ПУКОТИНЕ ПРИ САВИЈАЊУ ПЛОЧЕ РАДИ ЊИХОВОГ ЗАТВАРАЊА

Резиме. Разматран је проблем интеракције две паралелно померене пукотине при савијању плоче. Затварање пукотина истражено је у класичној дводимензионалној формулацији, користећи модел глатког контакта дуж линије. Утицај релативног положаја пукотина и додира њихових ивица на факторе силе и интензитета момента проучаван је методом сингуларних интегралних једначина.

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