

## NOTE ON A BALL ROLLING OVER A SPHERE: INTEGRABLE CHAPLYGIN SYSTEM WITH AN INVARIANT MEASURE WITHOUT CHAPLYGIN HAMILTONIZATION

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ABSTRACT. In this note we consider the nonholonomic problem of rolling without slipping and twisting of an  $n$ -dimensional balanced ball over a fixed sphere. This is a  $SO(n)$ -Chaplygin system with an invariant measure that reduces to the cotangent bundle  $T^*S^{n-1}$ . For the rigid body inertia operator  $\mathbb{I}\omega = I\omega + \omega I$ ,  $I = \text{diag}(I_1, \dots, I_n)$  with a symmetry  $I_1 = I_2 = \dots = I_r \neq I_{r+1} = I_{r+2} = \dots = I_n$ , we prove that the reduced system is integrable, general trajectories are quasi-periodic, while for  $r \neq 1, n-1$  the Chaplygin reducing multiplier method does not apply.

### 1. Introduction

**1.1.** Let  $(Q, L, \mathcal{D})$  be a nonholonomic system, where  $Q$  is a configuration space and  $\mathcal{D}$  a nonintegrable distribution of constraints. For simplicity, in the note we consider a Lagrangian  $L$  that is the kinetic energy determined by the Riemannian metric  $g$ . Assume that the Lie group  $G$  acts freely by isometries on  $(Q, g)$ , the quotient space  $N = Q/G$  is a manifold, and  $\mathcal{D}$  is  $G$ -invariant, transversal, and a complement to the  $G$ -orbits ( $\mathcal{D}$  is a principal connection of the bundle  $Q \rightarrow N = M/G$ ). Then the nonholonomic geodesic flow is  $G$ -invariant and reduces to the tangent bundle of the base manifold  $N$ . The *reduced Lagrange-d'Alembert* equations take the form

$$(1.1) \quad \left( \frac{\partial L_{\text{red}}}{\partial x} - \frac{d}{dt} \frac{\partial L_{\text{red}}}{\partial \dot{x}}, \eta \right) = \Sigma(\dot{x}, \dot{x}, \eta) \quad \text{for all } \eta \in T_x N,$$

where the reduced Lagrangian  $L_{\text{red}}$  is obtained from  $L|_{\mathcal{D}}$  by the identification  $TN = \mathcal{D}/G$ , and  $\Sigma(X, Y, Z)$  is a  $(0, 3)$ -tensor field on the base manifold  $N$ , skew-symmetric in  $Y$  and  $Z$ , which depends on the metric and the curvature of the connection. The system  $(Q, L, \mathcal{D}, G)$  is referred to as a  *$G$ -Chaplygin system*, as a generalization of classical Chaplygin systems with Abelian symmetries [2, 8, 10, 11, 24].

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For  $\Sigma \neq 0$  the equations are not Hamiltonian. Chaplygin proposed a remarkable Hamiltonization procedure as follows [10]. Consider the time substitution  $d\tau = \nu(x)dt$ , and denote  $x' = dx/d\tau = \dot{x}/\nu$ . Then the Lagrangian function transforms to  $L^*(x', x) = L_{\text{red}}(\dot{x}, x)|_{\dot{x}=\nu x'}$ . The factor  $\nu$  is a *Chaplygin reducing multiplier* if in the new time  $\tau$  the reduced system (1.1) transforms into the usual Euler–Lagrange equations of  $L^*(x', x)$ . In particular, in the cotangent bundle formulation, the original system has an invariant measure and it is conformally symplectic (see [4, 8, 11, 15]).

Usually, integrable nonholonomic  $G$ –Chaplygin problems are solved by using the Chaplygin reducing multiplier method (e.g., see [6, 15, 19]). The aim of this note is to provide an example of a solvable system with an invariant measure which does not allow a Chaplygin multiplier.

**1.2.** Following [19, 23], we consider the rolling without slipping and twisting of an  $n$ -dimensional ball of radius  $\rho$  over the outer surface of the  $(n-1)$ -dimensional fixed sphere of radius  $\sigma$  (the case (i), see Figure 1); over the inner surface of the sphere ( $\sigma > \rho$ , the case (ii), see Figure 2); rolling over the outer surface of the  $(n-1)$ -dimensional fixed sphere of radius  $\sigma$ , but the fixed sphere is within the rolling ball ( $\sigma < \rho$ , in this case, the rolling ball is actually a spherical shell, the case (iii), see Figure 2).

Consider the space frame  $\mathbb{R}^n(\mathbf{x})$  with the origin  $O$  at the center of the fixed sphere and the moving frame  $\mathbb{R}^n(\mathbf{X})$  with the origin  $C$  at the center of the rolling ball. The mapping from the moving to the space frame is given by  $\mathbf{x} = g\mathbf{X} + \mathbf{r}$ , where  $g \in SO(n)$  is a rotation matrix and  $\mathbf{r} = \overrightarrow{OC}$  is the position vector of the ball center  $C$  in the space frame (see Figure 1). The configuration space  $Q$  is the direct product of the Lie group  $SO(n)$  and the sphere  $\mathcal{S} = \{\mathbf{r} \in \mathbb{R}^n \mid (\mathbf{r}, \mathbf{r}) = (\sigma \pm \rho)^2\}$ , where we take ”+” for the case (i) and ”–” for the cases (ii) and (iii).

We additionally assume that the ball is balanced, i.e., its geometric center coincides with the mass center. Then the Lagrangian of the system is given by

$$L(\omega, \dot{\mathbf{r}}, g, \mathbf{r}) = \frac{1}{2}\langle \mathbb{I}\omega, \omega \rangle + \frac{1}{2}m(\dot{\mathbf{r}}, \dot{\mathbf{r}}),$$

where  $(\cdot, \cdot)$  is the Euclidean scalar product in  $\mathbb{R}^n$ ,  $\langle \cdot, \cdot \rangle = -\frac{1}{2}\text{tr}(\cdot \circ \cdot)$  is proportional to the Killing form on  $so(n)$ ,  $\omega = g^{-1}\dot{g}$  is the angular velocity of the ball in the moving frame,  $m$  is the mass of the ball, and  $\mathbb{I} : so(n) \rightarrow so(n)$  is the inertia operator. After [9], a balanced ball (with the inertia operator that is not proportional to the identity operator) is usually referred to as a *Chaplygin ball*.

The direction  $\overrightarrow{OA}/|OA|$  of the contact point  $A$  in the frame attached to the ball is given by the unit vector  $\gamma = \frac{1}{\sigma \pm \rho}g^{-1}\mathbf{r}$  (see Figure 1). It is invariant with respect to the diagonal *left*  $SO(n)$ -action:  $a \cdot (g, \mathbf{r}) = (ag, a\mathbf{r})$ ,  $a \in SO(n)$ . The action defines  $SO(n)$ -bundle

$$(1.2) \quad SO(n) \longrightarrow Q = SO(n) \times \mathcal{S} \xrightarrow{\pi} S^{n-1} = Q/SO(n)$$

with the submersion  $\pi$  given by  $\gamma = \pi(g, \mathbf{r})$ .

The contact point  $A$  of the ball in the moving frame is  $\mathbf{X}_A = -(\pm\rho\gamma)$ . The condition that the ball is rolling without slipping is that the velocity  $\dot{\mathbf{x}}_A$  of the

contact point in the space frame is equal to zero

$$0 = \dot{\mathbf{x}}_A = \frac{d}{dt}(g\mathbf{X}_A + \mathbf{r}) = \mp\rho\dot{g}\gamma + \dot{\mathbf{r}} = \mp\rho(\dot{g}g^{-1})g\gamma + \dot{\mathbf{r}} \quad (\dot{\mathbf{X}}_A = 0).$$

This leads to the constraint  $\dot{\mathbf{r}} = \pm\frac{\rho}{\sigma\pm\rho}\Omega\mathbf{r}$ , where  $\Omega = \text{Ad}_g\omega = \dot{g}g^{-1}$  is the angular velocity in the space frame. On the other hand, no twisting at the contact point can be written as  $\Omega \in \mathbf{r} \wedge \mathbb{R}^n$  e.g.,  $\omega \in \gamma \wedge \mathbb{R}^n$  (for more details, see [23]).

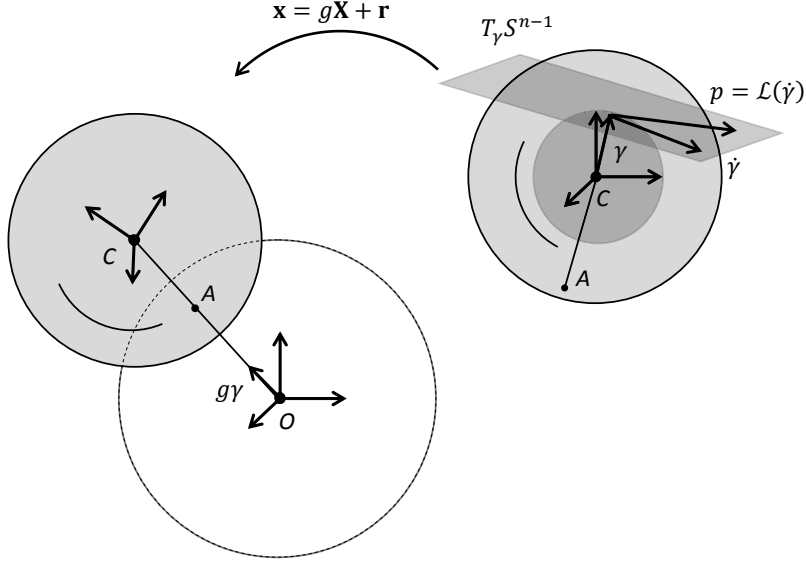


FIGURE 1. The rolling without slipping and twisting of an  $n$ -dimensional ball of radius  $\rho$  over the outer surface of the  $(n-1)$ -dimensional fixed sphere of radius  $\sigma$  (the case (i)).  $O$ ,  $C$ , and  $A$  denote the center of the fixed sphere, the center of the rolling ball, and the contact point, respectively. The reduced space, the unit sphere in the moving frame consisting of vectors  $\gamma = \frac{1}{\sigma+\rho}g^{-1}\mathbf{r}$ , for  $\rho > 1$ , is illustrated as well.

The constraints determine the  $(n-1)$ -dimensional constraint distribution  $\mathcal{D}$ , which is a principal connection of the bundle (1.2). The Lagrangian is  $SO(n)$ -invariant as well. Thus, it is a  $SO(n)$ -Chaplygin system and reduces to the tangent bundle  $TS^{n-1} \cong \mathcal{D}/SO(n)$ .

The reduced Lagrange–d’Alembert equations take the form (1.1), where [23]

$$\begin{aligned} L_{\text{red}}(\dot{\gamma}, \gamma) &= -\frac{1}{4\epsilon^2} \text{tr}(\mathbf{I}(\gamma \wedge \dot{\gamma}) \circ (\gamma \wedge \dot{\gamma})) = -\frac{1}{2\epsilon^2} (\mathbf{I}(\gamma \wedge \dot{\gamma})\gamma, \dot{\gamma}), \\ \Sigma_{\gamma}(X, Y, Z) &= \frac{2\epsilon-1}{2\epsilon^3} \text{tr}(\mathbf{I}(\gamma \wedge X) \circ (Y \wedge Z)) = \frac{2\epsilon-1}{\epsilon^3} (\mathbf{I}(\gamma \wedge X)Y, Z), \\ \mathbf{I} &= \mathbb{I} + D \cdot \text{Id}_{so(n)}, \quad D = m\rho^2, \quad \epsilon = \sigma/(\sigma \pm \rho). \end{aligned}$$

Note that when the radii of the sphere and the ball are equal ( $\epsilon = 1/2$ ), the curvature of  $\mathcal{D}$  vanishes and  $\Sigma \equiv 0$  – the reduced system is Hamiltonian without a time reparametrization (for  $n = 3$  see [7, 12]). Also, if  $\mathbb{I}$  is proportional to the identity operator then  $\Sigma \equiv 0$  (the reaction forces vanish although the curvature of  $\mathcal{D}$  is different from zero).

The system always has an invariant measure [20]. Moreover, for the inertia operator

$$(1.3) \quad \mathbb{I}(E_i \wedge E_j) = (a_i a_j - D)E_i \wedge E_j \quad \text{i.e.,} \quad \mathbf{I}(X \wedge Y) = AX \wedge AY,$$

where  $A = \text{diag}(a_1, \dots, a_n)$ , the function  $\nu(\gamma) = \epsilon(A\gamma, \gamma)^{\frac{1}{2\epsilon}-1}$  is a Chaplygin multiplier: under a time substitution  $d\tau = \nu(\gamma)dt$ , the reduced system becomes the geodesic flow of the metric defined by the Lagrangian [23]

$$(1.4) \quad L^*(\gamma', \gamma) = L_{\text{red}}(\dot{\gamma}, \gamma)|_{\dot{\gamma}=\nu(\gamma)\gamma'} = \frac{1}{2}(\gamma, A\gamma)^{\frac{1}{\epsilon}-2}((A\gamma', \gamma')(A\gamma, \gamma) - (A\gamma, \gamma')^2).$$

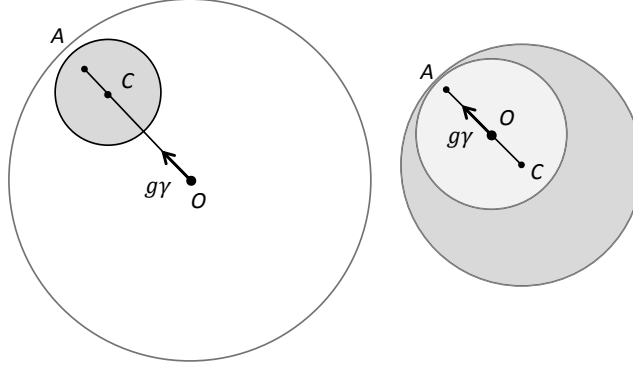


FIGURE 2. The cases (ii) and (iii) in the space frame: the rolling of a ball over the inner surface of the sphere and the rolling of a spherical shell over the outer surface of the fixed sphere.

The procedure of reduction and Hamiltonization for  $n = 3$  is given by Ehlers and Koiller [12], while Borisov and Mamaev proved the integrability for a specific ratio between the radii of the ball and the spherical shell (the case (iii), where  $\rho = 2\sigma$ , i.e.  $\epsilon = -1$ ), see [6]. The  $n$ -dimensional system with the inertia operator  $\mathbb{I}$  given by (1.3) is also integrable for  $\epsilon = -1$ , as well as for arbitrary  $\epsilon$  when the matrix  $A$  has only two distinct parameters [19].

**1.3.** Generally, for  $n \geq 4$ , the operator (1.3) is not a physical inertia operator of a multidimensional rigid body that has the form

$$(1.5) \quad \omega \mapsto I\omega + \omega I, \quad I = \text{diag}(I_1, \dots, I_n).$$

Here  $I$  is a positive definite matrix called the *mass tensor*, which is diagonal in the moving orthonormal base determined by the principal axes of inertia.<sup>1</sup>

<sup>1</sup>Let  $\varrho(\mathbf{X})$  be the density of the ball. The mass tensor  $I$  is defined by  $I = \int \varrho(\mathbf{X})\mathbf{X} \otimes \mathbf{X}d\mathbf{X}$ , e.g., see [16, 18].

Note that the case  $\epsilon = +1$  is the limit case, when the radius of the fixed sphere tends to infinity. Furthermore, then the associated equation (1.1) becomes the reduced equation of the Veselova problem with the inertia operator  $\mathbf{I}$  studied in [15]. Recently, the integrability of the Veselova problem with a physical inertia operator (1.5), where

$$(1.6) \quad I_1 = I_2 = \cdots = I_r \neq I_{r+1} = I_{r+2} = \cdots = I_n$$

without involving Chaplygin Hamiltonisation has been proved by Fasso, Garcia-Naranjo, and Montaldi [13]. It is a natural problem to consider the rolling ball over a sphere for the given rigid body inertia operator as well. In this note we prove that the reduced system is solvable and that it does not allow Chaplygin reducing multiplier for  $\epsilon \neq 1/2$  and  $r \neq 1, n-1$ .

The reduced system and an invariant measure are described in Theorem 2.1 (Section 2). In Section 3 we prove that for any symmetry  $I_i = I_j$  one can associate Noether type integral linear in momenta (Theorem 3.1), which allows us to obtain a simple "dynamical" proof for the nonexistence of a Chaplygin multiplier (Theorem 3.2). In the case of a  $SO(r) \times SO(n-r)$ -dynamical symmetry, the reduced system is integrable. For  $r \neq 1, n-1$ , generic motions are quasi-periodic over 3-dimensional invariant tori, while the Chaplygin reducing multiplier method does not apply. For  $r = 1$  or  $r = n-1$ , invariant tori are two-dimensional and the system has a Chaplygin multiplier (Theorem 4.1, Section 4).

## 2. Equations of motion and an invariant measure

**2.1.** First, we recall the cotangent formulation of the system (see [23]). Consider the Legendre transformation

$$(2.1) \quad \mathcal{L} : \quad p = \frac{\partial L_{\text{red}}}{\partial \dot{\gamma}} = -\frac{1}{\epsilon^2} \mathbf{I}(\gamma \wedge \dot{\gamma})\gamma.$$

The point  $(p, \gamma)$  belongs to the cotangent bundle of a sphere realized as a symplectic submanifold in the symplectic linear space  $(\mathbb{R}^{2n}(p, \gamma), dp_1 \wedge d\gamma_1 + \cdots + dp_n \wedge d\gamma_n)$ :

$$(2.2) \quad \begin{aligned} (\gamma, \gamma) &= 1, \\ (\gamma, p) &= 0. \end{aligned}$$

Note that here we identified the tangent bundle  $TS^{n-1}$  and the cotangent bundle  $T^*S^{n-1}$  by means of the Euclidean metric in  $\mathbb{R}^n$  (see Figure 1).

The reduced flow on  $T^*S^{n-1}$  takes the form

$$(2.3) \quad \dot{\gamma} = X_\gamma(p, \gamma), \quad \dot{p} = X_p(p, \gamma),$$

where  $X_\gamma = \mathcal{L}^{-1}(p)$  is the inverse of the Legendre transformation (2.1) and

$$(2.4) \quad X_p = \frac{(1-\epsilon)}{\epsilon^3} (\mathbf{I}(\gamma \wedge X_\gamma))X_\gamma + \frac{(\epsilon-1)}{\epsilon^3} ((\mathbf{I}(\gamma \wedge X_\gamma))X_\gamma, \gamma)\gamma - 2H\gamma$$

(see [23]). Here

$$H(p, \gamma) = (p, \dot{\gamma}) - L(\dot{\gamma}, \gamma)|_{\dot{\gamma}=X_\gamma} = \frac{1}{2}(X_\gamma, p)$$

is the Hamiltonian function of the system.

**2.2.** Now we assume that the inertia operator  $\mathbb{I}$  is of the form (1.5). Then we can write the modified operator  $\mathbf{I} = \mathbb{I} + D \cdot \text{Id}_{\mathfrak{so}(n)}$  as

$$(2.5) \quad \mathbf{I}\omega = \epsilon^2(J\omega + \omega J),$$

where

$$(2.6) \quad J = \text{diag}(J_1, \dots, J_n) = \frac{1}{\epsilon^2} \text{diag}\left(I_1 + \frac{D}{2}, \dots, I_n + \frac{D}{2}\right),$$

and the Legendre transformation (2.1) takes the form

$$(2.7) \quad p = -J(\gamma \wedge \dot{\gamma})\gamma - (\gamma \wedge \dot{\gamma})J\gamma = J\dot{\gamma} + (J\gamma, \gamma)\dot{\gamma} - (J\gamma, \dot{\gamma})\gamma.$$

It coincides with the Legendre transformation of the Veselova problem considered in [13] (see [13, Proposition 3.4]). That is why we can use the inverse

$$(2.8) \quad \dot{\gamma} = X_\gamma(p, \gamma) = C_\gamma\left(p - \frac{(p, C_\gamma(\gamma))}{(\gamma, C_\gamma(\gamma))}\gamma\right)$$

derived there. Here

$$C_\gamma = \text{diag}(J_1 + (\gamma, J\gamma), \dots, J_n + (\gamma, J\gamma))^{-1}$$

and (2.8) is the unique inverse of (2.7) for  $p, \gamma, \dot{\gamma}$  that satisfy (2.2) and  $(\dot{\gamma}, \gamma) = 0$ .

From (2.4), (2.5), the equation in  $p$  becomes

$$(2.9) \quad \dot{p} = X_p = \frac{1-\epsilon}{\epsilon}((X_\gamma, X_\gamma)J\gamma - (\gamma, JX_\gamma)X_\gamma - (X_\gamma, X_\gamma)(\gamma, J\gamma)\gamma) - 2H\gamma,$$

with the Hamiltonian  $H$

$$(2.10) \quad H(p, \gamma) = \frac{1}{2}(p, X_\gamma) = \frac{1}{2}(p, C_\gamma(p)) - \frac{1}{2} \frac{(p, C_\gamma(\gamma))^2}{(\gamma, C_\gamma(\gamma))}.$$

Let  $\mathbf{w}$  be the canonical symplectic form on  $T^*S^{n-1}$ :

$$\mathbf{w} = dp_1 \wedge d\gamma_1 + \dots + dp_n \wedge d\gamma_n|_{T^*S^{n-1}}.$$

**THEOREM 2.1.** *The reduced equations of the rolling of a ball over a sphere without slipping and twisting with the rigid body inertia operator (1.5) are given by (2.8), (2.9), whereas  $J$  is defined by (2.6). The reduced system has an invariant measure*

$$(2.11) \quad \mu(\gamma)\mathbf{w}^{n-1} = \left(\frac{\det C_\gamma}{(\gamma, C_\gamma(\gamma))}\right)^{1-\frac{1}{2\epsilon}}\mathbf{w}^{n-1}.$$

**PROOF.** In [23] we proved that the reduced equations (2.3), for arbitrary modified inertia operator  $\mathbf{I}$ , have an invariant measure

$$(2.12) \quad (\det \mathbf{I}|_{\mathbb{R}^n \wedge \gamma})^{\frac{1}{2\epsilon}-1}\mathbf{w}^{n-1}.$$

In particular, we can see that the density of a measure for  $\epsilon = +1$  is  $(\det \mathbf{I}|_{\mathbb{R}^n \wedge \gamma})^{-\frac{1}{2}}$ . It is also the density for the reduced Veselova system for the inertia operator  $\mathbf{I}$  (see also [15, Theorem 5.5] for  $r = 1$  and  $\mathcal{I} = \mathbf{I}$ ). On the other hand, for  $\mathbf{I}\omega = J\omega + \omega J$ ,

Fasso, Garcia-Naranjo, and Montaldi found the invariant measure of the Veselova system in the form (see Proposition 3.7, [13])

$$\left( \frac{\det C_\gamma}{(\gamma, C_\gamma(\gamma))} \right)^{1/2} \mathbf{w}^{n-1}.$$

Therefore, for  $\mathbf{I}$  given by (2.5), the invariant measure (2.12), up to multiplication by a constant, is equal to (2.11).  $\square$

### 3. Integrals and nonexistence of a Chaplygin multiplier

**3.1.** Assume  $I_i = I_j$ , i.e.,  $J_i = J_j$ . Then the Lagrangian  $L_{\text{red}}$  and the Hamiltonian  $H$  are invariant with respect to the rotations in the  $(\gamma_i, \gamma_j)$ -plane. We can expect the associated Noether type integral linear in momenta (e.g., see [1, 21]). We have

THEOREM 3.1. *Let  $J_i = J_j$ . Then*

$$(3.1) \quad \phi_{ij} = (J_i + (\gamma, J\gamma))^{\frac{1}{2\epsilon}-1} (p_i \gamma_j - p_j \gamma_i).$$

is the first integral of the reduced flow (2.8), (2.9).

PROOF. Let us denote  $\mathcal{J} = (\gamma, J\gamma)$ ,  $|\dot{\gamma}|^2 = (\dot{\gamma}, \dot{\gamma})$ . Then we have

$$\dot{\phi}_{ij} = 2 \left( \frac{1}{2\epsilon} - 2 \right) (J_i + \mathcal{J})^{\frac{1}{2\epsilon}-2} (\dot{\gamma}, J\gamma) (p_i \gamma_j - p_j \gamma_i) + (J_i + \mathcal{J})^{\frac{1}{2\epsilon}-1} \frac{d}{dt} (p_i \gamma_j - p_j \gamma_i)$$

and

$$\begin{aligned} \frac{d}{dt} (p_i \gamma_j - p_j \gamma_i) &= \frac{1-\epsilon}{\epsilon} (|\dot{\gamma}|^2 J_i \gamma_i - (\dot{\gamma}, J\gamma) \dot{\gamma}_i - |\dot{\gamma}|^2 \mathcal{J} \gamma_i) \gamma_j - 2H \gamma_i \gamma_j \\ &\quad - \frac{1-\epsilon}{\epsilon} (|\dot{\gamma}|^2 J_i \gamma_j - (\dot{\gamma}, J\gamma) \dot{\gamma}_j - |\dot{\gamma}|^2 \mathcal{J} \gamma_j) \gamma_i + 2H \gamma_j \gamma_i \\ &\quad + (J_i \dot{\gamma}_i - (\dot{\gamma}, J\gamma) \gamma_i + \mathcal{J} \dot{\gamma}_i) \dot{\gamma}_j - (J_i \dot{\gamma}_j - (\dot{\gamma}, J\gamma) \gamma_j + \mathcal{J} \dot{\gamma}_j) \dot{\gamma}_i \\ &= -\frac{1-\epsilon}{\epsilon} (\dot{\gamma}, J\gamma) (\dot{\gamma}_i \gamma_j - \dot{\gamma}_j \gamma_i) + (\dot{\gamma}, J\gamma) (\dot{\gamma}_i \gamma_j - \dot{\gamma}_j \gamma_i) \\ &= \left( 2 - \frac{1}{\epsilon} \right) (\dot{\gamma}, J\gamma) (\dot{\gamma}_i \gamma_j - \dot{\gamma}_j \gamma_i). \end{aligned}$$

Next, from (2.8) we have

$$\dot{\gamma}_i \gamma_j - \dot{\gamma}_j \gamma_i = \frac{1}{J_i + \mathcal{J}} (p_i \gamma_j - p_j \gamma_i),$$

which proves the statement.  $\square$

Note that the integral  $\phi_{ij}$  is not the integral given by the so-called *nonholonomic Noether theorem* (e.g., see [1, 14, 22]), since the generator of the associated  $SO(2)$ -action on the configuration space  $Q$  is not a section of the distribution  $\mathcal{D}$ .

**3.2.** Next we apply the Chaplygin reducing multiplier method: let  $d\tau = \nu(\gamma)dt$ ,  $\gamma' = d\gamma/d\tau = \dot{\gamma}/\nu$ ,  $L^*(\gamma', \gamma) = L_{\text{red}}(\dot{\gamma}, \gamma)|_{\dot{\gamma}=\nu\gamma'}$ . Then we have the new momenta  $\tilde{p} = \partial L^*/\partial\gamma' = \nu p$ . The factor  $\nu$  is a reducing multiplier if under the above time reparameterization the equations (2.3) become Hamiltonian with respect to the symplectic form

$$\begin{aligned}\tilde{\mathbf{w}} &= d\tilde{p}_1 \wedge d\gamma_1 + \cdots + d\tilde{p}_n \wedge d\gamma_n|_{T^*S^{n-1}} \\ &= \nu\mathbf{w} + d\nu \wedge (p_1d\gamma_1 + \cdots + p_nd\gamma_n)|_{T^*S^{n-1}}\end{aligned}$$

(e.g., see [4, 8, 10, 15]). In other words, the vector field  $X = (X_p, X_\gamma)$  is proportional to the Hamiltonian vector field  $\tilde{X}_H$ :

$$X = \nu \cdot \tilde{X}_H, \quad \text{where} \quad i_{\tilde{X}_H} \tilde{\mathbf{w}} = -dH, \quad H(\tilde{p}, \gamma) = H(p, \gamma)|_{p=\nu^{-1}\tilde{p}}.$$

As we mentioned, for the inertia operator (1.3), under the time substitution  $d\tau = \epsilon(A\gamma, \gamma)^{\frac{1}{2\epsilon}-1} dt$ , the reduced system becomes a Hamiltonian system describing a geodesic flow on  $S^{n-1}$  with the metric defined by the Lagrangian (1.4). For  $n = 3$  all inertia operators are of the form (1.3) and the above Hamiltonization reduces to the one given in [12]. Also, for  $\epsilon = 1/2$  the system is already Hamiltonian.

In considering integrable examples below, we will need the following statement.

**THEOREM 3.2.** *Assume that  $n \geq 4$ ,  $\epsilon \neq 1/2$ , and  $J_i = J_j \neq J_k = J_l$  for some mutually different indexes  $i, j, k, l$ . Then the reduced flow (2.8), (2.9) does not allow a Chaplygin multiplier.*

**PROOF.** The existence of a Chaplygin reducing multiplier  $\nu(\gamma)$  implies that the original system has an invariant measure  $\nu^{n-2}\mathbf{w}^{n-1}$  (e.g., see [15, Theorem 3.5]). From the expression of an invariant measure (2.11) we get that a possible Chaplygin multiplier should be proportional to

$$(3.2) \quad \nu(\gamma) = \left( \frac{\det C_\gamma}{(\gamma, C_\gamma(\gamma))} \right)^{\frac{2\epsilon-1}{2\epsilon(n-2)}}.$$

Assume that (3.2) is a Chaplygin multiplier. Then the function  $\nu(\gamma)$  and the Hamiltonian (2.10) in coordinates  $\tilde{p} = \nu p$

$$H(\tilde{p}, \gamma) = H(p, \gamma)|_{p=\tilde{p}/\nu} = \frac{1}{2\nu^2}(\tilde{p}, C_\gamma(\tilde{p})) - \frac{1}{2\nu^2} \frac{(\tilde{p}, C_\gamma(\gamma))^2}{(\gamma, C_\gamma(\gamma))}$$

are invariant with respect to the rotations in the  $(\gamma_i, \gamma_j)$  and  $(\gamma_k, \gamma_l)$  planes. Thus, the Hamiltonian flow of  $H$  has the Noether integrals  $\Phi_{ij} = \tilde{p}_i\gamma_j - \tilde{p}_j\gamma_i$  and  $\Phi_{kl} = \tilde{p}_k\gamma_l - \tilde{p}_l\gamma_k$  (e.g., see [1, 21]). The vector field  $X$  of the original system (2.8), (2.9) is proportional to the Hamiltonian vector field  $\tilde{X}_H$ , implying that it has the same integrals, in coordinates  $(p, \gamma)$  given by:

$$\Phi_{ij} = \nu(p_i\gamma_j - p_j\gamma_i), \quad \Phi_{kl} = \nu(p_k\gamma_l - p_l\gamma_k).$$

On the other hand, according to (3.1),  $f_1 = \phi_{ij}/\Phi_{ij}$  and  $f_2 = \phi_{kl}/\Phi_{kl}$  are integrals on an open dense set  $\Phi_{ij} \neq 0$ ,  $\Phi_{kl} \neq 0$ , and, by continuity

$$f_1(\gamma) = \frac{1}{\nu}(J_i + (\gamma, J\gamma))^{\frac{1}{2\epsilon}-1}, \quad f_2(\gamma) = \frac{1}{\nu}(J_k + (\gamma, J\gamma))^{\frac{1}{2\epsilon}-1}$$



are integrals on the whole phase space as well. Observe that if  $f$  is an integral of the reduced flow (2.8), (2.9) that depends only on  $\gamma$ , then  $f$  is a constant. Thus,  $f_1(\gamma)$  and  $f_2(\gamma)$  are integrals only if

$$\left(\frac{\det C_\gamma}{(\gamma, C_\gamma(\gamma))}\right)^{\frac{2\epsilon-1}{2\epsilon(n-2)}} = \text{const}_1 \cdot (J_i + (\gamma, J\gamma))^{\frac{1}{2\epsilon}-1} = \text{const}_2 \cdot (J_k + (\gamma, J\gamma))^{\frac{1}{2\epsilon}-1}.$$

Since  $\epsilon \neq 1/2$ , we get that  $J_i = J_k$ , which is a contradiction. □

The above considerations cannot be applied if we assume that  $n-1$  parameters  $I_i$  are equal, for example  $I_1 = I_2 = \dots = I_{n-1}$  ( $SO(n-1)$ -symmetric rigid body, *multidimensional Lagrange top*). Moreover, in that case  $\mathbb{I}$  is of the form (1.3), where  $A = \text{diag}(a_1, \dots, a_1, a_n)$  is defined by

$$I_1 = \dots = I_{n-1} = \frac{a_1^2 - D}{2}, \quad I_n = a_1 a_n - \frac{a_1^2 + D}{2}.$$

Then  $\epsilon(A\gamma, \gamma)^{\frac{1}{2\epsilon}-1}$  is a Chaplygin multiplier and we have the identities

$$\left(\frac{\det C_\gamma}{(\gamma, C_\gamma(\gamma))}\right)^{\frac{2\epsilon-1}{2\epsilon(n-2)}} = \text{const}_1 \cdot (J_1 + (\gamma, J\gamma))^{\frac{1}{2\epsilon}-1} = \text{const}_2 \cdot (A\gamma, \gamma)^{\frac{1}{2\epsilon}-1}.$$

Note that one can prove the theorem directly, by applying the time-reparametrisation  $d\tau = \nu(\gamma)dt$ , where  $\nu$  is given by (3.2), into the system (2.8), (2.9). However, the calculations are much more complicated.

#### 4. Integrability of a symmetric case

Firstly, note that in the case of  $SO(n)$ -symmetry, when the mass tensor  $I$  (i.e., the matrix  $J$ ) is proportional to the identity matrix, the  $(0, 3)$ -tensor  $\Sigma$  vanishes and the trajectories of (2.8), (2.9) are great circles for all  $\epsilon$ . Similarly as for the Veselova problem [13], we have the following statement.

**THEOREM 4.1.** *For the symmetric inertia operator (1.5), (1.6), the reduced system (2.8), (2.9) is solvable by quadratures and we have:*

- (i) *If  $r \neq 1, n-1$ , generic motions are quasi-periodic over 3-dimensional invariant tori that are level sets of integrals  $H, \phi_{ij}, \phi_{kl}, 1 \leq j < i \leq r, r < l < k \leq n$ .*
- (ii) *If  $r = n-1$ , generic motions are quasi-periodic over 2-dimensional invariant tori that are level sets of  $H, \phi_{ij}, 1 \leq j < i \leq n-1$  (similarly for  $r = 1$ ).*

**PROOF.** For  $\epsilon = 1/2$  the system is Hamiltonian and the proof follows from the Theorem on non-commutative integrability of the Hamiltonian systems (see [5]). In the case  $r = n-1$ , the function  $\nu(\gamma) = (J_1 + (\gamma, J\gamma))^{\frac{1}{2\epsilon}-1}$  is a Chaplygin multiplier and the integrability follows from the non-commutative integrability of the associated Hamiltonian system (see [19]).

According to Theorem 3.2, for  $\epsilon \neq 1/2$  and  $r \neq 1, n-1$ , the system does not have a Chaplygin multiplier. However, we can apply a variant of the reduction method used by Fasso, Garcia-Naranjo, and Montaldi in the case of the Veselova problem [13].

The system is  $SO(r) \times SO(n-r)$ -invariant. For any initial conditions  $(p_0, \gamma_0)$ , one can find a matrix  $R = \text{diag}(R_1, R_2) \in SO(r) \times SO(n-r)$ , such that the coordinates of  $R\gamma_0$  and  $Rp_0$  with indexes  $3, 4, \dots, n-3, n-2$  vanish. Therefore, the only non-zero values of the Noether type integrals (3.1) at  $(Rp_0, R\gamma_0)$  are  $\phi_{21}$  and  $\phi_{n,n-1}$  and the coordinates with indexes  $3, 4, \dots, n-3, n-2$  of the trajectory with the initial conditions  $(Rp_0, R\gamma_0)$  are zero. Therefore, without loss of generality it can be assumed that  $n = 4, r = 2$ . Since the system is  $SO(2) \times SO(2)$ -invariant, we can pass to the second reduced space  $P = T^*S^3/SO(2) \times SO(2)$ .

The regular compact connected components  $M_{c_1, c_2, h}$  of invariant varieties

$$\phi_{21} = c_1, \quad \phi_{43} = c_2, \quad H = h$$

are 3-dimensional and  $SO(2) \times SO(2)$ -invariant. Therefore, the reduced invariant sets

$$M_{c_1, c_2, h}/SO(2) \times SO(2) \subset P$$

are relative periodic orbits. Whence, we get that generic trajectories are quasi-periodic over 3-dimensional invariant tori  $M_{c_1, c_2, h}$  (for the reconstruction of relative periodic orbits, e.g. see [13, 17]).  $\square$

Thus, for  $r \neq 1, n-1$  and  $\epsilon \neq 1/2$ , the problem is solvable, has an invariant measure, and according to Theorem 3.2 does not allow the Chaplygin reducing multiplier. Closely related, let us note that the rolling of the ball over a horizontal plane without spinning and twisting, where the mass center does not coincide with the geometrical center, provides an example of the system such that the appropriate phase space is foliated by invariant tori, but the system itself does not have an analytic invariant measure and is not Hamiltonizable either (see [3]).

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**НАПОМЕНА О КОТРЉАЊУ КУГЛЕ ПО СФЕРИ:  
ИНТЕГРАБИЛНИ ЧАПЛИГИНОВ СИСТЕМ СА  
ИНВАРИАНТНОМ МЕРОМ БЕЗ ЧАПЛИГИНОВЕ  
ХАМИЛТОНИЗАЦИЈЕ**

РЕЗИМЕ. У овој ноти разматрамо нехолономни проблем котрљања  $n$ -димензионе балансиране кугле по непокретој сфери без клизања и ротације у тангентној равни додира. То је  $SO(n)$ -Чаплигинов систем са инваријантном мером који се своди на котангентно раслојење  $T^*S^{n-1}$ . За оператор инерције крутог тела  $\mathbb{L}\omega = I\omega + \omega I$ ,  $I = \text{diag}(I_1, \dots, I_n)$  са симетријом  $I_1 = I_2 = \dots = I_r \neq I_{r+1} = I_{r+2} = \dots = I_n$ , доказујемо да је редуковани систем интеграбилан, опште трајекторије су квази-периодичне, при чему се за  $r \neq 1, n-1$  Чаплигинова метода редуccionог множитеља не може применити.

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