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ON THE STABILITY OF AN EQUILIBRIUM AND THE SMALL MOTIONS OF A RIGID BODY CONTAINING A LIQUID, SUSPENDED IN A UNIFORM FLOW

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Abstract. In this paper, we consider a planar motion of a rigid body partially filled with an inviscid liquid and suspended in a uniform horizontal flow. At first, we write the equations of the problem, prove the existence of an equilibrium under a suitable condition and, using a first integral, we give a sufficient condition of stability of this one. Afterwards, we give the equations of the small oscillations of the system about its equilibrium position. Writing these equations in an operatorial form, we prove the existence of a denumerable infinity of complex conjugate pairs of eigenvalues having the infinity as a point of accumulation and obtain the characteristic equation permitting the calculation of the eigenvalues.

1. Introduction

The theory of unsteady motions of a profile in an incompressible inviscid fluid in irrotational motion was worked out independently by Sedov [11] and Couchet [2,4]. An account of these problems can be found in a paper by other authors [1].

On the other hand, the study of the stability of motion of a rigid body containing a liquid has been the subject of numerous works. The foundations of this theory can be found in a classic book [9]. Finally, the study of the small oscillations of a liquid in a container by means of the methods of functional analysis is of interest to numerous scientists. The theory and references can be found in [6,7,9,10].

The problem of the stability of an equilibrium position of a profile in a uniform flow was tackled by Couchet in the framework of his work on the critical velocity of a wing [3].

In this work, we consider the problem of the equilibrium and the small motions of a rigid body suspended by one of its points, having a cavity partially filled with an inviscid, incompressible liquid, and set in a uniform horizontal flow, restricting ourselves to the planar problem (like in the theory of profiles).

Such a problem can arise in naval hydrodynamics for a submerged body containing a liquid.

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In the first part of the paper, we write the equations of the problem, prove the existence of a vertical equilibrium position under suitable conditions and, using a first integral, we obtain a sufficient condition of stability of this one by means of the Rumiantsev's method [9].

In the second part, we write the linearized equations of the small oscillations about the equilibrium position. Writing these equations in operatorial form, we can prove the existence of a denumerable infinity of complex conjugate pairs of eigenvalues having the infinity as a point of accumulation.

Finally, we obtain the characteristic equation that permits studying the eigenvalues graphically.

2. Preliminary remarks

We consider a profile without a sharp edge, moving in an incompressible inviscid fluid, in irrotational motion, at rest at infinity. It is well-known [2, 4, 11] that, by using the theory of functions of a complex variable, it is possible to calculate the field of the fluid velocities, the pressure in the fluid and the aerodynamic forces exerted by the fluid on the profile.

Consequently, it is possible to obtain the equations of motion of the profile submitted to given forces.

It is convenient to use a reduced form of these equations [4, 8].

It can be found that particular orthogonal axes Oxy are rigidly fixed to the profile such that, under the following notations:

- ℓ, m are the components of the velocity of O on Ox, Oy,
- θ is the angle $\widehat{O_1x_1, Ox}$ between Ox and a fixed axis O_1x_1 (we set $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega$),
- X, Y are the components on Ox, Oy of the resultant of the forces different from the aerodynamic forces, acting on the profile,
- N is the moment about O of these forces,
- $\tilde{\Gamma}$ is the circulation about the profile, constant by virtue of Helmholtz's theorem,
- x_c, y_c are the coordinates of the centre C of the profile [2, 4],
- ρ is the constant density of the fluid,

the equations of the motion of the profile can be written

$$\begin{cases} A\frac{\mathrm{d}l}{\mathrm{d}t} - B\omega m = X - \rho\tilde{\Gamma}(m + \omega x_c), \\ B\frac{\mathrm{d}m}{\mathrm{d}t} + A\omega\ell = Y + \rho\tilde{\Gamma}(\ell - \omega y_c), \\ C\frac{\mathrm{d}\omega}{\mathrm{d}t} + (B - A)\ell m = N + \rho\tilde{\Gamma}(\ell x_c + m y_c), \end{cases} \tag{2.1}$$

where A, B, C are positive constants depending on the profile and on the external fluid ($B \geqslant A$). It can be shown that, if the profile lies in a uniform flow with the constant velocity V parallel to O_1x_1, the equations of its motion can be obtained by replacing ℓ and m by $\ell - V\cos\theta$, $m + V\sin\theta$ [2, 4].

Remark 2.1. The precedent theory is not valid for a profile with a sharp edge, because vortices appear behind the profile.

3. Position of the problem

We consider a planar body without a sharp edge, having a cavity partially filled with an incompressible inviscid liquid and moving in a uniform flow of constant horizontal velocity V (see Figure 1).

For simplicity, we suppose that the axis Oy is the axis of symmetry of the cavity.

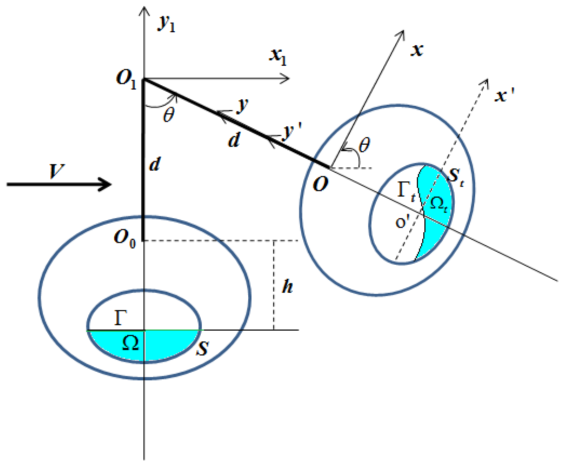

FIGURE 1. Model of the system.

We suppose that the axis Oy carries a rod OO_1 with negligible mass, which rotates about a fixed point O_1 ($OO_1 = d$). At the instant $t = 0$, O_1O occupies the position O_1O_0 vertical, directed downwards and we set $\widehat{\overrightarrow{O_1O_0}, \overrightarrow{O_1O}} = \theta$. We take as fixed axes O_1x_1 horizontal, O_1y_1 directed upwards.

We suppose that the motions of the external fluid are irrotational, with velocity $V\vec{x}_1$ at the infinity.

Under convenient conditions, the body has an equilibrium position which O_1O_0 is vertical, directed downwards; the liquid occupies a domain Ω bounded by the wetted part S of the boundary of the cavity and the horizontal free line Γ whose equation is $y_1 = -d - h$. Ω_t, S_t, Γ_t are the positions of Ω, S, Γ at the instant t.

We denote by $\vec{R}_0 = X_0\vec{x} + Y_0\vec{y}$ and $\vec{N}_0 = N_0\vec{z}$ ($\vec{z} = \vec{x} \times \vec{y}$) the resultant and the moment about O of the forces exerted by the rod O_1O on the body in O. Since the mass of O_1O is negligible, we have, taking the moment about O_1 of the forces acting on the rod:

$$\overrightarrow{O_1O} \times (-\vec{R}_0) + (-\vec{N}_0) = 0,$$

$$dX_0 + N_0 = 0. \tag{3.1}$$

4. Equations of motion of the system body-liquid

4.1. Equations of motion of the body. We denote by μ the mass of the body, (x_G, y_G) the coordinates of its centre of inertia, $\vec{R}_1 = X_1\vec{x}_1 + Y_1\vec{y}_1$ the resultant and $\vec{N}_1 = N_1\vec{z}$ the moment about O of the forces exerted by the internal liquid on the body.

Using the equations (2.1), we have the equations of motion of the body:

$$
(4.1)\quad \begin{cases} A\dfrac{\mathrm{d}(\ell - V\cos\theta)}{\mathrm{d}t} - B\omega(m + V\sin\theta) = -\mu g\sin\theta - \rho\tilde{\Gamma}(m + V\sin\theta + \omega x_c) \\ \qquad\qquad + X_1 + X_0, \\ B\dfrac{\mathrm{d}(m + V\sin\theta)}{\mathrm{d}t} + A\omega(\ell - V\cos\theta) = -\mu g\cos\theta + \rho\tilde{\Gamma}(\ell - V\cos\theta - \omega y_c) \\ \qquad\qquad + Y_1 + Y_0, \\ C\dfrac{\mathrm{d}\omega}{\mathrm{d}t} + (B - A)(\ell - V\cos\theta)(m + V\sin\theta) = -\mu g(x_G\cos\theta - y_G\sin\theta) \\ \qquad\qquad + \rho\tilde{\Gamma}[(\ell - V\cos\theta)x_c + (m + V\sin\theta)y_c] + N_1 + N_0. \end{cases}
$$

We are going to write the equations of the motion of the mass of the liquid.

If we denote by $\vec{\gamma}_{\mathrm{a}}$ the acceleration of a particle M of the liquid, by ρ^* its constant density, by μ_ℓ its mass, by (x_{G_ℓ}, y_{G_ℓ}) the (variable) coordinates of its centre of inertia, we have:

$$
(4.2)\quad \begin{cases} \int_{\Omega_t} \rho^*\gamma_{ax}\mathrm{d}\Omega_t = -\mu_\ell g\sin\theta - X_1; \\ \int_{\Omega_t} \rho^*\gamma_{ay}\mathrm{d}\Omega_t = -\mu_\ell g\cos\theta - Y_1; \\ \int_{\Omega_t} \rho^*(x\gamma_{ay} - y\gamma_{ax})\mathrm{d}\Omega_t = -\mu_\ell g(x_{G_\ell}\cos\theta - y_{G_\ell}\sin\theta) - N_1. \end{cases}
$$

Adding the equations (4.1) and (4.2), we obtain the equations of motion of the system body-liquid:

$$
(4.3)\quad \begin{cases} A\dfrac{\mathrm{d}(\ell - V\cos\theta)}{\mathrm{d}t} - B\omega(m + V\sin\theta) + \displaystyle\int_{\Omega_t} \rho^*\gamma_{ax}\mathrm{d}\Omega_t \\ \qquad = -(\mu + \mu_\ell)g\sin\theta - \rho\tilde{\Gamma}(m + V\sin\theta + \omega x_c) + X_0, \\ B\dfrac{\mathrm{d}(m + V\sin\theta)}{\mathrm{d}t} + A\omega(\ell - V\cos\theta) + \displaystyle\int_{\Omega_t} \rho^*\gamma_{ay}\mathrm{d}\Omega_t \\ \qquad = -(\mu + \mu_\ell)g\cos\theta + \rho\tilde{\Gamma}(\ell - V\cos\theta - \omega y_c) + Y_0, \\ C\dfrac{\mathrm{d}\omega}{\mathrm{d}t} + (B - A)(\ell - V\cos\theta)(m + V\sin\theta) + \displaystyle\int_{\Omega_t} \rho^*(x\gamma_{ay} - y\gamma_{ax})\mathrm{d}\Omega_t \\ \qquad = -\mu g(x_G\cos\theta - y_G\sin\theta) - \mu_\ell g(x_{G_\ell}\cos\theta - y_{G_\ell}\sin\theta) \\ \qquad\qquad + \rho\tilde{\Gamma}[(\ell - V\cos\theta)x_c + (m + V\sin\theta)y_c] + N_0. \end{cases}
$$

4.2. Equations of motion of the liquid. If we denote by P the pressure of the liquid, we have:

$$
(4.4)\quad \rho^*\vec{\gamma}_a = -\overrightarrow{\mathrm{grad}}\, P - \rho^* g\vec{y}_1 \quad \text{in} \quad \Omega_t.
$$

If $\vec{V}_r$ is the velocity of a particle of the liquid, with respect to the body, we have the incompressibility condition

$$
(4.5)\quad \operatorname{div} \vec{V}_r = 0 \quad \text{in} \quad \Omega_t,
$$

and the kinematic condition on the wetted boundary

$$
(4.6)\quad \vec{V}_r \cdot \vec{n} = 0 \quad \text{on} \quad S_t,
$$

$\vec{n}$ being the external unit vector normal to S_t.

Finally, if we suppose that the pressure is equal to zero above the free line, we have the dynamic condition:

$$P = 0 \quad \text{on} \quad \Gamma_t. \tag{4.7}$$

4.3. Equilibrium condition and first integral of motion.

4.3.1. *A deduced equation from the equation* (4.3). Obviously, we have

$$\ell = d\dot{\theta}; \quad m = 0,$$

so that the first and the third equations (4.3) can be written

$$\begin{cases} Ad\ddot{\theta} - (B-A)V\sin\theta\cdot\dot{\theta} + \displaystyle\int_{\Omega_t}\rho^*\gamma_{ax}\mathrm{d}\Omega_t = -(\mu+\mu_\ell)g\sin\theta \\ \qquad\qquad - \rho\tilde{\Gamma}(V\sin\theta + \dot{\theta}x_c) + X_0, \\ C\ddot{\theta} + (B-A)V\sin\theta\cdot\dot{\theta} - (B-A)V^2\sin\theta\cos\theta + \displaystyle\int_{\Omega_t}\rho^*(x\gamma_{ay} - y\gamma_{ax})\mathrm{d}\Omega_t \\ \qquad = -\mu g(x_G\cos\theta - y_G\sin\theta) - \mu_\ell g(x_{G_\ell}\cos\theta - y_{G_\ell}\sin\theta) \\ \qquad\qquad + \rho\tilde{\Gamma}[(d\dot{\theta} - V\cos\theta)x_c + V\sin\theta y_c] + N_0. \end{cases}$$

Eliminating X_0 and N_0 by means of the equation (3.1), we obtain

$$\begin{aligned} &(C+Ad^2)\ddot{\theta} - (B-A)V^2\sin\theta\cos\theta + \int_{\Omega_t}\rho^*[x\gamma_{ay} - (y-d)\gamma_{ax}]\mathrm{d}\Omega_t \\ &= -\mu g[d\sin\theta + x_G\cos\theta - y_G\sin\theta] - \mu_\ell g[d\sin\theta + x_{G_\ell}\cos\theta - y_{G_\ell}\sin\theta] \\ &\qquad + \rho\tilde{\Gamma}V[d\sin\theta + x_c\cos\theta - y_c\sin\theta]. \end{aligned} \tag{4.8}$$

4.3.2. *Equilibrium condition.* The equilibrium position that is directed downwards exists if the equation (4.8) is satisfied for: $\theta = 0$, $\vec{\gamma}_a = 0$, $\vec{V}_r = 0$.

We easily obtain, x_{G_ℓ} being equal to zero by symmetry:

$$\mu g x_G + \rho\tilde{\Gamma}Vx_c = 0, \tag{4.9}$$

which gives the circulation $\tilde{\Gamma}$ if V is given.

4.3.3. *A few results.* Multiplying the sides of the equation (4.8) by $\dot{\theta}$ and taking into account $\ell = d\dot{\theta}$, $m = 0$, we obtain the following results:

(a)
$$\begin{aligned} -\mu g[d\sin\theta + x_G\cos\theta - y_G\sin\theta] &= -\mu g[(\ell - \omega y_G)\sin\theta + (m + \omega x_G)\cos\theta] \\ &= -\mu g\vec{V}_a(G)\cdot\vec{y}_1 \\ &= -\mu g\frac{\mathrm{d}}{\mathrm{d}t}(\overrightarrow{O_1G}\cdot\vec{y}_1). \end{aligned}$$

(b)
$$\begin{aligned} -\mu_\ell g[d\sin\theta + x_{G_\ell}\cos\theta - y_{G_\ell}\sin\theta] &= -\mu_\ell g[(\ell - \omega y_{G_\ell})\sin\theta + (m + \omega x_{G_\ell})\cos\theta] \\ &= -\mu_\ell g\vec{V}_\ell(G_\ell)\cdot\vec{y}_1, \end{aligned}$$

$\vec{V}_\ell(G_\ell)$ being the velocity of transport of G_ℓ.

(**c**)
$$-\rho\tilde{\Gamma}V[d\sin\theta + x_c\cos\theta - y_c\sin\theta] = -\rho\tilde{\Gamma}V\cdot\vec{V}_a(C)\cdot\vec{y}_1 = -\rho\tilde{\Gamma}V\frac{\mathrm{d}}{\mathrm{d}t}(\overrightarrow{O_1C}\cdot\vec{y}_1).$$

(**d**)
$$\int_{\Omega_t}\rho^*[(d-y)\gamma_{ax} + x\gamma_{ay}]\mathrm{d}\Omega_t\cdot\dot{\theta} = \int_{\Omega_t}\rho^*[(\ell-\omega y)\gamma_{ax} + (m+\omega x)\gamma_{ay}]\mathrm{d}\Omega_t = \int_{\Omega_t}\rho^*\vec{V}_\ell\cdot\vec{\gamma}_a\mathrm{d}\Omega_t$$

where $\vec{V}_\ell$ is the velocity of transport of a particle of the liquid.

Consequently, we obtain the equation

$$(C + Ad^2)\cdot\dot{\theta}\ddot{\theta} - (B-A)V^2\sin\theta\cos\theta\cdot\dot{\theta} + \int_{\Omega_t}\rho^*\vec{V}_\ell\cdot\vec{\gamma}_a\mathrm{d}\Omega_t = -\mu g\frac{\mathrm{d}}{\mathrm{d}t}[\overrightarrow{O_1G}\cdot\vec{y}_1] - \mu_\ell g\vec{V}_\ell(G_\ell)\cdot\vec{y}_1 - \rho\tilde{\Gamma}V\frac{\mathrm{d}}{\mathrm{d}t}[\overrightarrow{O_1C}\cdot\vec{y}_1]. \tag{4.10}$$

4.3.4. *Other formula.* From the Euler's equation (4.4), we deduce

$$\int_{\Omega_t}\rho^*\vec{\gamma}_a\cdot\vec{V}_r\mathrm{d}\Omega_t = -\int_{\Omega_t}\overrightarrow{\mathrm{grad}}P\cdot\vec{V}_r\mathrm{d}\Omega_t - \int_{\Omega_t}\rho^*g\vec{y}_1\cdot\vec{V}_r\mathrm{d}\Omega_t.$$

But, we have

$$\int_{\Omega_t}\overrightarrow{\mathrm{grad}}P\cdot\vec{V}_r\mathrm{d}\Omega_t = \int_{\Omega_t}[\mathrm{div}(P\vec{V}_r) - P\,\mathrm{div}\,\vec{V}_r]\mathrm{d}\Omega_t = \int_{\Gamma_t+S_t}P\cdot\vec{V}_r\cdot\vec{n}\mathrm{d}(\partial\Omega_t) = 0,$$

by virtue of the conditions (4.5), (4.6), (4.7).

Consequently, we obtain

$$\int_{\Omega_t}\rho^*\vec{\gamma}_a\cdot\vec{V}_r\mathrm{d}\Omega_t = -\mu_\ell g\vec{V}_\ell(G_\ell)\cdot\vec{y}_1. \tag{4.11}$$

4.3.5. *The first integral of motion.* Adding the equations (4.10) and (4.11), we have

$$\int_{\Omega_t}\rho^*\vec{V}_\ell\cdot\vec{\gamma}_a\mathrm{d}\Omega_t + \int_{\Omega_t}\rho^*\vec{V}_r\cdot\vec{\gamma}_a\mathrm{d}\Omega_t = \int_{\Omega_t}\rho^*\vec{V}_a\cdot\vec{\gamma}_a\mathrm{d}\Omega_t = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\int_{\Omega_t}\rho^*\vec{V}_a^2\mathrm{d}\Omega_t\Big),$$

$$-\mu_\ell g\vec{V}_\ell(G_\ell)\cdot\vec{y}_1 - \mu_\ell g\vec{V}_r(G_\ell)\cdot\vec{y}_1 = -\mu_\ell g\vec{V}_a(G_\ell)\cdot\vec{y}_1 = -\frac{\mathrm{d}}{\mathrm{d}t}\big[-\mu_\ell g\overrightarrow{O_1G_\ell}\cdot\vec{y}_1\big],$$

so that we obtain the equation

$$(C + Ad^2)\cdot\dot{\theta}\ddot{\theta} - (B-A)V^2\sin\theta\cos\theta\cdot\dot{\theta} + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\frac{\rho^*}{2}\vec{V}_a^2\mathrm{d}\Omega_t = -\frac{\mathrm{d}}{\mathrm{d}t}\big[(\mu g\overrightarrow{O_1G} + \rho\tilde{\Gamma}V\overrightarrow{O_1C} + \mu_\ell g\overrightarrow{O_1G_\ell})\cdot\vec{y}_1\big],$$

and consequently the first integral (energy integral)

$$\frac{1}{2}(C+Ad^2)\dot{\theta}^2 - \frac{1}{2}(B-A)V^2\sin^2\theta + \frac{1}{2}\int_{\Omega_t}\rho^*\vec{V}_a^2\mathrm{d}\Omega_t = -\big(\mu g\overrightarrow{O_1G} + \rho\tilde{\Gamma}V\overrightarrow{O_1C} + \mu_\ell g\overrightarrow{O_1G_\ell}\big)\cdot\vec{y}_1 + Cte. \tag{4.12}$$

4.3.6. *A reduced form of the first integral.* The first integral (4.12) can be written

$$\mathscr{C} + W = Cte,$$

with

$$\mathscr{C} = \frac{1}{2}(C+Ad^2)\dot{\theta}^2 + \frac{1}{2}\int_{\Omega_t}\rho^*\vec{V}_a^2\mathrm{d}\Omega_t,$$

quadratic functional positive definite with respect to the velocities,

$$W = -\frac{1}{2}(B-A)V^2\sin^2\theta + \mu g y_{1G} + \rho\tilde{\Gamma}V y_{1C} + \mu_\ell g y_{1G_\ell};$$

functional that depends on θ and on the configuration of the liquid in the cavity.

5. A sufficient stability condition of the equilibrium via Rumiantsev's theory

5.1. Transformation of W. We transform the last terms of W by means of the formula

$$y_1 = -d\cos\theta + x\sin\theta + y\cos\theta.$$

Using the equilibrium condition (4.9), we obtain

$$\mu g y_{1G} + \rho\tilde{\Gamma}V y_{1C} = -(\mu g + \rho\tilde{\Gamma}V)d\cos\theta + (\mu g y_G + \rho\tilde{\Gamma}V y_C)\cos\theta.$$

On the other hand, we write

$$\begin{aligned}\mu_\ell g y_{G_\ell} &= -\mu_\ell g d\cos\theta + \mu_\ell g(x_{G_\ell}\sin\theta + y_{G_\ell}\cos\theta)\\ &= -\mu_\ell g d\cos\theta + \rho^* g\int_{\Omega_t}(x\sin\theta + y\cos\theta)\mathrm{d}\Omega_t,\end{aligned}$$

so that W takes the form

$$\begin{aligned}W = &-\frac{(B-A)V^2\sin^2\theta}{2} + [\mu g(y_G - d)\\ &+ \rho\tilde{\Gamma}V(y_c - d) - \mu_\ell g d]\cos\theta + \rho^* g\int_{\Omega_t}(x\sin\theta + y\cos\theta)\mathrm{d}\Omega_t.\end{aligned}$$

5.2. Calculation of W_0 and $W - W_0$. In order to study the stability of the equilibrium, we use the *Rumiantsev's theory* [9].

The value W_0 of W in the equilibrium is obtained by taking $\theta = 0$.

$$W_0 = \mu g(y_G - d) + \rho\tilde{\Gamma}V(y_c - d) - \mu_\ell g d + \mu_\ell g y_{G_{\ell(e)}},$$

where $y_{G_{\ell(e)}}$ is the ordinate of G_ℓ in the equilibrium position.

Consequently, we have

$$\begin{aligned} W - W_0 = &- \frac{(B-A)V^2 \sin^2\theta}{2} \\ &- [\mu g(y_G - d) + \rho\tilde{\Gamma} V(y_c - d) - \mu_\ell g d](1-\cos\theta) \\ &- \mu_\ell g y_{G_{\ell(e)}} + \rho^* g \int_{\Omega_t} (x\sin\theta + y\cos\theta)\mathrm{d}\Omega_t. \end{aligned}$$

5.3. The second variation of W and stability condition. We consider the system in the vicinity of the equilibrium position, where the equation of the free line Γ_t is $y = -h + \zeta(x,t)$, ζ and its derivatives being of the first order with respect to θ.

We calculate $W - W_0$ at the second order.

Let $\hat{\Omega}$ be the position of Ω at the instant t (see Figure 2).

We have

$$\int_{\Omega_t} \rho^*(x\sin\theta + y\cos\theta)\mathrm{d}\Omega_t = \int_{\hat{\Omega}} \rho^*(x\sin\theta+y\cos\theta)\mathrm{d}\hat{\Omega} + \int_{\Omega_t - \hat{\Omega}} \rho^*(x\sin\theta + y\cos\theta)\mathrm{d}\tau.$$

We can write

$$\int_{\hat{\Omega}} \rho^*(x\sin\theta + y\cos\theta)\mathrm{d}\hat{\Omega} = \mu_\ell y_{G_{\ell(e)}} \cos\theta \quad (\text{since } x_{G_{\ell(e)}} = 0).$$

$$\begin{aligned} \int_{\Omega_t - \hat{\Omega}} \rho^*(x\sin\theta + y\cos\theta)\mathrm{d}\tau &= \rho^* \int_\Gamma \left[\int_{-h}^{-h+\zeta} (x\sin\theta + y\cos\theta)\mathrm{d}y \right] \mathrm{d}\Gamma \\ &= \rho^* \int_\Gamma \left(x\zeta\cos\theta + \frac{\zeta^2}{2}\cos\theta - h\zeta\cos\theta \right)\mathrm{d}\Gamma. \end{aligned}$$

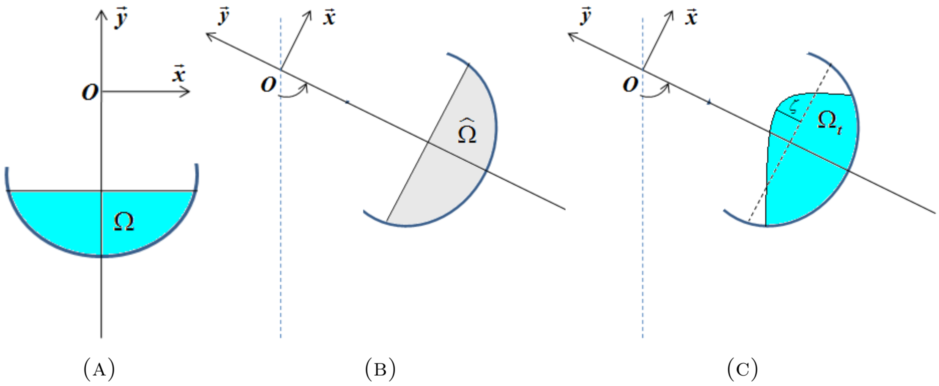

FIGURE 2. Configurations of Ω, $\hat{\Omega}$ and Ω_t

Since $\int_\Gamma \zeta\,\mathrm{d}\Gamma = 0$ by virtue of the incompressibility of the liquid, we obtain, at the second order:

$$\int_{\Omega_t} \rho^*(x\sin\theta + y\cos\theta)\mathrm{d}\Omega_t = \mu_\ell y_{G_{\ell(e)}} \left(1 - \frac{\theta^2}{2}\right) + \rho^*\theta \int_\Gamma x\zeta\,\mathrm{d}\Gamma + \frac{\rho^*}{2}\int_\Gamma \zeta^2 \mathrm{d}\Gamma,$$

and consequently

$$W - W_0 = \frac{X}{2}\theta^2 + \rho^* g\theta \int_\Gamma x\zeta \,\mathrm{d}\Gamma + \frac{\rho^* g}{2}\int_\Gamma \zeta^2 \,\mathrm{d}\Gamma,$$

where we have set

$$X = -[(B-A)V^2 + \mu g(y_G - d) + \rho\tilde{\Gamma}V(y_c - d) + \mu_\ell g(y_{G_{\ell(e)}} - d)].$$

(There are no terms of the first order because they are equal to zero in the equilibrium position).

So, the second variation of W is:

$$\delta^2 W = \frac{1}{2}\left[X\theta^2 + 2\rho^* g\theta \int_\Gamma x\zeta \,\mathrm{d}\Gamma + \rho^* g \int_\Gamma \zeta^2 \mathrm{d}\Gamma\right].$$

By virtue of the Schwarz iniquality, we have

$$\left|\theta \int_\Gamma x\zeta \,\mathrm{d}\Gamma\right| \leqslant \sqrt{I_0}|\theta|\|\zeta\|,$$

where $\|\cdot\|$ is the norm of the space $\tilde{L}^2(\Gamma) = \{f \in L^2(\Gamma),\ \int_\Gamma f \,\mathrm{d}\Gamma = 0\}$ and where we have set $I_0 = \int_\Gamma x^2 \mathrm{d}\Gamma$.

Consequently, we have

$$\delta^2 W \geqslant \tfrac{1}{2}\left[X|\theta|^2 - 2\rho^* g\sqrt{I_0}|\theta|\|\zeta\| + \rho^* g\|\zeta\|^2\right].$$

The right-hand side is a quadratic form of $|\theta|$, $\|\zeta\|$, which is positive definite if

$$X > \rho^* g I_0.$$

Under this condition, according to the Rumiantsev's theory, the equilibrium position is stable with respect to $|\theta|$, $\|\zeta\|$, $\omega = \dot{\theta}$ and the absolute kinetic energy of the liquid $\frac{1}{2}\int_{\Omega_t} \rho^* \vec{V}_a^2 \mathrm{d}\Omega_t$.

In the following, we suppose that this sufficient equilibrium condition is satisfied.

6. Linearized operatorial equations of the small motion of the system about its equilibrium position

6.1. Dynamic condition on the free line Γ_t. We introduce the (small) displacement $\vec{U}$ of a particle M of the liquid with respect to the axes Oxy. The absolute velocity of M is:

$$\vec{V}_a = \vec{V}_O + \dot{\theta}\vec{z} \times \overrightarrow{OM} + \dot{\vec{U}}.$$

Its components on Ox, Oy are

$$\vec{V}_a = \begin{cases} (d-y)\dot{\theta} + \dot{U}_x \\ x\dot{\theta} + \dot{U}_y \end{cases}.$$

The pressure in the equilibrium position is

$$P_{st} = -\rho^* g(y_1 + h + d),$$

so that the pressure P in the moving liquid is, if p is the dynamic pressure,

$$P = -\rho^* g(\theta x + y + h) + p.$$

Since $P = 0$ on Γ_t, the dynamic condition on Γ_t is

$$p_{|\Gamma} = \rho^* g(\theta x + \zeta).$$

6.2. A first linearized equation: Euler's equation. The Euler's equation can be written

$$\rho^* \vec{\gamma}_a = -\overrightarrow{\text{grad}}\, p.$$

Then, there is an acceleration potential, and, consequently, in the condition of Lagrange's theorem, a velocity potential $\Phi(x, y, t)$:

$$\vec{V}_a = \overrightarrow{\text{grad}}\, \Phi.$$

We have in linear theory:

$$\rho^* \frac{\partial \Phi}{\partial t} = -p + \mathcal{C}(t), \tag{6.1}$$

where $\mathcal{C}(t)$ is a function of the time to determinate.

On the other hand, by virtue of the incompressibility of the liquid, Φ is a harmonic function

$$\Delta \Phi = 0; \quad \text{in } \Omega.$$

On the boundary $\partial\Omega = S \cup \Gamma$ of Ω, we must have for the normal derivative of Φ

$$\frac{\partial \Phi}{\partial n} = \vec{V}_a \cdot \vec{n} = [(d - y)n_x + x n_y]\dot{\theta} + \vec{V}_r \cdot \vec{n}$$

or

$$\frac{\partial \Phi}{\partial n} = \begin{cases} [(d - y)n_x + x n_y]\dot{\theta} & \text{on } S \\ [(d - y)n_x + x n_y]\dot{\theta} + \dot{\zeta} & \text{on } \Gamma \ (n_{x|\Gamma} = 0, n_{y|\Gamma} = 1). \end{cases}$$

Then, Φ is solution to a Neumann problem.

We seek Φ in the form

$$\Phi(x, y, t) = \varphi(x, y, t) + \dot{\theta}\Phi_\theta(x, y),$$

with

$$\Delta\varphi = 0 \quad \text{in } \Omega; \quad \frac{\partial \varphi}{\partial n}_{|S} = 0; \quad \frac{\partial \varphi}{\partial n}_{|\Gamma} = \dot{\zeta}; \quad \int_\Gamma \varphi_{|\Gamma} \mathrm{d}\Gamma = 0,$$

$$\Delta\Phi_\theta = 0 \quad \text{in } \Omega; \quad \frac{\partial \Phi}{\partial n}_{|\partial\Omega} = (d - y)n_x + x n_y; \quad \int_\Gamma \Phi_{\theta|\Gamma} \mathrm{d}\Gamma = 0.$$

These Neumann problems have one and only one solution. $\Phi_\theta(x, y)$ is a Joukowski's potential [9] depending on the form of Ω.

On the other hand, it is well-known [5] that

$$\varphi_{|\Gamma} = K\dot{\zeta},$$

where the operator K is bounded, selfadjoint, positive definite, compact from $\tilde{L}^2(\Gamma)$ into $\tilde{L}^2(\Gamma)$. Then, we have

$$p = -\rho^*(\dot{\varphi} + \ddot{\theta}\Phi_\theta) + \mathcal{C}(t),$$

That replice the Euler's Equation. We obtain a first linearized equation by using the condition (6.1); we have

$$\rho^* g(\theta x_{|\Gamma} + \zeta) = -\rho^*(K\ddot{\zeta} + \ddot{\theta}\Phi_{\theta|\Gamma}) + \mathcal{C}(t).$$

Since ζ and $x_{|\Gamma}$ (by symmetry) belong to $\tilde{L}^2(\Gamma)$, we obtain, by integrating on Γ, $\mathcal{C}(t) = 0$, and consequently:

$$\ddot{\theta}\Phi_{\theta|\Gamma} + K\ddot{\zeta} + gx_{|\Gamma}\theta + g\zeta = 0. \tag{6.2}$$

6.3. A second linearized equation. We are going to obtain a second equation between θ and ζ linearizing the equation (4.8), i.e., taking into account the equilibrium equation (4.9):

$$\begin{aligned}(C + Ad^2)\ddot{\theta} - (B - A)V^2 \sin\theta\cos\theta + \int_\Omega \rho^*[(d-y)\gamma_{ax} + x\gamma_{ay}]\mathrm{d}\Omega \\ = -\mu g(d - y_G)\sin\theta - \rho\tilde{\Gamma}V(d - y_c)\sin\theta - \mu_\ell g[(d - y_{G_\ell})\sin\theta + x_{G_\ell}\cos\theta].\end{aligned}$$

We have, in linear theory:

$$\vec{\gamma}_a = \overrightarrow{\mathrm{grad}}\frac{\partial\Phi}{\partial t}, \quad \text{denoted by } \overrightarrow{\mathrm{grad}}\dot{\Phi},$$

and then

$$\int_\Omega \rho^*[(d-y)\gamma_{ax} + x\gamma_{ay}]\mathrm{d}\Omega = \int_\Omega \rho^*\Big[x\frac{\partial\dot{\Phi}}{\partial y} - (y-d)\frac{\partial\dot{\Phi}}{\partial x}\Big]\mathrm{d}\Omega.$$

We have, at first

$$\begin{aligned}-\mu_\ell g x_{G_\ell} &= -\rho^* g\int_{\Omega_t} x\,\mathrm{d}\Omega_t = -\rho^* g\Big[\int_{\hat{\Omega}} x\,\mathrm{d}\hat{\Omega} + \int_{\Omega_t - \hat{\Omega}} x\,\mathrm{d}\tau\Big] \\ &= -\rho^* g\int_\Gamma\Big[\int_{-h}^{-h+\zeta}\mathrm{d}y\Big]x\,\mathrm{d}\Gamma\end{aligned}$$

and then

$$-\mu_\ell g x_{G_\ell} = -\rho^* g\int_\Gamma x\zeta\,d\Gamma.$$

Afterwards, using the Green's formula and since Φ and Φ_θ are harmonic, we can write:

$$\begin{aligned}\rho^*\int_\Omega\Big[x\frac{\partial\dot{\Phi}}{\partial y} - (y-d)\frac{\partial\dot{\Phi}}{\partial x}\Big]\mathrm{d}\Omega &= \rho^*\int_\Omega\Big\{\frac{\partial}{\partial y}(x\dot{\Phi}) - \frac{\partial}{\partial x}[(y-d)\dot{\Phi}]\Big\}\mathrm{d}\Omega \\ &= \rho^*\int_{\partial\Omega}[xn_y - (y-d)n_x]\dot{\Phi}\,\mathrm{d}(\partial\Omega) \\ &= \rho^*\int_{\partial\Omega}\frac{\partial\Phi_\theta}{\partial n}\dot{\Phi}\,\mathrm{d}(\partial\Omega) \\ &= \rho^*\int_{\partial\Omega}\frac{\partial\dot{\Phi}}{\partial n}\,\Phi_\theta\mathrm{d}(\partial\Omega) \\ &= \rho^*\int_{\partial\Omega}\Big(\frac{\partial\dot{\varphi}}{\partial n} + \ddot{\Phi}\frac{\partial\Phi_\theta}{\partial n}\Big)\Phi_\theta\mathrm{d}(\partial\Omega).\end{aligned}$$

Or

$$\begin{cases} \frac{\partial \dot\varphi}{\partial n} = \ddot\zeta & \text{on} \quad \Gamma; \\ \frac{\partial \dot\varphi}{\partial n} = 0 & \text{on} \quad S, \end{cases}$$

so that we have

$$\rho^* \int_\Omega \Big[x \frac{\partial \dot\Phi}{\partial y} - (y-d)\frac{\partial \dot\Phi}{\partial x} \Big] \mathrm{d}\Omega = \rho^* \int_\Gamma \Phi_\theta \ddot\zeta \, \mathrm{d}\Gamma + \rho^* \ddot\theta \int_\Omega \overrightarrow{\mathrm{grad}}^2 \, \Phi_\theta \mathrm{d}\Omega.$$

We finally obtain the linearized equation

$$\Big[C + Ad^2 + \rho^* \int_\Omega \overrightarrow{\mathrm{grad}}^2 \, \Phi_\theta \mathrm{d}\Omega \Big] \ddot\theta + \rho^* \int_\Gamma \Phi_{\theta|\Gamma} \ddot\zeta \, \mathrm{d}\Gamma + X\theta + \rho^* g \int_\Gamma x\zeta \, d\Gamma = 0. \tag{6.3}$$

6.4. Operatorial equation of the problem. Setting

$$J = C + Ad^2 + \rho^* \int_\Omega \overrightarrow{\mathrm{grad}}^2 \, \Phi_\theta \mathrm{d}\Omega,$$

and

$$u = \begin{pmatrix} \zeta \\ \theta \end{pmatrix} \in \tilde L^2(\Gamma) \times \mathbb{C},$$

we write the equations (6.2), (6.3) in the operatorial form:

$$\mathcal{A}\ddot u + \mathcal{B} u = 0; \tag{6.4}$$

with

$$\mathcal{A} = \begin{pmatrix} \rho^* K & \rho^* \Phi_{\theta|\Gamma} \\ \rho^* \int_\Gamma \Phi_{\theta|\Gamma} \mathrm{d}\Gamma & J I_{\mathbb{C}} \end{pmatrix}; \quad \mathcal{B} = \begin{pmatrix} \rho^* g I_{\tilde L^2(\Gamma)} & \rho^* g x_{|\Gamma} \\ \rho^* g \int_\Gamma x_{|\Gamma} \, \mathrm{d}\Gamma & X I_{\mathbb{C}} \end{pmatrix}$$

6.5. Study of the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathscr{C}$. In this subsection, we give a few properties of these operators.

a) $\mathcal{A}$ is obviously selfadjoint and compact; it is positive definite.

Indeed, we have

$$\begin{aligned}(\mathcal{A}\dot u, \dot u)_{\tilde L^2(\Gamma)\times\mathbb{C}} &= \rho^* (K\dot\zeta, \dot\zeta)_{\tilde L^2(\Gamma)} + 2\rho^* \dot\theta \int_\Gamma \Phi_{\theta|\Gamma} \dot\zeta \, \mathrm{d}\Gamma + J\dot\theta^2 \\ &= \rho^* \int_\Gamma \varphi_{|\Gamma} \frac{\partial \varphi}{\partial n}_{|\Gamma} \mathrm{d}\Gamma + 2\rho^* \dot\theta \int_\Gamma \Phi_{\theta|\Gamma} \frac{\partial \varphi}{\partial n}_{|\Gamma} \mathrm{d}\Gamma + J\dot\theta^2,\end{aligned}$$

and using the boundary conditions for φ:

$$(\mathcal{A}\dot u, \dot u)_{\tilde L^2(\Gamma)\times\mathbb{C}} = \rho^* \int_\Omega \overrightarrow{\mathrm{grad}}^2 (\varphi + \dot\theta \Phi_\theta) \mathrm{d}\Omega + (C + Ad^2)\dot\theta^2 \geqslant 0.$$

We have $(\mathcal{A}\dot u, \dot u) = 0$ only for $\dot\theta = 0$, $\varphi = constant$, then for $\dot\zeta = 0$ and finally for $\dot u = 0$.

b) $\mathcal{B}$ is selfadjoint and not compact.

We have

$$(\mathcal{B}u, u)_{\tilde L^2(\Gamma)\times\mathbb{C}} = \rho^* g \|\zeta\|_{\tilde L^2(\Gamma)} + 2\rho^* g \int_\Gamma x_{|\Gamma} \zeta \, \mathrm{d}\Gamma \cdot \theta + X|\theta|^2 = 2\delta^2 W.$$

Under the stability condition $X > \rho^* g I_0$, $\mathcal{B}$ is strongly positive.

c) Introducing the operator $\mathscr{C} = \mathcal{B}^{-1/2}\mathcal{A}\mathcal{B}^{-1/2}$.

Therefore, we can set

$$\mathcal{B}^{1/2}u = v \in \tilde{L}^2(\Gamma) \times \mathbb{C},$$

and the differential equation (6.4) becomes:

$$\mathscr{C}\ddot{v} + v = 0,$$

with $\mathscr{C} = \mathcal{B}^{-1/2}\mathcal{A}\mathcal{B}^{-1/2}$ selfadjoint, positive definite, and compact.

Seeking the solution depending on t to the law $e^{\lambda t}$, $\lambda \in \mathbb{C}$, we obtain the equation

$$\mathscr{C}v = \nu v, \quad \nu = -\lambda^{-2}.$$

$\mathscr{C}$ has eigenvalues $\nu_1, \nu_2, \dots, \nu_n, \dots$ positive and having zero as a point of accumulation.

The eigenvalues of our problem are $\pm i\nu_n^{1/2}$ $(n = 1, 2, \dots)$ and have the infinity of point of accumulation.

7. On the calculation of the eigenvalues

7.1. We call k_n^2 $(n = 1, 2, \dots)$ the eigenvalues of the operator K $(k_1^2 \geqslant k_2^2 \geqslant \dots \geqslant k_n^2 \geqslant \dots \to 0$ when $n \to +\infty)$ and Ψ_n $(n = 1, 2, \dots)$ the corresponding eigenfunctions that form a complete orthonormal system in $\tilde{L}^2(\Gamma)$; we have

$$K\Psi_n = k_n^2\Psi_n \quad (n = 1, 2, \dots).$$

Since $\zeta, x_{|\Gamma}, \Phi_{\theta|\Gamma}$ belong to $\tilde{L}^2(\Gamma)$, we can write

$$\zeta(x,t) \sim \sum_n \zeta_n(t)\Psi_n; \quad x_{|\Gamma} \sim \sum_n x_{|\Gamma}^n\Psi_n; \quad \Phi_{\theta|\Gamma} \sim \sum_n \Phi_{\theta|\Gamma}^n\Psi_n,$$

where the symbol $\sim$ indicates that the series are convergent in the norm of $\tilde{L}^2(\Gamma)$.

On the other hand, we have

$$K\zeta \sim \sum_n k_n^2\zeta_n\Psi_n,$$

and the equalities

$$\int_\Gamma x_{|\Gamma}\zeta \,\mathrm{d}\Gamma = \sum_n x_{|\Gamma}^n\zeta_n; \quad \int_\Gamma \Phi_{\theta|\Gamma}\zeta \,\mathrm{d}\Gamma = \sum_n \Phi_{\theta|\Gamma}^n\zeta_n.$$

7.2. The equation (6.3) can be written

$$\rho^* \sum_n \Phi_{\theta|\Gamma}^n\ddot{\zeta}_n + J\ddot{\theta} + \rho^* g \sum_n x_{|\Gamma}^n\zeta_n + X\theta = 0,$$

and the equation (6.2) becomes

$$\sum_n (k_n^2\ddot{\zeta}_n + \Phi_{\theta|\Gamma}^n\ddot{\theta} + g\zeta_n + gx_{|\Gamma}^n\theta)\Psi_n = 0,$$

and can be replaced by the system

$$k_n^2\ddot{\zeta}_n + \Phi_{\theta|\Gamma}^n\ddot{\theta} + g\zeta_n + gx_{|\Gamma}^n\theta = 0 \quad (n = 1, 2, \dots).$$

7.3. Seeking the solutions depending on the time according to the law $e^{\lambda t}$, we obtain

$$\sum_n \rho^*(\Phi^n_{\theta|\Gamma}\lambda^2 + gx^n_{|\Gamma})\zeta_n + (J\lambda^2 + X)\theta = 0,$$

$$(k_n^2\lambda^2 + g)\zeta_n + (\Phi^n_{\theta|\Gamma}\lambda^2 + gx^n_{|\Gamma})\theta = 0 \quad n = 1, 2, \dots.$$

Eliminating the ζ_n, we obtain the characteristic equation

$$J\lambda^2 + X = \rho^* \sum_{n=1}^{\infty} \frac{(\Phi^n_{\theta|\Gamma}\lambda^2 + gx^n_{|\Gamma})^2}{k_n^2\lambda^2 + g}.$$

The eigenvalues being pure imaginary, we set $\lambda^2 = -\xi$, ξ real>0, and we can graphically solve the equation

$$X - J\xi = \rho^* \sum_{n=1}^{\infty} \frac{(gx^n_{|\Gamma} - \Phi^n_{\theta|\Gamma}\xi)^2}{g - k_n^2\xi}, \quad \xi > 0.$$

But, the coefficients depend on the geometry of the domain Ω.

Therefore, we are only going to give a few results. The curve

$$\mathcal{Y} = \rho^* \sum_n \frac{(gx^n_{|\Gamma} - \Phi^n_{\theta|\Gamma}\xi)^2}{g - k_n^2\xi}$$

has the asymptotes

$$\xi = \frac{g}{k_n^2} \quad (n = 1, 2, \dots).$$

For $\xi < \frac{g}{k_1^2}$, $\mathcal{Y}$ is positive. The curve intersects the axis $\mathcal{Y}$ at the point $\mathcal{Y} = \rho^* g \sum_n x^{n^2}_{|\Gamma} = \rho^* g \int_\Gamma x^2_{|\Gamma} \mathrm{d}\Gamma < X$ by virtue of the stability condition.

The equation has at least one root ξ_1, with $0 < \xi_1 < \frac{g}{k_1^2}$.

When $\xi \to \frac{g}{k_n^2}$ to the left, $\mathcal{Y} \to +\infty$; when $\xi \to \frac{g}{k_n^2}$ to the right, $\mathcal{Y} \to -\infty$.

$\mathcal{Y}$ being continuous in the intervals $\left]\frac{g}{k_n^2}, \frac{g}{k_{n+1}^2}\right[$ $(n = 1, 2, \dots)$, the equation has at least one root in each interval. For instance, we have the following figure:

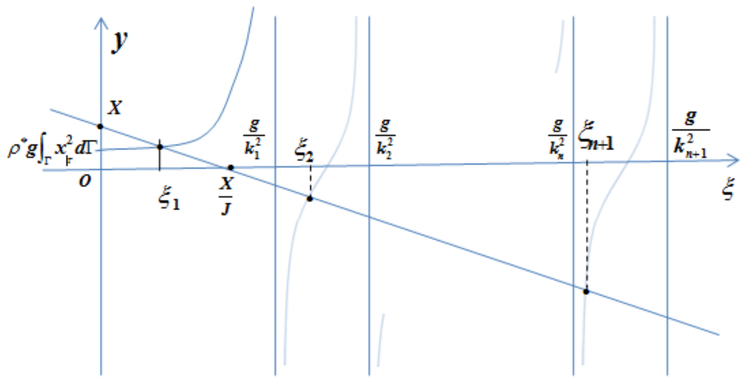

FIGURE 3. Graphical representation of ξ_k $(k = 1, 2, \cdots)$.

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О СТАБИЛНОСТИ РАВНОТЕЖНОГ ПОЛОЖАЈА И МАЛИМ КРЕТАЊИМА КРУТОГ ТЕЛА КОЈЕ САДРЖИ ТЕЧНОСТ, СМЕШТЕНОГ У РАВНОМЕРНОМ ТОКУ

Резиме. У овом раду разматрамо раванско кретање крутог тела делимично испуњеног течношћу и смештеног у равномерном хоризонталном току. Прво су изведене једначине проблема, под одређеним условима доказано постојање положаја равнотеже и, корисшћењем првог интеграла, дат је довољан услов његове стабилности. Након тога дате су једнчине малих осцилација система око равнотежног положаја. Пишући ове једначине помођу оператора, доказујемо постојање непребројиво коњуговано комплексних парова сопствених вредности које имају бесконачност као тачку нагомилавања и добијамо карактеристичну једначину која омогућава израчунавање сопствених вредности.

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