THEORETICAL AND APPLIED MECHANICS
Volume 46 (2019) Issue 1, 89–96 DOI: https://doi.org/10.2298/TAM190116002N

# ADMISSIBILITY OF A SOLUTION TO GENERALIZED CHAPLYGIN GAS

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Abstract. It is known that there is a solution to the Riemann problem for generalized Chaplygin gas model and that it contains the Dirac delta function in some cases. In some cases, usual admissible criteria can not extract a unique weak solution as it was shown in [4]. The aim of this paper is to use a solution to perturbed generalized Chaplygin model by a small constant $\varepsilon > 0$ and obtain a its unique limit. A weak solution to the unperturbed system that equals that limit is called admissible. The perturbation is made by using the modified model of Chaplygin gas defined in [5].

## 1. Introduction

The original Chaplygin system

$$\begin{aligned}\partial_t\rho + \partial_x(\rho u) &= 0\\ \partial_t(\rho u) + \partial_x\Big(\rho u^2 - \frac{1}{\rho}\Big) &= 0\end{aligned}$$

was introduced as a model for a fluid passing by an obstacle (see [2]). The model of generalized Chaplygin gas appears in a number of cosmology theories as a compressible fluid with a pressure inversely proportional to a gas energy density, $p = -C/\rho^\alpha$, $C > 0$, $0 < \alpha < 1$, see [1]. It is used as a model for the dark energy in the Universe. (We will use $C = 1$ in the rest of the paper for simplicity.) The system consists of the mass and momentum conservation laws

$$\begin{aligned}\partial_t\rho + \partial_x(\rho u) &= 0\\ \partial_t(\rho u) + \partial_x\Big(\rho u^2 - \frac{1}{\rho^\alpha}\Big) &= 0.\end{aligned} \tag{1.1}$$

Let us briefly give the properties of the system. It is a strictly hyperbolic system with the eigenvalues $\lambda_1 = u - \sqrt{\alpha}\rho^{-\frac{1+\alpha}{2}}$, $\lambda_2 = u + \sqrt{\alpha}\rho^{-\frac{1+\alpha}{2}}$ and appropriate eigenvectors $r_1 = \big(-1, -u + \sqrt{\alpha}\rho^{-\frac{1+\alpha}{2}}\big)^T$ and $r_2 = \big(1, u + \sqrt{\alpha}\rho^{-\frac{1+\alpha}{2}}\big)^T$. Both fields are genuinely nonlinear.

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2010 *Mathematics Subject Classification:* 35L67; 35L67; 76L05.
*Key words and phrases:* generalized Chaplygin model, delta shocks, fluc perturbation.

Using the standard procedures one can find rarefaction curves:

$$R_1 : u = u_0 + \frac{2\sqrt{\alpha}}{1+\alpha}\left(\rho^{-\frac{1+\alpha}{2}} - \rho_0^{-\frac{1+\alpha}{2}}\right), \quad \rho < \rho_0$$
$$R_2 : u = u_0 - \frac{2\sqrt{\alpha}}{1+\alpha}\left(\rho^{-\frac{1+\alpha}{2}} - \rho_0^{-\frac{1+\alpha}{2}}\right), \quad \rho > \rho_0,$$

and shock ones:

$$S_1 : u = u_0 - \sqrt{\frac{\rho-\rho_0}{\rho_0}\Big(\frac{1}{\rho_0^\alpha} - \frac{1}{\rho^\alpha}\Big)}, \quad \rho > \rho_0,$$

$$S_2 : u = u_0 - \sqrt{\frac{\rho-\rho_0}{\rho_0}\Big(\frac{1}{\rho_0^\alpha} - \frac{1}{\rho^\alpha}\Big)}, \quad \rho < \rho_0.$$

Shock speeds for points at both curves $S_1$ and $S_2$ are given by

$$c = u_0 \pm \sqrt{\frac{\rho^{1-\alpha}}{\rho_0^{1+\alpha}}\frac{\rho^\alpha - \rho_0^\alpha}{\rho - \rho_0}}.$$

where $+$ sign is for $S_2$ and $-$ for $S_1$. A solution to the Riemann problem

$$(\rho, u)|_{t=0} = \begin{cases} (\rho_0, u_0), & x < 0 \\ (\rho_1, u_1), & x > 0 \end{cases}. \tag{1.2}$$

for (1.1) is given as a combination of the elementary waves for the points $(\rho_1, q_1)$ above and on the curve

$$\Gamma_{ss} = \Gamma_{ss}(\rho_0, q_0) : q = \Big(\frac{q_0}{\rho_0} - \rho^{-\frac{1+\alpha}{2}} - \rho_0^{-\frac{1+\alpha}{2}}\Big)\rho. \tag{1.3}$$

Below that line there are no classical solutions. One can use shadow waves [3] in order to solve the problem there. These waves are approximate solutions to balance law systems. In [4] the following lemma is proved

Lemma 1.1. *There exists a simple shadow wave written in the form*

$$(\rho, u)(x,t) = \begin{cases} (\rho_0, u_0), & x < (c-\varepsilon)t \\ (\rho_{0,\varepsilon}, u_{0,\varepsilon}), & (c-\varepsilon)t < x < ct \\ (\rho_{1,\varepsilon}, u_{1,\varepsilon}), & ct < x < (c+\varepsilon)t \\ (\rho_1, u_1), & x > (c+\varepsilon)t, \end{cases} \tag{1.4}$$

*that solves* (1.1) *with the initial data* (1.2) *if and only if*

$$(u_0\rho_0\rho_1 - u_1\rho_0\rho_1)^2 > (\rho_0 - \rho_1)\Big(\frac{1}{\rho_1^\alpha} - \frac{1}{\rho_0^\alpha}\Big)\rho_0\rho_1. \tag{1.5}$$

*The speed $c$ is given by*

$$c = \frac{[\rho u] + \sqrt{[\rho u]^2 - [\rho]\Big[\frac{(\rho u)^2 - \rho^{1-\alpha}}{\rho}\Big]}}{[\rho]},$$

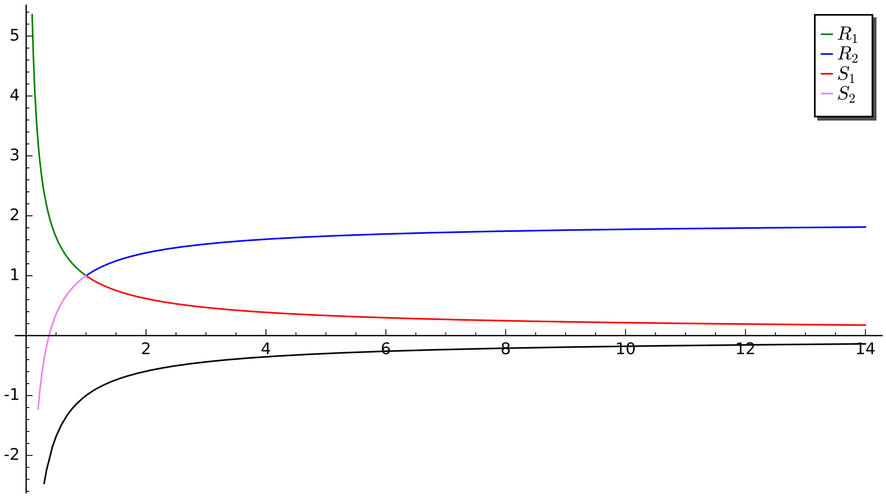


FIGURE 1. Classical waves

*where* $[x]$ *denotes a jump* $x_1 - x_0$. *The strength of the shadow wave equals*

$$\sigma = \sqrt{(u_0 - u_1)^2 \rho_0 \rho_1 - (\rho_0 - \rho_1)\Big(\frac{1}{\rho_1^\alpha} - \frac{1}{\rho_0^\alpha}\Big)}.$$

*The solution* (1.4) *converges to*

$$(\bar\rho, \bar u)(x,t) = \begin{cases} (\rho_0, u_0), & x < ct \\ (\rho_1, u_1), & x > ct, \end{cases} + \sigma t, \tag{1.6}$$

*in the distributional sense as* $\varepsilon \to 0$.

"Solve" here means that a distributional limit of the left–hand sides in (1.1) with $(\rho, u)$ substituted by a net (1.4) equals zero as $\varepsilon \to 0$.

In order to get a unique solution to the given Riemann problem, one has to exclude all points $(\rho_1, u_1)$ satisfying (1.5) above the line $\Gamma_{ss}$ by an admissibility criterion. The usual one, overcompressibility, $\lambda_2(\rho_0, u_0) \geqslant \lambda_1(\rho_0, u_0) \geqslant c \geqslant \lambda_2(\rho_1, u_1) \geqslant \lambda_1(\rho_1, u_1)$, is not enough. In [4], one can find a better condition made by using Lax entropy pairs, but there are still points above $\Gamma_{ss}$ for which one cannot know whether they are admissible or not.

Our aim is to perturb the second equation in (1.1) by adding a small term depending on $\varepsilon$ such that we get a modified Chaplygin model described in [5]. Then, we expect that $\varepsilon \to 0$ will recover the above shadow wave solution only below the curve $\Gamma_{ss}$. That could be used as a new admissibility criterion. One can see a similar procedure for the original Chaplygin system in [6].

## 2. Modified Chaplygin gas model

We will use the following perturbation of the system (1.1) based on the model from [5] by letting the positive parameter in the flux function vanish

$$\begin{aligned}\partial_t\rho+\partial_x(\rho u)&=0\\ \partial_t(\rho u)+\partial_x\Big(\rho u^2+\varepsilon\rho-\frac{1}{\rho^\alpha}\Big)&=0.\end{aligned}\tag{2.1}$$

Let us assume initial data (1.2) for the system. Such Riemann problem has a unique solution in the physical domain $\rho>0$, $u\in\mathbb{R}$ given as a combination of elementary waves. Let us briefly describe those solutions.

We have a strictly hyperbolic system with the eigenvalues $\lambda_1=u-\sqrt{\varepsilon+\alpha\rho^{-(1+\alpha)}}$, $\lambda_2=u+\sqrt{\varepsilon+\alpha\rho^{-(1+\alpha)}}$ and appropriate eigenvectors $r_1=\big(-1,-u+\sqrt{\varepsilon+\alpha\rho^{-(1+\alpha)}}\big)^T$ and $r_2=\big(1,u+\sqrt{\varepsilon+\alpha\rho^{-(1+\alpha)}}\big)^T$. Both fields are genuinely nonlinear for $\alpha\in(0,1)$ and $\varepsilon$ small enough.

Perturbed rarefaction curves are given by

$$\begin{aligned}R_{1,\varepsilon}:u&=u_0-\int_{\rho_0}^{\rho}s^{-1}\sqrt{\varepsilon+\alpha s^{-(1+\alpha)}}ds,\quad \rho<\rho_0\\ R_{2,\varepsilon}:u&=u_0+\int_{\rho_0}^{\rho}s^{-1}\sqrt{\varepsilon+\alpha s^{-(1+\alpha)}}ds,\quad \rho>\rho_0.\end{aligned}\tag{2.2}$$

Perturbed shock curves are

$$\begin{aligned}S_{1,\varepsilon}:u&=u_0-\frac{1}{\sqrt{\rho\rho_0}}((\rho_0^{-\alpha}-\rho^{-\alpha})+\varepsilon(\rho-\rho_0)),\quad \rho>\rho_0,\\ S_{2,\varepsilon}:u&=u_0-\frac{1}{\sqrt{\rho\rho_0}}((\rho_0^{-\alpha}-\rho^{\alpha})+\varepsilon(\rho-\rho_0)),\quad \rho<\rho_0.\end{aligned}\tag{2.3}$$

A shock speed for a point $(\rho,u)$ at the curve $S_1$ or $S_2$ is given by

$$c_{1,\varepsilon}=u_0-\sqrt{\frac{\rho^{1-\alpha}}{\rho_0^{1+\alpha}}\Big(\frac{\rho^\alpha-\rho_0^\alpha}{\rho-\rho_0}+\varepsilon\Big)},\quad \text{or}\quad c_{2,\varepsilon}=u_0+\sqrt{\frac{\rho^{1-\alpha}}{\rho_0^{1+\alpha}}\Big(\frac{\rho^\alpha-\rho_0^\alpha}{\rho-\rho_0}+\varepsilon\Big)}.$$

A solution to the Riemann problem

$$(\rho,u)|_{t=0}=\begin{cases}(\rho_0,u_0), & x<0\\ (\rho_1,u_1), & x>0\end{cases}$$

for (2.1) is given as a combination of the elementary waves for all points $(\rho,u)\in\mathbb{R}_+\times\mathbb{R}$ contrary to the case of generalized Chaplygin gas (1.1). That is the main difference between (1.1) and (2.1). One can find an illustration in Figure 2: grey lines represent non-perturbed rarefaction and shock curves, the black line is $\Gamma_{ss}$. Note the important fact: $S_{1,\varepsilon}$ crosses the line $\Gamma_{ss}$.

One could see that all perturbed $R$ and $S$–curves tend to unperturbed ones, but with one significant difference. The perturbed $S_2$–curve lies sufficiently below the critical curve $\Gamma_{ss}$ so one could expect that it could be possible to obtain a $S_1+S_2$ solution to (2.1) below the critical curve. One can see an illustration in Figure 3. Unlike the original system we have the following lemma.

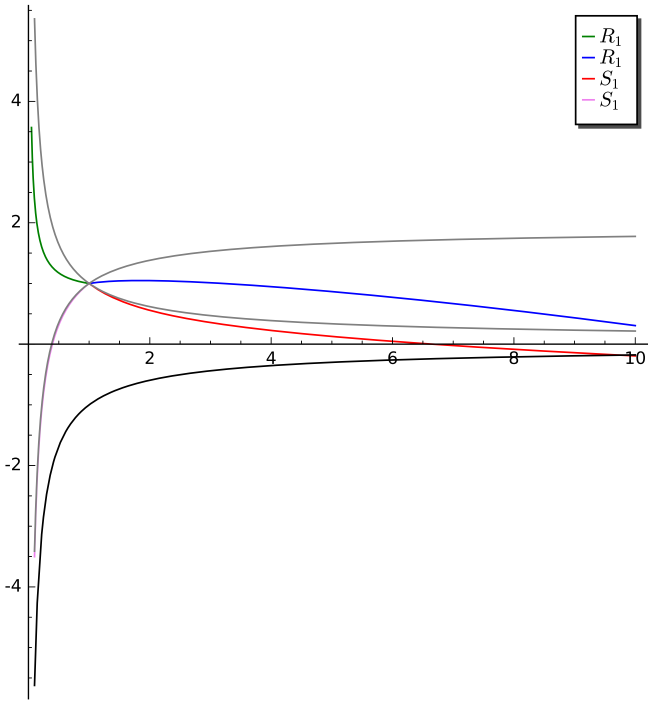


FIGURE 2. Perturbed elementary waves

THEOREM 2.1. *The Riemann problem* (2.1), (1.2) *has a unique entropic solution consisting of a combination of shocks and rarefaction waves.*

*As* $\varepsilon \to 0$ *the solution tends to the one to* (1.1)*. Additionally, it converges to the shadow wave solution* (1.4) *if and only if* $(\rho_1, u_1)$ *lies below the curve* $\Gamma_{ss}$.

PROOF. Proof for all areas but the one between $S_{1,\varepsilon}$ and $S_{2,\varepsilon}$ curve is almost the same as for the original system (1.1). We will present a proof for that area here.

First, we will prove that any point $(\rho_1, u_1)$ between these lines can be connected to $(\rho_0, u_0)$ by a combination of two shocks. Let us denote a middle state by $(\rho_\varepsilon, u_\varepsilon)$. It is a solution to the following system of equations

$$u_\varepsilon = u_0 - \frac{1}{\sqrt{\rho_1\rho_\varepsilon}}\sqrt{(\rho_\varepsilon - \rho_0)((\rho_0^{-\alpha} - \rho_\varepsilon^{-\alpha}) + \varepsilon(\rho_\varepsilon - \rho_0))}, \quad \rho_\varepsilon > \rho_0,$$
$$u_1 = u_\varepsilon - \frac{1}{\sqrt{\rho_1\rho_\varepsilon}}\sqrt{(\rho_1 - \rho_\varepsilon)((\rho_\varepsilon^{-\alpha} - \rho_1^{-\alpha}) + \varepsilon(\rho_1 - \rho_\varepsilon))}, \quad \rho_1 < \rho_\varepsilon.$$

Thus, the value $\rho_\varepsilon$ is a solution to the equation $f_1(\rho) = f_2(\rho)$, where

$$f_1(\rho) := u_0 - \sqrt{\Big(\frac{1}{\rho_0} - \frac{1}{\rho}\Big)\Big(\frac{1}{\rho_0^\alpha} - \frac{1}{\rho^\alpha}\Big) + \varepsilon\Big(\frac{\rho}{\rho_0} + \frac{\rho_0}{\rho} - 2\Big)}$$

and

$$f_2(\rho) := u_1 + \sqrt{\Big(\frac{1}{\rho_1} - \frac{1}{\rho}\Big)\Big(\frac{1}{\rho_1^\alpha} - \frac{1}{\rho^\alpha}\Big) + \varepsilon\Big(\frac{\rho}{\rho_1} + \frac{\rho_1}{\rho} - 2\Big)}.$$

One can see that $f_1'(\rho) < 0$, $\rho > \rho_0$, $f_1(\infty) = -\infty$, $f_2'(\rho) > 0$, $\rho > \rho_1$, and $f_1(\infty) = \infty$ so there exists a unique solution $\rho_\varepsilon$ to the equation $u_\varepsilon = f_1(\rho_\varepsilon) = f_2(\rho_\varepsilon)$. Immediately, one sees that $\rho_\varepsilon$ is increasing and goes to infinity like const $/\varepsilon$ as $\varepsilon \to 0$. The value of $u_\varepsilon$ is then uniquely defined, too.

We will now prove that a distributional limit of a shock combination $S_{1,\varepsilon} + S_{2,\varepsilon}$ is the same as a limit of the shadow wave solution to the unperturbed system (1.1).

Denote by $c_{1,\varepsilon}$ a speed of a shock wave connecting $(\rho_0, u_0)$ with $(\rho_\varepsilon, u_\varepsilon)$ and by $c_{2,\varepsilon}$ a speed of a shock wave connecting $(\rho_\varepsilon, u_\varepsilon)$ with $(\rho_1, u_1)$. Using the relations

$$c_{1,\varepsilon} = \frac{\rho_\varepsilon u_\varepsilon - \rho_0 u_0}{\rho_\varepsilon - \rho_0}, \quad c_{2,\varepsilon} = \frac{\rho_\varepsilon u_\varepsilon - \rho_1 u_1}{\rho_\varepsilon - \rho_1},$$

we have

$$\begin{aligned}\bar\sigma := \lim_{\varepsilon\to 0}(c_{2,\varepsilon} - c_{1,\varepsilon})\rho_\varepsilon &= \frac{\rho_\varepsilon}{(\rho_\varepsilon - \rho_0)(\rho_\varepsilon - \rho_1)}((\rho_0 u_0 - \rho_1 u_1)\rho_\varepsilon + (\rho_1 - \rho_0)\rho_\varepsilon u_\varepsilon) \\ &= (\rho_1 - \rho_0)\lim_{\varepsilon\to 0} u_\varepsilon - (\rho_1 u_1 - \rho_0 u_0)\end{aligned}$$

Letting $\varepsilon \to 0$, we get $\lim_{\varepsilon\to 0} u_\varepsilon = \lim_{y\to\infty} f_1(\rho_\varepsilon) = u_0 - \rho_0^{-1/2}(\rho_0^{-\alpha} + y)^{1/2}$ and $\lim_{\varepsilon\to 0} u_\varepsilon = \lim_{y\to\infty} f_2(\rho_\varepsilon) = u_1 - \rho_1^{-1/2}(\rho_1^{-\alpha} + y)^{1/2}$, where $y = \lim_{\varepsilon\to 0}\varepsilon\rho_\varepsilon$. One can find $y$ from these relations, $y = \xi^2 - \rho_0^{-\alpha}$, where

$$\xi = \frac{\rho_0\rho_1}{\rho_1 - \rho_0}\Big(\frac{u_0 - u_1}{\sqrt{\rho_0}} - \frac{1}{\sqrt{\rho_1}}\sqrt{(u_0 - u_1)^2 - \Big(\frac{1}{\rho_0} - \frac{1}{\rho_1}\Big)\Big(\frac{1}{\rho_0^\alpha} - \frac{1}{\rho_1^\alpha}\Big)}\Big).$$

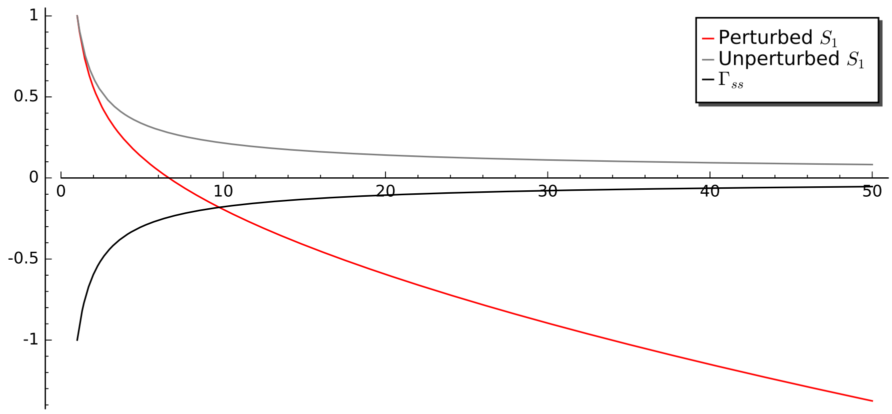


Figure 3. Perturbed elementary waves

Finally, that gives

$$\bar{\sigma} = \sqrt{\rho_0\rho_1}\sqrt{(u_0 - u_1)^2 + \big(\frac{1}{\rho_0} - \frac{1}{\rho_1}\big)\big(\frac{1}{\rho_0^\alpha} - \frac{1}{\rho_1^\alpha}\big)}$$

that equals $\sigma$ from (1.6). Also $\lim_{\varepsilon\to 0} c_{i,\varepsilon} = \lim_{\varepsilon\to 0} u_\varepsilon = c$, $i = 1, 2$, with $c$ from (1.6). That means that the distributional limit of the solution to (2.1) for $(\rho_1, u_1)$ below $\Gamma_{ss}$ equals the distributional limit of (1.4). Above that curve the limit is $S_1 + S_2$. □

Thus, this theorem could be used as an admissibility condition for eliminating unwanted shadow waves above the curve $\Gamma_{ss}$.

An approximate solution to (1.1) is admissible if and only if a classical solution to (2.1) converges to the same distribution.

**Acknowledgement.** The work is partially supported by the projects OI174024, III44006 (Ministry of Science, RS), 142-451-3652 (Provincial secretariat for science APV).

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# ПРИХВАТЉИВОСТ РЕШЕЊА УОПШТЕНОГ ЧАПЛИГИНОВОГ ГАСА

Резиме. Познато је да постоје решења Римановог проблема за уопштени Чаплигинов гас која садрже делта функцију. У неким ситуацијама не могу да се примене уобичајене методе бирања јединственог слабог решења, како је то показано у [4]. Циљ овог рада је да се искористи ограничено решење пертурбваног уопштеног Чаплигиновог модела малом константом $\varepsilon > 0$ и нађе јединствени лимит таквог решења када $\varepsilon \to 0$. Слабо решење непертурбованог система којем тежи то решење пертурбованог система ће бити прихватљиво. Пертурбација је урађена коришћењем модела модификованог Чаплигиновог гаса дефинисаном у [5].

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(Received 16.01.2019.)
(Available online 22.04.2019.)