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HAMILTONISATION, MEASURE PRESERVATION AND FIRST INTEGRALS OF THE MULTI-DIMENSIONAL RUBBER ROUTH SPHERE

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Dedicated to S. A. Chaplygin on the occasion of his 150th anniversary

ABSTRACT. We consider the multi-dimensional generalisation of the problem of a sphere, with axi-symmetric mass distribution, that rolls without slipping or spinning over a plane. Using recent results from García-Naranjo [21] and García-Naranjo and Marrero [22], we show that the reduced equations of motion possess an invariant measure and may be represented in Hamiltonian form by Chaplygin's reducing multiplier method. We also prove a general result on the existence of first integrals for certain Hamiltonisable Chaplygin systems with internal symmetries that is used to determine conserved quantities of the problem.

1. Introduction

An important contribution of S. A. Chaplygin to the field of nonholonomic systems was the introduction of the so-called *Chaplygin's reducing multiplier method* [9]. It is concerned with a certain class of nonholonomic systems with symmetry, commonly referred to today as *nonholonomic Chaplygin systems*, whose reduced equations of motion have the form of a classical mechanical system subjected to extra gyroscopic forces. Chaplygin's method consists of searching for a *time reparametrisation*, hoping that in the new time variable, and after a momentum rescaling, the extra forces vanish and the resulting system is Hamiltonian. If successful, this process is often referred to as *Chaplygin Hamiltonisation*. It is also common to say that the Chaplygin system at hand, in the original time variable, is *conformally Hamiltonian*. The subclass of Chaplygin systems allowing a Chaplygin Hamiltonisation is quite remarkable, and substantial effort has been devoted to their study and characterisation (see e.g. [3,6,9,10,16,18,21,25,36] and references therein). The purpose of this paper is to provide a new non-trivial example within

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this category, and to prove a considerably general Noether-type of result for these systems, which links their internal symmetries with first integrals, and which we apply to our example.

ϕ-simple Chaplygin systems: Hamiltonisation and first integrals. Our approach to Chaplygin Hamiltonisation relies on the notion of *ϕ-simple Chaplygin systems* introduced recently in García-Naranjo and Marrero [22] inspired by the results of the author in [21]. These systems form an exceptional subclass of non-holonomic Chaplygin systems that always possess an invariant measure and allow a Chaplygin Hamiltonisation.

The definition of ϕ-simple Chaplygin systems relies on a certain tensor field $\mathcal{T}$ of type $(1,2)$ defined on the *shape space*[1] S of the system, which measures the interplay between the kinetic energy and the non-integrability of the constraint distribution. This tensor field already appears in the works of Koiller [30] and Cantrijn et al [8], and, following the terminology of [21, 22], will be called the *gyroscopic tensor*. A Chaplygin system is said to be *ϕ-simple* if there exists a function $\phi \in C^\infty(S)$ such that the gyroscopic tensor $\mathcal{T}$ satisfies[2]

$$\mathcal{T}(Y,Z) = Z[\phi]Y - Y[\phi]Z, \tag{1.1}$$

for any two vector fields Y, Z on S. The above condition was obtained as the coordinate-free formulation of the recent results on Chaplygin Hamiltonisation given by the author [21]. It is shown in [22] that the condition to be ϕ-simple is equivalent to the verification of certain sufficient conditions for Chaplygin Hamiltonisation given previously by Stanchenko [36] and Cantrijn et al [8]. The advantage of the formulation in [21] and [22] with respect to these references is that condition (1.1) can be systematically examined in concrete examples.

The statement that a ϕ-simple Chaplygin system allows a Chaplygin Hamiltonisation is independent of the number of degrees of freedom of the problem, and may be interpreted as a generalisation of the celebrated Chaplygin's Reducing Multiplier Theorem [9] whose applicability is restricted to systems whose shape space has dimension 2 – see the discussion in [21] and [22].

The criterion of ϕ-simplicity has already been used in [21] and [22] to establish the Hamiltonisation of non-trivial examples. Among them is the multi-dimensional Veselova problem, whose Hamiltonisation was first proven by Fedorov and Jovanović [16,17] by a direct application of Chaplygin's method to the reduced equations of motion. In Section 4 of this paper we prove that the multi-dimensional rubber Routh sphere (introduced below) is also ϕ-simple. This allows us to prove that the system allows a Chaplygin Hamiltonisation, and to give a closed formula for its invariant measure, without writing the equations of motion. Our results

[1] The shape space is the quotient manifold $S = Q/G$ where Q is the configuration manifold of the system and G is the underlying symmetry group of the Chaplygin system.

[2] Throughout the paper we denote by $Y[f]$ the action of the vector field Y on the scalar function f.

seem to support the thesis that ϕ-simplicity is the relevant mechanism behind the Chaplygin Hamiltonisation of concrete examples.[3]

In Section 3 we prove a general result that shows how extra symmetries of ϕ-simple Chaplygin systems lead to the existence of conserved quantities (Theorem 3.1). This result is applied to find first integrals of the multi-dimensional rubber Routh sphere in Section 4.3. This contributes to the recent efforts to understand the mechanisms responsible for the existence of first integrals that are linear in velocities in nonholonomic mechanics (see e.g. [1, 11, 12, 24]).

The multi-dimensional rubber Routh sphere. Routh [35] considered the problem of a sphere whose distribution of mass is axially symmetric, and which rolls without slipping on the plane. Later, Borisov and Mamaev [5, 7] considered the problem under an additional *rubber*[4] constraint that forbids spinning. In this paper we consider the multi-dimensional generalisation of this system. Our term *multi-dimensional rubber Routh sphere* is supposed to indicate the presence of a no-spin constraint in the word *rubber*, and the axi-symmetric assumption on the mass distribution of the sphere with the mention of *Routh*'s name. A closely related problem is the multi-dimensional rubber Chaplygin sphere considered by Jovanović [27].

The study of multi-dimensional systems in nonholonomic mechanics goes back to Fedorov and Kozlov [15], and has received wide attention as a source of interesting examples for integrability, Hamiltonisation and other types of dynamical features [13, 14, 16, 17, 19, 26, 28, 29, 37]. Our analysis of the multi-dimensional rubber Routh sphere contributes to enlarging this family of examples.

Structure of the paper. In Section 2 we present a quick review of the recent constructions in [21, 22]. This summary includes the definition of the gyroscopic tensor and its expression in local coordinates. We also recall the notion of ϕ-simplicity (described above) and, in Theorem 2.1, we indicate its precise relationship with measure preservation and Hamiltonisation. Section 3 is completely devoted to Theorem 3.1, which relates internal symmetries of ϕ-simple Chaplygin systems to first integrals. Section 4 is concerned with the multidimensional rubber Routh sphere. To simplify the reading, we first treat the 3D system in subsection 4.1 and then proceed to the nD generalisation in subsection 4.2. The ϕ-simplicity of the system is presented in Theorem 4.2 and the consequential measure preservation and Hamiltonisation properties in Corollary 4.1. The results of Section 3 are then applied to determine first integrals of the problem in subsection 4.3. The paper finishes with Appendix A that contains the proof of a technical lemma needed in the proof of Theorem 4.2.

[3] This statement is meant within the framework of Hamiltonisation of Chaplygin systems. Other examples, like the remarkable Hamiltonisation of the Chaplygin sphere obtained by Borisov and Mamaev [4], involve a further symmetry reduction [6] and other geometric mechanisms come into play to ensure that invariant first integrals descend to the quotient space as Casimir functions [20].

[4] The *rubber* terminology for constraints that prohibit spinning goes back to Ehlers et al [10] and Koiller and Ehlers [31] and is now quite standard in the field.

2. Preliminaries: a review of ϕ-simple Chaplygin systems and their measure preservation and Hamiltonisation properties

In this section we briefly recall the notion of nonholonomic Chaplygin systems and, more specifically, ϕ-simple Chaplygin systems introduced in García-Naranjo and Marrero [22], together with their measure preservation and Hamiltonisation properties.

2.1. Nonholonomic Chaplygin systems and the gyroscopic tensor. For our purposes, a *nonholonomic system* is a triple (Q, D, L). Here Q is an n-dimensional smooth manifold modelling the configuration space of the system. $D \subset TQ$ is a vector sub-bundle whose fibres define a non-integrable distribution on Q of constant rank $r \geqslant 2$, which models $n-r$ linear nonholonomic constraints as follows: a curve $q(t)$ on Q is said to satisfy the constraints if and only if $\dot{q}(t) \in D_{q(t)}$ for all t. Finally, $L\colon TQ \to \mathbb{R}$ is the Lagrangian of the system that is assumed to be of *mechanical type*, namely

$$L = K - U,$$

where the kinetic energy K defines a Riemannian metric $\langle\!\langle \cdot , \cdot \rangle\!\rangle$ on Q, and $U\colon Q \to \mathbb{R}$ is the potential energy.

The triple (Q, D, L) contains all the information for the evolution of the system in accordance with the *Lagrange–D'Alembert principle* of ideal constraints. The (velocity) phase space of the system is D and the dynamics is described by the flow of a uniquely defined vector field $X_{nh} \in \mathfrak{X}(D)$. An intrinsic definition of this vector field may be found, for instance, in [32].

Definition 2.1. The nonholonomic system (Q, D, L) is said to be a *Chaplygin system* if there exists an $(n-r)$-dimensional Lie group G acting freely and properly on Q and satisfying the following properties:

(1) G acts by isometries with respect to the kinetic energy metric $\langle\!\langle \cdot , \cdot \rangle\!\rangle$, and the potential energy U is invariant,
(2) D is invariant in the sense that $D_{g\cdot q} = Tg(D_q)$ for all $g \in G$ and $q \in Q$,
(3) for all $q \in Q$ the following direct sum splitting holds

$$T_qQ = \mathfrak{g} \cdot q \oplus D_q,$$

where $\mathfrak{g}$ denotes the Lie algebra of q, and $\mathfrak{g} \cdot q$ is the tangent space to the orbit through q at q.

Remark 2.1. Chaplygin systems as defined above are also referred to in the literature as *non-abelian Chaplygin systems* [30], *generalised Chaplgyin systems* [8, 16] or the *principal kinematic case of a nonholonomic system with symmetries* [2].

The smooth r-dimensional manifold $S := Q/G$ associated to a Chaplygin system is called the *shape space*. As a consequence of the first and second conditions in Definition 2.1, the vector field X_{nh} describing the dynamics is equivariant (with respect to the G-action on D defined by the restriction of the tangent lifted action of G to TQ) and the system admits a G-reduction. The reduced dynamics is described by the flow of the *reduced vector field* $\bar{X}_{nh}$ on the orbit space D/G. For a Chaplygin system, the reduced phase space D/G is naturally identified with the tangent bundle TS, or cotangent bundle T^*S, using the (reduced) Legendre transform. The reduced equations of motion have the form of a mechanical system on the shape space S subject to gyroscopic forces. Geometrically, they may be written in almost symplectic form, i.e. in Hamiltonian-like form, $\mathbf{i}_{\bar{X}_{nh}}\Omega_{nh} = dH$, where H is the reduced Hamiltonian (energy), but where the non-degenerate 2-form Ω_{nh} on T^*S fails, in general, to be closed. See section 2.3 below or references [10,16,22,23] for more details.

We now recall the definition of the *gyroscopic tensor* from García-Naranjo and Marrero [22]. To do this, we first note that the kinetic energy metric defines the orthogonal decomposition $TQ = D \oplus D^\perp$. We shall denote by

$$\mathcal{P} \colon TQ \to D,$$

the bundle projection associated to such decomposition. Next, we note that, as was first pointed out by Koiller [30], the second and third conditions in Definition 2.1 imply that the fibres of D are the horizontal spaces of a principal connection on the principal bundle $\pi \colon Q \to Q/G = S$. As is well known, corresponding to such a principal connection, there is a well defined *horizontal lift* that associates to any vector field $Y \in \mathfrak{X}(S)$ an equivariant vector field $\mathrm{hor}(Y) \in \mathfrak{X}(Q)$ taking values on D, and that is π-related to Y. We are now ready to present:

Definition 2.2. The *gyroscopic tensor* $\mathcal{T}$ is the $(1,2)$ skew-symmetric tensor field on S determined by assigning to the vector fields $Y, Z \in \mathfrak{X}(S)$, the vector field $\mathcal{T}(Y,Z) \in \mathfrak{X}(S)$, given by

$$\mathcal{T}(Y,Z)(s) = (T_q\pi)(\mathcal{P}[\mathrm{hor}(Y),\, \mathrm{hor}(Z)](q)) - [Y,Z](s), \tag{2.1}$$

for $s \in S$, and where $q \in Q$ is any point such that $\pi(q) = s$, and where $[\cdot\,,\cdot]$ denotes the Jacobi–Lie bracket of vector fields.

That $\mathcal{T}$ is a well-defined $(1,2)$ tensor field on S is shown in [22]. It is also shown in this reference that the gyroscopic tensor $\mathcal{T}$ coincides with other tensor fields that have been considered before by Koiller [30] and Cantrijn et al [8].

2.2. Local expressions for the gyroscopic tensor. Let $s = (s^1, \dots, s^r)$ be local coordinates on the shape space S. Then we may write

$$\mathcal{T}\Big(\frac{\partial}{\partial s^i}, \frac{\partial}{\partial s^j}\Big) = \sum_{k=1}^r C_{ij}^k \frac{\partial}{\partial s^k},$$

for certain s-dependent coefficients C_{ij}^k, which, in view of the skew-symmetry of $\mathcal{T}$, are skew-symmetric with respect to the lower indices, i.e. $C_{ij}^k = -C_{ji}^k$. Following the terminology introduced in García-Naranjo [21], we refer to C_{ij}^k as the *gyroscopic coefficients*.

Fix $i, j \in \{1, \dots r\}$. The gyroscopic coefficients C_{ij}^k, $k = 1, \dots, r$, may be computed in practice by solving the following linear system of equations:

$$\sum_{k=1}^r K_{kl} C_{ij}^k = \langle\!\langle [h_i, h_j], h_l \rangle\!\rangle, \quad l = 1, \dots, r, \tag{2.2}$$

where we recall that $\langle\!\langle \cdot\, , \cdot \rangle\!\rangle$ is the kinetic energy metric on Q, and we have denoted

$$h_k := \operatorname{hor}\left(\frac{\partial}{\partial s^k}\right), \qquad K_{kl} := \langle\!\langle h_k, h_l \rangle\!\rangle, \quad k, l = 1, \dots, r. \tag{2.3}$$

Note that the matrix K_{kl} is invertible by linear independence of $\{h_1, \dots, h_l\}$. That (2.2) holds is a direct consequence of the definition (2.1) of the gyroscopic tensor since $\left[\frac{\partial}{\partial s^i}, \frac{\partial}{\partial s^j}\right] = 0$.

2.3. The reduced equations of motion. We now present the reduced equations of motion of a Chaplygin system. Our exposition mainly follows García-Naranjo [21].

The distribution D interpreted as a principal connection on the principal bundle $\pi\colon Q \to S$, induces a Riemannian metric on S that will be denoted by $\langle\!\langle \cdot\, , \cdot \rangle\!\rangle^S$. For $v_1, v_2 \in T_sS$ it is defined by

$$\langle\!\langle v_1, v_2 \rangle\!\rangle^S_s := \langle\!\langle \operatorname{hor}_q(v_1), \operatorname{hor}_q(v_2) \rangle\!\rangle_q, \qquad q \in \pi^{-1}(s),$$

and is locally given by $\sum_{i,j=1}^r K_{ij} ds^i \otimes ds^j$ with K_{kl} defined by (2.3). Similarly, the invariance of the potential energy U induces a reduced potential $U_S \in C^\infty(S)$ such that $U = U_S \circ \pi$. Therefore, there is a well defined *reduced Lagrangian* $\mathcal{L}\colon TS \to \mathbb{R}$, of mechanical type, defined by

$$\mathcal{L}(s, \dot{s}) = \tfrac{1}{2} \langle\!\langle \dot{s}, \dot{s} \rangle\!\rangle^S_s - U_S(s). \tag{2.4}$$

Locally we have $\mathcal{L}(\dot{s}, s) = \frac{1}{2}\sum_{k,l=1}^r K_{kl}\dot{s}^k\dot{s}^l - U_S(s)$. As was announced above, the reduced equations of motion take the form of a mechanical system on S which is subject to *gyroscopic forces*. These may be written in terms of the gyroscopic coefficients C_{ij}^k as (see e.g. [21]):

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{s}^i}\right) - \frac{\partial \mathcal{L}}{\partial s^i} = -\sum_{j,k=1}^r C_{ij}^k \dot{s}^j \frac{\partial \mathcal{L}}{\partial \dot{s}^i}, \quad i = 1, \dots, r. \tag{2.5}$$

Now use the standard *Legendre transformation*:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{s}^i}, \quad i = 1, \dots, r, \tag{2.6}$$

and define the *reduced Hamiltonian* $H\colon T^*S \to \mathbb{R}$

$$H(s, p) := \sum_{j=1}^r p_j \frac{\partial \mathcal{L}}{\partial \dot{s}^j} - \mathcal{L} = \sum_{k,l=1}^r K^{kl} p_k p_l + U_S(s), \tag{2.7}$$

where K^{kl} are the entries of the inverse matrix of K_{kl}. It is a standard exercise to show that Eqs. (2.5) are equivalent to the following first order system on T^*S:

$$\dot{s}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial s^i} - \sum_{j,k=1}^r C_{ij}^k p_k \frac{\partial H}{\partial p_j}, \qquad i = 1, \dots, r. \tag{2.8}$$

Eqs. (2.8) give the local expression of the reduced vector field $\bar{X}_{nh}$ on T^*S. As mentioned above, this vector field satisfies $\mathbf{i}_{\bar{X}_{nh}}\Omega_{nh} = dH$, where the almost-symplectic 2-form

$$\Omega_{nh} = \sum_{j=1}^{r} ds^j \wedge dp_j + \sum_{i<j}\sum_{k=1}^{r} C_{ij}^k p_k \, ds^i \wedge ds^j.$$

We refer the reader to [22] for an intrinsic construction of Ω_{nh} using the gyroscopic tensor. A different, yet equivalent, approach to the construction of Ω_{nh} is taken in [10].

2.4. ϕ-simple Chaplygin systems, measure preservation and Hamiltonisation. In this subsection we recall the recent results on Chaplygin Hamiltonisation from García-Naranjo [21] and García-Naranjo and Marrero [22][5]. We begin with the following:

Definition 2.3. [22]A non-holonomic Chaplygin system is said to be *ϕ-simple* if there exists a function $\phi \in C^\infty(S)$ such that the gyroscopic tensor $\mathcal{T}$ satisfies

$$\mathcal{T}(Y,Z) = Z[\phi]Y - Y[\phi]Z, \tag{2.9}$$

for all $Y, Z \in \mathfrak{X}(S)$.

The class of ϕ-simple Chaplygin systems is quite special. As it turns out, their reduced equations on T^*S always possess an invariant measure and allow a Hamiltonisation by Chaplygin's reducing multiplier method. More precisely:

Theorem 2.1. [22]

1. *The reduced equations of motion (2.8) of a ϕ-simple Chaplygin system possess the invariant measure $\mu = \exp(\sigma)\nu$, where ν is the Liouville measure on T^*S and $\sigma = (r-1)\phi$.*
2. *The reduced equations of motion (2.8) of a ϕ-simple Chaplygin system become Hamiltonian after the time reparametrisation $dt = \exp(-\phi(s))d\tau$.*

The implication of item (i) of the theorem is clear. If the Chaplygin system under consideration is ϕ-simple, then Eqs. (2.8) preserve the volume form on T^*S whose local expression is

$$\mu = \exp((1-r)\phi \, ds^1 \wedge \cdots \wedge ds^r \wedge dp_1 \wedge \cdots \wedge dp_r.$$

One interpretation of item (ii)-followed in [22]-is that ϕ-simplicity implies that the 2-form Ω_{nh} is conformally symplectic with conformal factor $\exp(\phi(s))$. In other words, the 2-form $\bar{\Omega} := \exp(\phi(s))\Omega_{nh}$ is closed and hence symplectic. Hence, the rescaled vector field $Z := \exp(-\phi(s))\bar{X}_{nh}$ satisfies $\mathbf{i}_Z\bar{\Omega} = dH$ and is therefore Hamiltonian (with respect to the symplectic structure $\bar{\Omega}$).

Another, equivalent, interpretation of item (ii) of Theorem 2.1–followed in [21]–is obtained by defining the momentum rescaling:

$$\tilde{p}_i = \exp(\phi(s))p_i, \quad i = 1, \dots, r,$$

[5]As shown in [22], these results are equivalent to certain sufficient conditions for Hamiltonisation given first by Stanchenko [36] and Cantrijn et al. [8].

and writing the reduced Hamiltonian in the new variables

$$\tilde{H}(s,\tilde{p}) = H(s, \exp(-\phi(s))\tilde{p}).$$

Item (ii) of Theorem 2.1 states that, for a ϕ-simple Chaplygin system, Eqs. (2.8) are written in the $(s,\tilde{p})$ variables in *conformally Hamiltonian* form

$$\frac{ds^i}{dt} = \exp(\phi(s))\frac{\partial \tilde{H}}{\partial \tilde{p}_i}, \quad \frac{d\tilde{p}_i}{dt} = -\exp(\phi(s))\frac{\partial \tilde{H}}{\partial s^i}, \qquad i = 1,\dots,r. \tag{2.10}$$

The conformal factor $\exp(\phi(s))$ may be absorbed in the time reparametrisation $dt = \exp(-\phi(s))\,d\tau$ leading to the Hamiltonian system:

$$\frac{ds^i}{d\tau} = \frac{\partial \tilde{H}}{\partial \tilde{p}_i}, \quad \frac{d\tilde{p}_i}{d\tau} = -\frac{\partial \tilde{H}}{\partial s^i}, \qquad i = 1,\dots,r.$$

3. Noether's Theorem for ϕ-simple Chaplygin systems

We now show how additional–sometimes called internal–symmetries of ϕ-simple Chaplygin systems lead to first integrals. This is a consequence of the conformally Hamiltonian structure of their reduced equations. As we show below, the conserved quantities are simply a rescaling by the conformal factor of the standard momentum map for Hamiltonian systems.

We begin by recalling some standard notation. Suppose that the Lie group A, with Lie algebra $\mathfrak{a}$, acts on S. For $\xi \in \mathfrak{a}$ we denote by $\xi_S \in \mathfrak{X}(S)$ the *infinitesimal generator* of ξ. Namely, ξ_S is the vector field on S defined by

$$\xi_S(s) := \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)\cdot s \in T_sS.$$

THEOREM 3.1. *Consider a ϕ-simple Chaplygin system and suppose that the Lie group A acts on the shape space S and leaves ϕ invariant.*

(1) *If the reduced Lagrangian $\mathcal{L}\colon TS \to \mathbb{R}$ defined by (2.4) is invariant under the tangent lifted action of A to TS, then the* rescaled tangent bundle momentum map

$$\mathcal{J}\colon TS \to \mathfrak{a}^*, \quad \textit{defined by} \quad \mathcal{J}(s,\dot{s})(\xi) = \exp(\phi(s))\langle\!\langle \dot{s}, \xi_S(s)\rangle\!\rangle_s^S, \qquad \xi \in \mathfrak{a}, \tag{3.1}$$

is constant along the flow of the reduced equations (2.5).

(2) *If the reduced Hamiltonian $H : T^*S \to \mathbb{R}$ defined by (2.7) is invariant under the cotangent lifted action of A to T^*S, then the* rescaled cotangent bundle momentum map

$$\mathcal{J}\colon T^*S \to \mathfrak{a}^*, \quad \textit{defined by} \quad \mathcal{J}(s,p) = \exp(\phi(s))\langle p, \xi_S(s)\rangle, \qquad \xi \in \mathfrak{a},$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between T^*S and TS, is constant along the flow of the reduced equations (2.8).

(3) *Items (i) and (ii) are equivalent via the Legendre transformation (2.6).*

PROOF. Let $\xi \in \mathfrak{a}$ and suppose that in local coordinates $\xi_S = \sum_{i=1}^r \xi^j(s)\frac{\partial}{\partial s^j}$. The rescaled tangent bundle momentum map is locally given by:

$$\mathcal{J}(s^i,\dot{s}^i)(\xi) = \exp(\phi(s))\sum_{i=1}^r \xi^j(s)\frac{\partial \mathcal{L}}{\partial \dot{s}^j},$$

which, in view of the Legendre transformation (2.6), coincides with

$$\mathcal{J}(s^i, p_i)(\xi) = \exp(\phi(s)) \sum_{i=1}^{r} \xi^j(s) p_j, \tag{3.2}$$

which is the local expression for the rescaled cotangent bundle momentum map. So the definitions of $\mathcal{J}$ in items (ii) and (iii) are indeed matched by the Legendre transformation. The equivalence between the invariance assumptions on L and H-with respect to the appropriate lifted action of A-is quite standard (see e.g. [33]). We complete the proof by showing that, under the cotangent lift invariance assumption on H, $\mathcal{J}$ given by (3.2) is indeed a first integral of Eqs. (2.8). We begin by using the assumption of ϕ-simplicity to rewrite Eqs. (2.8) as

$$\frac{ds^i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{d}{dt}(\exp(\phi) p_i) = \exp(\phi)\Big(-\frac{\partial H}{\partial s^i} + \frac{\partial \phi}{\partial s^i} \sum_{j=1}^{r} \frac{\partial H}{\partial p_j} p_j \Big), \qquad i = 1, \dots, r.$$

Indeed, a calculation based on the chain rule shows that the above system is equivalent to Eqs. (2.10). Therefore, using the above equations, we compute

$$\frac{d}{dt}\left(\exp(\phi) \sum_{i=1}^{r} \xi^i p_i \right) = \exp(\phi) \Bigg[\sum_{i=1}^{r} \Big(-\frac{\partial H}{\partial s^i} \xi^i + p_i \sum_{j=1}^{r} \frac{\partial \xi^i}{\partial s^j} \frac{\partial H}{\partial p_j} \Big) + \Big(\sum_{j=1}^{r} \frac{\partial H}{\partial p_j} p_j \Big) \Big(\sum_{i=1}^{r} \frac{\partial \phi}{\partial s^i} \xi^i \Big) \Bigg]. \tag{3.3}$$

On the other hand, the A-invariance of ϕ implies

$$\xi_S[\phi] = \sum_{i=1}^{r} \xi^i \frac{\partial \phi}{\partial s^i} = 0. \tag{3.4}$$

Moreover, recall (see e.g. [33]) that the cotangent lift of $\xi_S = \sum_{i=1}^{r} \xi^j(s) \frac{\partial}{\partial s^j}$ is the vector field $\xi_S^{T^*S}$ on T^*S expressed in bundle coordinates as $\xi_S^{T^*S} = \sum_{i=1}^{r} \xi^i \frac{\partial}{\partial s^i} - \sum_{i,j=1}^{r} \frac{\partial \xi^i}{\partial s^j} p_i \frac{\partial}{\partial p_j}$. Therefore, the assumption that H is invariant under the cotangent lift of A to T^*S implies

$$\xi_S^{T^*S}[H] = \sum_{i=1}^{r} \xi^i \frac{\partial H}{\partial s^i} - \sum_{i,j=1}^{r} \frac{\partial \xi^i}{\partial s^j} p_i \frac{\partial H}{\partial p_j} = 0. \tag{3.5}$$

Substitution of (3.4) and (3.5) into (3.3) proves the result. □

4. The rubber Routh sphere

Routh considered the motion of a sphere whose distribution of mass is axially symmetric, and which rolls without slipping on the plane. Here we enforce an additional *rubber* constraint that forbids spinning and consider the multi-dimensional generalisation of the system. The 3D version of the problem has already been considered by Borisov and coauthors in the works [5, 7] which treat more general problems of 3D bodies that roll without slipping or spinning over a surface.

4.1. The 3D case. Consider a sphere that rolls without slipping or spinning on the plane. The orientation of the sphere is determined by an orthogonal matrix $R \in \mathrm{SO}(3)$ that relates a *body* fixed frame $\{E_1, E_2, E_3\}$ to an inertial or *space* frame $\{e_1, e_2, e_3\}$. We assume that the space frame is chosen in such a way that the plane where the rolling takes place is spanned by e_1 and e_2, and will denote the space coordinates of the geometric centre O by $\boldsymbol{x} = (x_1, x_2, b)^t$, where the constant $b > 0$ is the sphere's radius. The position and orientation of the sphere is hence completely determined by the pair $(R, (x_1, x_2)) \in \mathrm{SO}(3) \times \mathbb{R}^2$ so the configuration space of the system is $Q = \mathrm{SO}(3) \times \mathbb{R}^2$. To simplify the exposition, we will denote an element $q \in Q$ as a pair $q = (R, x) \in \mathrm{SO}(3) \times \mathbb{R}^3$ with $x_3 = b$. This amounts to the identification of Q with the embedded submanifold of $\mathrm{SO}(3) \times \mathbb{R}^3$ defined by the holonomic constraint $x_3 = b$.

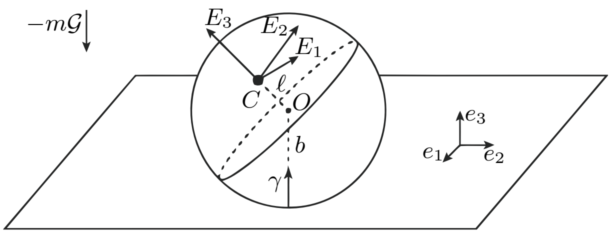

FIGURE 1. Rubber Routh sphere rolling on the plane.

Denote by $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^t \in \mathbb{R}^3$ the angular velocity vector of the sphere written in the *space* frame, and by $\boldsymbol{\Omega} = R^{-1}\boldsymbol{\omega} = (\Omega_1, \Omega_2, \Omega_3)^t \in \mathbb{R}^3$ the same vector written in the *body* frame. As is well known, these vectors correspond to the right and left trivialisations of the tangent velocity vector $\dot{R} \in T_R\mathrm{SO}(3)$ as follows

$$\dot{R}R^{-1} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \in \mathfrak{so}(3), \quad R^{-1}\dot{R} = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \in \mathfrak{so}(3),$$

where $\mathfrak{so}(3)$, the space of skew-symmetric real 3×3 matrices, is the Lie algebra of $\mathrm{SO}(3)$.

The no-slip rolling constraint is written as

$$\dot{x} = b\boldsymbol{\omega} \times e_3, \quad \text{or, equivalently,} \quad \dot{x} = bR(\Omega \times \gamma), \tag{4.1}$$

where $\dot{x} = (\dot{x}_1, \dot{x}_2, 0)^t$, $\times$ denotes the vector product in $\mathbb{R}^3$ and $\gamma := R^{-1}e_3$ denotes the so-called *Poisson vector* that gives body coordinates of the vector e_3 that is normal to the plane where the rolling takes place. The sphere is also subject to the no-spin or *rubber* constraint

$$\omega_3 = 0, \quad \text{or, equivalently,} \quad (\Omega, \gamma) = 0, \tag{4.2}$$

where $(\cdot\,, \cdot)$ denotes the scalar euclidean product in $\mathbb{R}^3$. The first two components of Eq. (4.1) together with Eq. (4.2) define 3 independent nonholonomic constraints that determine a rank 2 distribution D on Q.

Inspired by Routh [35], we assume that the mass distribution of the sphere is axially symmetric. The body frame $\{E_1, E_2, E_3\}$ is chosen with origin at the centre of mass C and with E_3 aligned with the axis of symmetry. This choice of body frame implies that the inertia tensor of the body is represented by a 3×3 matrix of the form $\mathbb{I} = \operatorname{diag}(I_1, I_1, I_3)$, with principal moments of inertia $I_1, I_3 > 0$. We denote by ℓ the distance between C and the geometric centre O and assume that the coordinates of O in the body frame are $(0, 0, -\ell)^t$, see Figure 1. The space coordinates of C are hence given by the vector $\boldsymbol{u} = x + \ell R E_3$. Considering that $\|\dot{\boldsymbol{u}}\|^2 = \|R^{-1}\dot{\boldsymbol{u}}\|^2$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^3$, the Lagrangian of the system $L\colon TQ \to \mathbb{R}$, given by the kinetic minus the potential energy, may be written as

$$L(R, \Omega, x, \dot{x}) = \frac{1}{2}(\mathbb{I}\boldsymbol{\Omega}, \boldsymbol{\Omega}) + \frac{m}{2}\|R^{-1}\dot{x} + \ell\Omega \times E_3\|^2 - m\mathcal{G}\ell\gamma_3, \tag{4.3}$$

where m is the mass of the sphere, $\mathcal{G}$ is the gravitational constant, and γ_3 denotes the third component of γ, i.e. $\gamma_3 = (\gamma, E_3)$.

In Eq. (4.3), and in what follows, we write a generic element of TQ as the quadruple $(R, \Omega, x, \dot{x}) \in \mathrm{SO}(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$, which is possible by the identification of $T\mathrm{SO}(3)$ with $\mathrm{SO}(3) \times \mathbb{R}^3$ via the left trivialisation, and the embedding $T\mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \times \mathbb{R}^3$, $((x_1, x_2), (\dot{x}_1, \dot{x}_2)) \mapsto ((x_1, x_2, b), (\dot{x}_1, \dot{x}_2, 0))$, induced from the holonomic constraint $x_3 = b$.

The evolution of the system is clearly independent of horizontal translations and rotations of the space frame about e_3. This corresponds to a symmetry action of the euclidean group $G = \mathrm{SE}(2)$ on the configuration space $Q = \mathrm{SO}(3) \times \mathbb{R}^2$ as we now show. We represent the group $G = \mathrm{SE}(2)$ as the Lie subgroup of $\mathrm{GL}(4, \mathbb{R})$ consisting of matrices of the form

$$g = \left(\begin{array}{ccc|c} & h & & y \\ \hline 0 & 0 & 0 & 1 \end{array}\right), \quad \text{where} \quad y = \begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix} \in \mathbb{R}^3,$$

$$\text{and} \quad h = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\tilde{h}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0 & 0 & 1 \end{array}\right) \in \mathrm{SO}(3), \quad \text{with} \quad \tilde{h} \in \mathrm{SO}(2).$$

The action of $g \in \mathrm{SE}(2)$ given above on an element $(R, x) \in \mathrm{SO}(3) \times \mathbb{R}^3$ is

$$g \cdot (R, x) = (hR, hx + y). \tag{4.4}$$

This action restricts to Q since it preserves the holonomic constraint $x_3 = b$.

Proposition 4.1. *The problem of the rubber Routh sphere that rolls without slipping or spinning on the plane is a Chaplgyin system with $G = \mathrm{SE}(2)$ acting on Q via Eq. (4.4).*

Remark 4.1. Proposition 4.1 is valid even if the sphere fails to be axially symmetric. In fact, it continues to hold for the problem of an arbitrary rubber smooth convex body that rolls without slipping or spinning on the plane.

PROOF. The tangent lifted action is

$$g \cdot (R, \Omega, x, \dot{x}) = (hR, \Omega, hx + y, h\dot{x}).$$

Moreover, given that $h^{-1}e_3 = e_3$, it follows that the Poisson vector γ is invariant. Using this, and the above expression for the tangent lift, it is immediate to see that both the rolling (4.1) and rubber (4.2) constraints are invariant. Similarly, one checks that the kinetic and potential energies of the Lagrangian (4.3) are invariant so the conditions (i) and (ii) in Definition 2.1 hold. In order to check that the condition (iii) in Definition 2.1 also holds, note that the Lie algebra $\mathfrak{se}(2)$ in our representation is spanned by the 4×4 matrices

$$\xi_1 = \left(\begin{array}{c|c} 0 & \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \\ \hline 0 & 0 \end{array} \right), \quad \xi_2 = \left(\begin{array}{c|c} 0 & \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \\ \hline 0 & 0 \end{array} \right), \quad \xi_3 = \left(\begin{array}{ccc|c} 0 & -1 & 0 & \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \\ \hline & 0 & & 0 \end{array} \right).$$

The infinitesimal generators of ξ_1 and ξ_2 are vector fields on Q that correspond to pure translations along the x_1 and x_2 axes respectively, which violate the rolling constraint (4.1). On the other hand, the infinitesimal generator of ξ_3 is a vector field on Q having constant $\omega_3 = 1$, which violates the rubber constraint (4.1). Hence, the group orbit is transversal to the constraint distribution and, by a dimension count, the condition (iii) in Definition 2.1 is also verified. □

The shape space $S = (\mathrm{SO}(3) \times \mathbb{R}^2)/\mathrm{SE}(2)$ is diffeomorphic to the two dimensional sphere S^2 and the orbit projection is

$$\pi \colon \mathrm{SO}(3) \times \mathbb{R}^2 \to \mathrm{S}^2, \quad (R, x) \mapsto \gamma, \tag{4.5}$$

where we recall that $\gamma = R^{-1}e_3 \in \mathbb{R}^3$ is the Poisson vector. Note that we realise

$$\mathrm{S}^2 = \left\{ \gamma = (\gamma_1, \gamma_2, \gamma_3)^t \in \mathbb{R}^3 \, : \, \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \right\}.$$

It follows from our discussion in section 2 that the reduced equations of motion are defined on the cotangent bundle $T^*\mathrm{S}^2$.

THEOREM 4.1. *The problem of the rubber Routh sphere that rolls without slipping or spinning on the plane is ϕ-simple with $\phi \colon \mathrm{S}^2 \to \mathbb{R}$ given by*

$$\phi(\gamma) = -\tfrac{1}{2} \ln(I_1\gamma_3^2 + I_3(1 - \gamma_3^2) + m(b + \ell\gamma_3)^2).$$

It follows from item (i) in Theorem 2.1 that the reduced equations on $T^*\mathrm{S}^2$ possess the invariant measure:

$$\mu = \frac{1}{\sqrt{I_1\gamma_3^2 + I_3(1 - \gamma_3^2) + m(b + \ell\gamma_3)^2}} \nu,$$

where ν is the Liouville volume form on $T^*\mathrm{S}^2$. Additionally, item (ii) in Theorem 2.1 implies that the reduced system on $T^*\mathrm{S}^2$ is conformally Hamiltonian with time reparametrisation:

$$dt = \sqrt{I_1\gamma_3^2 + I_3(1 - \gamma_3^2) + m(b + \ell\gamma_3)^2} d\tau.$$

Remark 4.2. The invariant measure for the problem was first given by Borisov and Mamaev in [5] (see also [7]). The Hamiltonisation of the system may be deduced as a consequence of the celebrated Chaplygin's Reducing Multiplier Theorem [9] since the shape space S^2 has dimension 2. For the multi-dimensional version of the problem considered below, these properties can no longer be deduced from known results and we will rely on Theorem 2.1.

We do not present a proof of Theorem 4.1 since it is a particular instance of Theorem 4.2 below.

4.2. The nD case. We consider a multi-dimensional generalisation of the problem considered in the previous section. Namely, an n-dimensional rigid body of spherical shape, with axially symmetric distribution of mass, that rolls without slipping or spinning on a horizontal (with respect to gravity) hyperplane on $\mathbb{R}^n$.

The orientation of the sphere is determined by a rotation matrix $R \in \mathrm{SO}(n)$ that specifies the attitude of the sphere by relating a body fixed frame $\{E_1, \dots, E_n\}$ and a space frame $\{e_1, \dots, e_n\}$. In analogy with the 3D case, we assume that the rolling takes place on the hyperplane spanned by $\{e_1, \dots, e_{n-1}\}$ and that the geometric centre O of the sphere has space coordinates $x = (x_1, \dots, x_{n-1}, b)$, where the constant $b > 0$ is the sphere's radius. We will also assume, as in the 3D case, that the body frame has its origin at the centre of mass C and E_n is aligned with the symmetry axis of the sphere. The orientation of E_n is such that the body coordinates of O are $(0, \dots, 0, -\ell)$. The configuration space of the problem is $Q = \mathrm{SO}(n) \times \mathbb{R}^{n-1}$. In analogy to the 3D case, we will work with the embedding of Q in $\mathrm{SO}(n) \times \mathbb{R}^n$ defined by the holonomic constraint $x_n = b$.

As is well known, for $n > 3$ the angular velocity can no longer be represented as a vector, but rather as an element in the Lie algebra $\mathfrak{so}(n)$ of $\mathrm{SO}(n)$. We denote by $\omega \in \mathfrak{so}(n)$ the representation of the angular velocity in the *space* frame and by $\Omega \in \mathfrak{so}(n)$ its representation in the *body* frame. These are related to the right and left trivialisation of the tangent vector $\dot{R} \in T_R\mathrm{SO}(n)$ by

$$\omega = \dot{R}R^{-1} \in \mathfrak{so}(n),$$
$$\Omega = R^{-1}\dot{R} \in \mathfrak{so}(n),$$

and satisfy $\omega = \mathrm{Ad}_R\Omega$, where $\mathrm{Ad}_R : \mathfrak{so}(n) \to \mathfrak{so}(n)$ is the adjoint operator.

The constraint of rolling without slipping is that the contact point of the sphere with the hyperplane $x_n = 0$ has zero velocity at every time, and is expressed as the following natural generalisation of (4.1):

$$\dot{x} = b\boldsymbol{\omega} e_n, \qquad \text{or, equivalently,} \quad \dot{x} = bR\Omega\gamma, \tag{4.6}$$

where $\dot{x} = (\dot{x}_1, \dots, \dot{x}_{n-1}, 0)^t$, and the Poisson vector $\gamma = (\gamma_1, \dots, \gamma_n)^t \in \mathbb{R}^n$ is now given by $\gamma := R^{-1}e_n$. On the other hand, the generalisation of the no-spin rubber constraint (4.2) is that the space representation of the angular velocity satisfies

$$\omega_{ij} = 0, \quad \text{for all} \quad i, j = 1, \dots, n-1. \tag{4.7}$$

In other words, ω has the form

$$\omega = \left(\begin{array}{c|c} \mathbf{0} & \begin{matrix} \omega_{1n} \\ \vdots \\ \omega_{n-1} \end{matrix} \\ \hline -\omega_{1n} \quad \dots \quad -\omega_{n-1} & 0 \end{array} \right),$$

where $\mathbf{0}$ above denotes the $(n-1) \times (n-1)$ zero matrix. The constraints (4.7) were considered by Jovanović [27] in the treatment of the multi-dimensional rubber Chaplygin sphere. They generalise the 3D rubber constraint (4.2) in the following sense: rotations of the sphere that occur on 2-dimensional planes that do not contain the normal vector e_n to the hyperplane where the rolling takes place are forbidden.

Our next step is to give a multi-dimensional generalisation of the Lagrangian (4.3). For this matter we recall that for an n-dimensional rigid body the inertia tensor $\mathbb{I}$ of the body is an operator

$$\mathbb{I}\colon \mathfrak{so}(n) \to \mathfrak{so}(n), \qquad \mathbb{I}(\Omega) = \mathbb{J}\Omega + \Omega\mathbb{J}, \tag{4.8}$$

where $\mathbb{J}$ is the so-called mass tensor of the body, which is a symmetric and positive definite $n \times n$ matrix (see e.g. [34]). Our assumption that the mass distribution is axially symmetric, and that the E_n axis of the body frame is aligned with the symmetry axis, implies that, with respect to our choice of body frame, the mass tensor has the form

$$\mathbb{J} = \operatorname{diag}(J_1, \dots, J_1, J_n), \quad J_1, J_n > 0. \tag{4.9}$$

Similarly to our treatment of the 3D case, we shall represent elements of $TQ = T(\mathrm{SO}(n) \times \mathbb{R}^{n-1})$ as quadruples $(R, \Omega, x, \dot{x}) \in \mathrm{SO}(n) \times \mathfrak{so}(n) \times \mathbb{R}^n \times \mathbb{R}^n$ with $x_n = b$ and $\dot{x}_n = 0$. This is done by identifying $T\mathrm{SO}(n) = \mathrm{SO}(n) \times \mathfrak{so}(n)$ via the left trivialisation, and by embedding $T\mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n \times \mathbb{R}^n$, putting $x_n = b$ and $\dot{x}_n = 0$. The Lagrangian of the multi-dimensional system $L\colon TQ \to \mathbb{R}$ is

$$L(R, \Omega, x, \dot{x}) = \frac{1}{2}(\mathbb{I}\Omega, \Omega)_\kappa + \frac{m}{2}\|R^{-1}\dot{x} + \ell\Omega E_n\|^2 - m\mathcal{G}\ell\gamma_n, \tag{4.10}$$

where $\|\cdot\|$ is the euclidean norm in $\mathbb{R}^n$, $\dot{x} = (\dot{x}_1, \dots, \dot{x}_{n-1}, 0)^t$ and $(\cdot,\cdot)_\kappa$ is the Killing metric in $\mathfrak{so}(n)$:

$$(\xi, \eta)_\kappa = -\frac{1}{2}\operatorname{tr}(\xi\eta).$$

In (4.10) we continue to denote by ℓ the distance of the centre of mass C to the geometric centre O.

In analogy to the 3D case, there is a symmetry action of the group $G = \mathrm{SE}(n-1)$ which we represent as the Lie subgroup of $\mathrm{GL}(n+1, \mathbb{R})$ consisting of matrices of the form

$$g = \left(\begin{array}{ccc|c} & h & & y \\ \hline 0 & 0 & 0 & 1 \end{array} \right), \qquad \text{where} \qquad y = \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \\ 0 \end{pmatrix} \in \mathbb{R}^n,$$

and $\quad h = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$\tilde{h}$}} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \in \mathrm{SO}(n), \qquad$ with $\quad \tilde{h} \in \mathrm{SO}(2).$

The action of $g \in \mathrm{SE}(n-1)$ given above on an element $(R, x) \in \mathrm{SO}(n) \times \mathbb{R}^n$ looks identical to Eq. (4.4), namely

$$g \cdot (R, x) = (hR, hx + y). \tag{4.11}$$

As in the 3D case, the action restricts to Q since the holonomic constraint $x_n = b$ is invariant. In analogy with Proposition 4.1 we have:

PROPOSITION 4.2. *The n-dimensional generalisation of the problem of the rubber Routh sphere that rolls without slipping or spinning on a hyperplane is a Chaplgyin system with $G = \mathrm{SE}(n-1)$ acting on Q via Eq. (4.11).*

The proof is analogous to that of Proposition 4.1 and we omit the details. Also, in analogy with Remark 4.1, we mention that the conclusion of Proposition 4.2 is independent of our symmetry assumptions on the mass distribution of the sphere and also applies to general rubber multi-dimensional convex rigid bodies that roll without slipping or spinning on a horizontal hyperplane in $\mathbb{R}^n$.

The shape space of the system $S = (\mathrm{SO}(n) \times \mathbb{R}^{n-1})/\mathrm{SE}(n-1)$ is diffeomorphic to the $n-1$ dimensional sphere S^{n-1}, and the orbit projection (4.5), valid in 3D, generalises automatically to

$$\pi : \mathrm{SO}(n) \times \mathbb{R}^{n-1} \to \mathrm{S}^{n-1}, \quad (R, x) \mapsto \gamma,$$

where we recall that in the Poisson vector $\gamma = R^{-1}e_n \in \mathbb{R}^n$, and we realise S^{n-1} by its embedding in $\mathbb{R}^n$:

$$\mathrm{S}^{n-1} = \left\{ \gamma = (\gamma_1, \dots, \gamma_n)^t \in \mathbb{R}^n \, : \, \gamma_1^2 + \dots + \gamma_n^2 = 1 \right\}. \tag{4.12}$$

As a consequence of our discussion in section 2, the $G = \mathrm{SE}(n-1)$-reduced equations of motion live on the cotangent bundle $T^*\mathrm{S}^{n-1}$. We now state our main result which, in view of Theorem 2.1, implies that the multi-dimensional rubber Routh sphere has an invariant measure and allows a Hamiltonisation.

THEOREM 4.2. *The n-dimensional rubber Routh sphere that rolls without slipping or spinning on a horizontal hyperplane is ϕ-simple with $\phi \colon \mathrm{S}^{n-1} \to \mathbb{R}$ given by*

$$\phi(\gamma) = -\tfrac{1}{2} \ln(2J_1 + (J_n - J_1)\gamma_n^2 + m(b + \ell\gamma_n)^2). \tag{4.13}$$

REMARK 4.3. The conclusion of the above theorem is consistent with Theorem 4.1 by noting that, in the 3D case, the principal moments of inertia I_1, I_3 are related to the entries J_1, J_3 of the mass tensor $\mathbb{J}$ by the relations $I_1 = J_1 + J_3$ and $I_3 = 2J_1$.

As a direct consequence of Theorem 2.1 and Theorem 4.2 we obtain:

COROLLARY 4.1. *The $\mathrm{SE}(n-1)$-reduced equations on $T^*\mathrm{S}^{n-1}$ of the n-dimensional rubber Routh sphere that rolls without slipping or spinning on a horizontal hyperplane possess the invariant measure:*

$$\mu = (2J_1 + (J_n - J_1)\gamma_n^2 + m(b + \ell\gamma_n)^2)^{\frac{2-n}{2}} \nu,$$

where ν is the Liouville volume form on $T^*\mathrm{S}^{n-1}$. Moreover, such reduced system is conformally Hamiltonian with time reparametrisation:

$$dt = \sqrt{2J_1 + (J_n - J_1)\gamma_n^2 + m(b+\ell\gamma_n)^2}d\tau.$$

Proof. The conclusion about the invariant measure follows from Eq. (4.13) and item (i) of Theorem 2.1 (putting $r = n-1$). The conformally Hamiltonian structure of the equations of motion follows from item (ii) of Theorem 2.1. □

The rest of the paper is devoted to proving Theorem 4.2 via a coordinate calculation. The strategy is to follow the steps outlined in section 2.2 to compute the gyroscopic coefficients C_{ij}^k. We will work with the local coordinates[6] $(s_1, \dots, s_{n-1})$ valid on the northern S_+^{n-1}, or southern S_-^{n-1}, hemispheres of S^{n-1} by the relations:

$$\gamma_1 = s_1, \quad \dots, \quad \gamma_{n-1} = s_{n-1}, \quad \gamma_n = \pm\sqrt{1 - s_1^2 - \dots - s_{n-1}^2}.$$

Associated to the embedding of S^{n-1} in $\mathbb{R}^n$, there is an embedding of $T\mathrm{S}^{n-1}$ in $\mathbb{R}^n \times \mathbb{R}^n$ given by

$$T\mathrm{S}^{n-1} = \big\{(\gamma, v) \in \mathbb{R}^n \times \mathbb{R}^n \, : \, \|\gamma\| = 1, \quad (\gamma, v)_{\mathbb{R}^n} = 0\big\}, \tag{4.14}$$

where $(\cdot\,,\cdot)_{\mathbb{R}^n}$ is the Euclidean scalar product in $\mathbb{R}^n$. Under the above identification, and regardless of the hemisphere under consideration, the coordinate vector fields $\frac{\partial}{\partial s_i}$ are given in terms of the canonical vectors $E_1, \dots, E_n$, by

$$\frac{\partial}{\partial s_i} = E_i - \frac{\gamma_i}{\gamma_n} E_n, \quad i = 1, \dots, n-1. \tag{4.15}$$

For the rest of the section, given $(R, x) \in \mathrm{SO}(n) \times \mathbb{R}^n$, we identify

$$T_{(R,x)}(\mathrm{SO}(n) \times \mathbb{R}^n) = \mathfrak{so}(n) \times \mathbb{R}^n,$$

using the left trivialisation of $T_R\mathrm{SO}(n)$ and the usual identification $T_x\mathbb{R}^n = \mathbb{R}^n$. Therefore, a vector field on Q is represented as an assignment that to a pair $(R, x) \in \mathrm{SO}(n) \times \mathbb{R}^n$ with $x_n = b$, associates a pair $(\xi(R,x), V(R,x)) \in \mathfrak{so}(n) \times \mathbb{R}^n$ that satisfies $V_n(R,x) = 0$. We will also find it useful to denote

$$y \wedge z := yz^t - zy^t \in \mathfrak{so}(n), \quad \text{for column vectors} \quad y, z \in \mathbb{R}^n.$$

In accordance with the notation of section 2.2 we denote by h_i the horizontal lift of the coordinate vector field $\frac{\partial}{\partial s_i}$. Namely,

$$h_i(R,x) := \mathrm{hor}_{(R,x)}\left(\frac{\partial}{\partial s_i}\right).$$

The following proposition gives an explicit expression for $h_i(R,x)$.

Proposition 4.3. *Let $\gamma \in \mathrm{S}_\pm^{n-1}$ and $(R,x) \in \pi^{-1}(\gamma)$, (i.e. $\gamma = R^{-1}e_n$). For $i = 1, \dots, n-1$, the horizontal lift*

$$h_i(R,x) = \left(\gamma \wedge \Big(E_i - \frac{\gamma_i}{\gamma_n}E_n\Big), bR\Big(-E_i + \frac{\gamma_i}{\gamma_n}E_n\Big)\right) \in \mathfrak{so}(n) \times \mathbb{R}^n. \tag{4.16}$$

[6] Throughout this section, and in contrast with our notation in section 2, we use sub-indices instead of super-indices on the coordinates on S.

PROOF. The rubber constraints (4.7) imply $\omega = e_n \wedge y$ for a vector $y \in \mathbb{R}^n$ that may be assumed to be perpendicular to e_n. Hence,

$$\Omega = \mathrm{Ad}_{R^{-1}}(e_n \wedge y) = (R^{-1}e_n) \wedge (R^{-1}y) = \gamma \wedge v,$$

where $v = R^{-1}y$ is perpendicular to γ. On the other hand, differentiating $\gamma = R^{-1}e_n$ gives $\dot{\gamma} = -\Omega\gamma$. Whence, $\dot{\gamma} = -(\gamma \wedge v)\gamma = v$ and we conclude that

$$\Omega = \gamma \wedge \dot{\gamma}. \tag{4.17}$$

The rolling constraint (4.6) then implies $\dot{x} = bR(\gamma \wedge \dot{\gamma})\gamma = -bR\dot{\gamma}$. Therefore, we get the following expression for the horizontal lift

$$\mathrm{hor}_{(R,x)}(\dot{\gamma}) = (\gamma \wedge \dot{\gamma}, -bR\dot{\gamma}) \in \mathfrak{so}(n) \times \mathbb{R}^n, \tag{4.18}$$

where $\dot{\gamma} \in T_\gamma \mathrm{S}^{n-1}$ and $(R, x) \in \pi^{-1}(\gamma)$. The result then follows by using (4.15). □

The following lemma gives expressions that involve the horizontal lifts h_i and the kinetic energy metric $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$ that will be used below to compute the gyroscopic coefficients C_{ij}^k. Its proof is postponed to Appendix A.

LEMMA 4.1. *For $i, j, k, l \in \{1, \dots, n-1\}$ we have*

$$\begin{aligned} K_{kl} = \langle\!\langle h_k, h_l \rangle\!\rangle &= (2J_1 + (J_n - J_1)\gamma_n^2 + m(b + \ell\gamma_n)^2)\delta_{kl} \\ &\quad + \Big(\frac{J_1 + J_n}{\gamma_n^2} + J_n - J_1 + m\Big(\Big(\frac{b}{\gamma_n} + \ell\Big)^2 + \frac{\ell^2}{\gamma_n^2}\Big)\Big)\gamma_k\gamma_l, \end{aligned} \tag{4.19}$$

$$\langle\!\langle [h_i, h_j], h_l \rangle\!\rangle = \Big(J_n - J_1 + m\ell\Big(\ell + \frac{b}{\gamma_n}\Big)\Big)(\gamma_j\delta_{il} - \gamma_i\delta_{jl}), \tag{4.20}$$

where δ_{ij} is the Kronecker delta.

We are now ready to prove the following lemma that gives explicit expressions for the gyroscopic coefficients C_{ij}^k in our coordinates.

LEMMA 4.2. *For $i, j, k \in \{1, \dots, n-1\}$ we have*

$$C_{ij}^k = \frac{\big(J_n - J_1 + m\ell\big(\ell + \frac{b}{\gamma_n}\big)\big)(\gamma_j\delta_{ik} - \gamma_i\delta_{jk})}{2J_1 + (J_n - J_1)\gamma_n^2 + m(b + \ell\gamma_n)^2}. \tag{4.21}$$

PROOF. Using

$$\sum_{k=1}^{n-1}(\gamma_j\delta_{ik} - \gamma_i\delta_{jk})\delta_{kl} = \gamma_j\delta_{il} - \gamma_i\delta_{jl} \quad \text{and} \quad \sum_{k=1}^{n-1}(\gamma_j\delta_{ik} - \gamma_i\delta_{jk})\gamma_k\gamma_l = 0,$$

it follows, in view of (4.19) and (4.20), that C_{ij}^k as given by (4.21) satisfy

$$\sum_{k=1}^{n-1} K_{kl}C_{ij}^k = \langle\!\langle [h_i, h_j], h_l \rangle\!\rangle, \quad i, j, k, l \in \{1, \dots, n-1\}.$$

In other words, the expressions (4.21) for C_{ij}^k are the unique solution to the system (2.2) that determines the gyroscopic coefficients. □

We are now ready to present:

Proof of Theorem 4.2. Lemma 4.2 implies

$$\mathcal{T}\Big(\frac{\partial}{\partial s_i},\frac{\partial}{\partial s_j}\Big)=\frac{\big(J_n-J_1+m\ell\big(\ell+\frac{b}{\gamma_n}\big)\big)}{2J_1+(J_n-J_1)\gamma_n^2+m(b+\ell\gamma_n)^2}\Big(s_j\frac{\partial}{\partial s_i}-s_i\frac{\partial}{\partial s_j}\Big), \tag{4.22}$$

$1\leqslant i,j\leqslant n-1$.

On the other hand, considering that $\gamma_n=\pm\sqrt{1-s_1^2-\cdots-s_{n-1}^2}$, we have $\partial\gamma_n/\partial s_k=-s_k/\gamma_n$ for $1\leqslant k\leqslant n-1$, and hence, from the expression (4.13) for $\phi\colon \mathrm{S}^{n-1}\to\mathbb{R}$ we compute

$$\frac{\partial\phi}{\partial s_k}=\left(\frac{J_n-J_1+m\ell(\ell+\frac{b}{\gamma_n})}{2J_1+(J_n-J_1)\gamma_n^2+m(b+\ell\gamma_n)^2}\right)s_k,\quad 1\leqslant k\leqslant n-1.$$

Therefore, Eq. (4.22) may be rewritten as

$$\mathcal{T}\Big(\frac{\partial}{\partial s_i},\frac{\partial}{\partial s_j}\Big)=\frac{\partial\phi}{\partial s_j}\frac{\partial}{\partial s_i}-\frac{\partial\phi}{\partial s_i}\frac{\partial}{\partial s_j},\quad 1\leqslant i,j\leqslant n-1.$$

The above expression, together with the tensorial properties of $\mathcal{T}$, shows that the ϕ-simplicity condition (2.9) holds on the open dense subset of S^{n-1} where $\gamma_n\neq 0$. By continuity, it holds on all of S^{n-1}. □

Remark 4.4. We note that the notion of ϕ-simplicity, and hence also our conclusions about measure preservation and Hamiltonisation, only depend on the kinetic energy and the constraints and do not involve the gravitational potential. This is a consequence of the *weak Noetherianity* of these concepts (see [22]).

4.3. First integrals. In this section we use Theorem 3.1 to prove that

$$F_{ij}:=\sqrt{2J_1+(J_n-J_1)\gamma_n^2+m(b+\ell\gamma_n)^2}\,\Omega_{ij},\quad 1\leqslant i,j\leqslant n-1, \tag{4.23}$$

are first integrals of the system. In 3D, there is only one such integral whose existence has been proven by Borisov and Mamaev [5] and, considering that $T^*\mathrm{S}^2$ has 2 degrees of freedom, it is sufficient to conclude integrability of the problem. The question of integrability in nD will be addressed in a forthcoming publication.

We begin by noting that, in view of expression (4.18) for the horizontal lift of $\dot\gamma\in T_\gamma\mathrm{S}^{n-1}$ and the expression for the Lagrangian (4.10), the reduced Lagrangian $\mathcal{L}\colon T\mathrm{S}^{n-1}\to\mathbb{R}$ is given by

$$\mathcal{L}(\gamma,\dot\gamma)=\tfrac12(\mathbb{I}(\dot\gamma\wedge\gamma),\dot\gamma\wedge\gamma)_\kappa+\frac{m}{2}\|(b+\ell\gamma_n)\dot\gamma-\ell\dot\gamma_n\gamma\|^2-m\ell\mathcal{G}\gamma_n,$$

which, using the specific form of the inertia tensor $\mathbb{I}$ given by Eqs. (4.8) and (4.9), simplifies to

$$\begin{aligned}\mathcal{L}(\gamma,\dot\gamma)=\frac12(2J_1+(J_n-J_1)\gamma_n^2+m(b+\ell\gamma_n)^2)\|\dot\gamma\|^2\\+\frac{m}{2}(J_n-J_1+m\ell^2)\dot\gamma_n^2-m\ell\mathcal{G}\gamma_n,\end{aligned} \tag{4.24}$$

where we have repeatedly used the condition $(\gamma,\dot\gamma)_{\mathbb{R}^n}=0$, which holds in view of our realisation (4.14) of the tangent bundle $T\mathrm{S}^{n-1}$.

Apart from the $G=\mathrm{SE}(n-1)$ action that allows us to reduce the dynamics to $T^*\mathrm{S}^{n-1}$, the system possesses additional symmetries due to our assumptions on the mass distribution of the sphere. These correspond to rotations of the body frame that preserve the symmetry axis E_n. The symmetry group is hence $A = \mathrm{SO}(n-1)$ and the action of $\tilde{a} \in \mathrm{SO}(n-1)$ on $(R, x) \in (\mathrm{SO}(n), x)$ is given by

$$\tilde{a} \cdot (R, x) = (Ra^{-1}, x), \quad \text{where} \quad a := \left(\begin{array}{c|c} \tilde{a} & 0 \\ \hline 0 & 1 \end{array}\right) \in \mathrm{SO}(n). \tag{4.25}$$

The tangent lift of this action on $(R, \Omega, x, \dot{x}) \in \mathrm{SO}(n) \times \mathfrak{so}(n) \times \mathbb{R}^n \times \mathbb{R}^n$ is

$$\tilde{a} \cdot (R, \Omega, x, \dot{x}) = (Ra^{-1}, \mathrm{Ad}_a \Omega, x, \dot{x}),$$

Using that $a^{-1}\mathbb{J}a = \mathbb{J}$ in view of Eq. (4.9), one may check that the Lagrangian (4.10) is invariant. The same is true about the constraints (4.6) and (4.7) so the dynamics is A-equivariant.

A crucial observation is that the A-action defined by Eq. (4.25) commutes with the G-action defined by Eq. (4.4), so there is a well defined A-action on the shape space S^{n-1}. As may be easily shown from Eq. (4.25) and the definition of $\gamma = R^{-1}e_n$, such an action is by rotations of the sphere S^{n-1} that fix the vertical axis. Namely, with the same notation for a and $\tilde{a}$ as above: $\tilde{a} \cdot \gamma = a\gamma$, $\gamma \in \mathrm{S}^{n-1}$, where we recall that S^{n-1} is realised by its embedding in $\mathbb{R}^n$ (4.12). In particular, this action fixes the north and south pole of S^{n-1} and therefore is non-free.

The tangent lift of this action to $T\mathrm{S}^{n-1}$ is $\tilde{a}\cdot(\gamma, \dot{\gamma}) = (a\gamma, a\dot{\gamma})$ and it is immediate to check that it leaves the reduced Lagrangian (4.24) invariant. It is also clear that the function ϕ given by (4.13) is A-invariant so the hypothesis to apply Theorem 3.1 holds.

The Lie algebra $\mathfrak{a} = \mathfrak{so}(n-1)$ is naturally identified with the set of $n \times n$ skew-symmetric matrices $\xi \in \mathfrak{so}(n)$ such that $\xi E_n = 0$. The infinitesimal generator of $E_i \wedge E_j \in \mathfrak{a}$, $1 \leqslant i, j \leqslant n-1$, is the vector field on S^{n-1} given by

$$(E_i \wedge E_j)_{\mathrm{S}^{n-1}}(\gamma) = (E_i \wedge E_j)\gamma \in T_\gamma \mathrm{S}^{n-1}.$$

Using the expression (4.24) for the reduced Lagrangian, we compute the action of the rescaled tangent bundle momentum map $\mathcal{J} \colon T\mathrm{S}^{n-1} \to \mathfrak{so}(n-1)^*$ defined by (3.1) on $E_i \wedge E_j \in \mathfrak{so}(n-1)$ to be given by

$$\begin{aligned} \mathcal{J}(\gamma, \dot{\gamma})(E_i \wedge E_j) &= \exp(\phi)\big[(2J_1 + (J_n - J_1)\gamma_n^2 + m(b + \ell\gamma_n)^2)((E_i \wedge E_j)\gamma, \dot{\gamma})_{\mathbb{R}^n} \\ &\qquad + m(J_n - J_1 + m\ell^2)\dot{\gamma}_n((E_i \wedge E_j)\gamma, E_n)_{\mathbb{R}^n}\big] \\ &= \sqrt{2J_1 + (J_n - J_1)\gamma_n^2 + m(b + \ell\gamma_n)^2}(\dot{\gamma}_i\gamma_j - \dot{\gamma}_j\gamma_i), \end{aligned}$$

where $(\cdot\,,\cdot)_{\mathbb{R}^n}$ denotes the euclidean inner product in $\mathbb{R}^n$ and in the second equality we have used the specific form (4.13) of the function ϕ. The quantity $\dot{\gamma}_i\gamma_j - \dot{\gamma}_j\gamma_i$ is the i-j entry of the matrix $-\gamma \wedge \dot{\gamma}$, which by Eq. (4.17) coincides with $-\Omega$. Therefore, by Theorem 3.1, the functions F_{ij} given by (4.23) are first integrals of the system as claimed.

Appendix A. Proof of Lemma 4.1

A.1. Proof of (4.19). The proof is a calculation for which we outline the details. Taking into account the form of the kinetic energy metric of the Lagrangian (4.10), and the expressions (4.16) for the horizontal lifts h_i, it follows that for $k, l \in \{1, \dots, n-1\}$ we may write

$$\langle\langle h_k, h_l \rangle\rangle = A_{kl} + B_{kl}, \tag{A.1}$$

where

$$\begin{aligned} A_{kl} :&= \left(\mathbb{I}\Big(\gamma \wedge \Big(E_k - \frac{\gamma_k}{\gamma_n}E_n\Big)\Big), \gamma \wedge \Big(E_l - \frac{\gamma_l}{\gamma_n}E_n\Big)\right)_\kappa, \\ B_{kl} :&= m\Big(b\Big(-E_k + \frac{\gamma_k}{\gamma_n}E_n\Big) + \ell\gamma \wedge \Big(E_k - \frac{\gamma_k}{\gamma_n}E_n\Big)E_n \\ &\qquad b\Big(-E_l + \frac{\gamma_l}{\gamma_n}E_n\Big) + \ell\gamma \wedge \Big(E_l - \frac{\gamma_l}{\gamma_n}E_n\Big)E_n\Big)_{\mathbb{R}^n}. \end{aligned}$$

We first simplify the expression of A_{kl}. Using the expressions (4.8) and (4.9) for the inertia tensor, one verifies that

$$\mathbb{I}\Big(\gamma \wedge \Big(E_k - \frac{\gamma_k}{\gamma_n}E_n\Big)\Big) = 2J_1\Big(\gamma + \frac{(J_n - J_1)\gamma_n}{2J_1}E_n\Big) \wedge \Big(E_k + \frac{(J_1 + J_n)\gamma_k}{2J_1\gamma_n}E_n\Big), \tag{A.2}$$

$j = 1, \dots, n-1$. The above expression, together with the general identity

$$(u_1 \wedge v_1, u_2 \wedge v_2)_\kappa = (u_1, u_2)_{\mathbb{R}^n}(v_1, v_2)_{\mathbb{R}^n} - (u_1, v_2)_{\mathbb{R}^n}(u_2, v_1)_{\mathbb{R}^n}, \tag{A.3}$$

that holds for $u_1, v_1, u_2, v_2 \in \mathbb{R}^n$, leads to

$$A_{kl} = (2J_1 + (J_n - J_1)\gamma_n^2)\delta_{kl} + \Big(\frac{J_1 + J_n}{\gamma_n^2} + J_n - J_1\Big)\gamma_k\gamma_l. \tag{A.4}$$

On the other hand, we have

$$\gamma \wedge \Big(E_j - \frac{\gamma_j}{\gamma_n}E_n\Big)E_n = \gamma_j E_n - \gamma_n E_j - \frac{\gamma_j}{\gamma_n}\gamma, \tag{A.5}$$

so we may write

$$\begin{aligned} B_{kl} &= m\Big(\Big(\ell + \frac{b}{\gamma_n}\Big)\gamma_k E_n - (b + \ell\gamma_n)E_k - \ell\frac{\gamma_j}{\gamma_n}\gamma, \\ &\qquad \Big(\ell + \frac{a}{\gamma_n}\Big)\gamma_l E_n - (a + \ell\gamma_n)E_l - \ell\frac{\gamma_l}{\gamma_n}\gamma\Big)_{\mathbb{R}^n} \\ &= m(b + \ell\gamma_n)^2\delta_{kl} + m\Big(\Big(\frac{b}{\gamma_n} + \ell\Big)^2 + \frac{\ell^2}{\gamma_n^2}\Big)\gamma_k\gamma_l. \end{aligned} \tag{A.6}$$

Substitution of (A.4) and (A.6) into (A.1) proves (4.19).

A.2. Proof of (4.20). The crucial part of the proof is to obtain the following expression for the Jacobi-Lie bracket of the vector fields h_i and h_j:

$$[h_i, h_j](R, x) = \Big(\Big(E_i - \frac{\gamma_i}{\gamma_n}E_n\Big) \wedge \Big(E_j - \frac{\gamma_j}{\gamma_n}E_n\Big), 0\Big) \in \mathfrak{so}(n) \times \mathbb{R}^n, \tag{A.7}$$

$i, j = 1, \dots, n-1$. Accept that this is the case for the moment. Then, similar to the calculation performed in section A.1, we have

$$\langle\langle [h_i, h_j], h_l \rangle\rangle = \tilde{A}_{ijl} + \tilde{B}_{ijl}, \tag{A.8}$$

where

$$\tilde{A}_{ijl} = \left(\mathbb{I}\Big(\gamma \wedge \Big(E_l - \frac{\gamma_l}{\gamma_n}E_n\Big)\Big), \Big(E_i - \frac{\gamma_i}{\gamma_n}E_n\Big) \wedge \Big(E_j - \frac{\gamma_j}{\gamma_n}E_n\Big)\right)_\kappa,$$

and

$$\tilde{B}_{ijl} = m\Big(\ell\Big(E_i - \frac{\gamma_i}{\gamma_n}E_n\Big)\wedge\Big(E_j - \frac{\gamma_j}{\gamma_n}E_n\Big)E_n,\\ b\Big(-E_l + \frac{\gamma_l}{\gamma_n}E_n\Big) + \ell\gamma\wedge\Big(E_l - \frac{\gamma_l}{\gamma_n}E_n\Big)E_n\Big)_{\mathbb{R}^n}.$$

On the one hand, using again (A.2) and (A.3), one may simplify

$$\tilde{A}_{ijl} = (J_n - J_1)(\gamma_j\delta_{il} - \gamma_i\delta_{jl}). \tag{A.9}$$

On the other hand, using

$$\Big(E_i - \frac{\gamma_i}{\gamma_n}E_n\Big)\wedge\Big(E_j - \frac{\gamma_j}{\gamma_n}E_n\Big)E_n = \frac{1}{\gamma_n}(\gamma_i E_j - \gamma_j E_i),$$

together with (A.5), allows one to write

$$\begin{aligned}\tilde{B}_{ijl} &= \frac{m\ell}{\gamma_n}\Big(\gamma_i E_j - \gamma_j E_i, \Big(\ell + \frac{b}{\gamma_n}\Big)\gamma_l E_n - \Big(b + \ell\gamma_n\Big)E_l - \ell\frac{\gamma_l}{\gamma_n}\gamma\Big)_{\mathbb{R}^n}\\ &= m\ell\Big(\ell + \frac{b}{\gamma_n}\Big)\Big(\gamma_j\delta_{il} - \gamma_i\delta_{lj}\Big).\end{aligned}$$

Substitution of the above expression, together with (A.9), onto (A.8) proves (4.20).

Hence, to complete the proof, it only remains to establish the validity of (A.7). The formula clearly holds for $i = j$, so below we assume that $i \neq j$. As a consequence of the independence of h_i on x, we have

$$[h_i, h_j](R, x) = \Big(\Big[\gamma\wedge\Big(E_i - \frac{\gamma_i}{\gamma_n}E_n\Big), \gamma\wedge\Big(E_j - \frac{\gamma_j}{\gamma_n}E_n\Big)\Big]_{\mathrm{SO}(n)}, W_{ij}\Big),$$

where $[\cdot\,,\cdot]_{\mathrm{SO}(n)}$ is the Lie bracket of vector fields on $\mathrm{SO}(n)$ (written in the left trivialisation as usual) and $W_{ij} \in \mathbb{R}^n$ has components

$$W_{ij}^{(k)} = b\gamma\wedge\Big(E_i - \frac{\gamma_i}{\gamma_n}E_n\Big)\Big(-R_{kj} + \frac{\gamma_j}{\gamma_n}R_{kn}\Big)\\ - b\gamma\wedge\Big(E_j - \frac{\gamma_j}{\gamma_n}E_n\Big)\Big(-R_{ki} + \frac{\gamma_i}{\gamma_n}R_{kn}\Big), \tag{A.10}$$

$k = 1, \dots, n-1$, $W_{ij}^{(n)} = 0$, where R_{kl} denotes the k-l entry of the matrix $R \in \mathrm{SO}(n)$.

On the one hand, García-Naranjo and Marrero [22, Lemma 4.4] compute:

$$\Big[\gamma\wedge\Big(E_i - \frac{\gamma_i}{\gamma_n}E_n\Big), \gamma\wedge\Big(E_j - \frac{\gamma_j}{\gamma_n}E_n\Big)\Big]_{\mathrm{SO}(n)} = \Big(E_i - \frac{\gamma_i}{\gamma_n}E_n\Big)\wedge\Big(E_j - \frac{\gamma_j}{\gamma_n}E_n\Big),$$

which establishes the correctness of the first entry of (A.7).

On the other hand, we shall prove that

$$\gamma\wedge\Big(E_i - \frac{\gamma_i}{\gamma_n}E_n\Big)\Big[-R_{kj} + \frac{\gamma_j}{\gamma_n}R_{kn}\Big] = \frac{R_{kn}}{\gamma_n^3}\gamma_i\gamma_j, \quad k = 1, \dots, n-1. \tag{A.11}$$

Considering that a similar formula holds when the roles of i and j are interchanged, it follows from (A.10) that W_{ij} vanishes and Eq. (A.7) indeed holds. The calculations to establish (A.11) rely on the following identity whose proof may be found in García-Naranjo and Marrero [22, Lemma B.1]:

$$E_i\wedge E_j[R_{kl}] = R_{ki}\delta_{jl} - R_{kj}\delta_{il}, \quad i, j, k, l \in \{1, \dots, n\}. \tag{A.12}$$

Using (A.12), and writing $\gamma = \sum_{l=1}^{n} \gamma_l E_l$ and $\gamma_l = R_{nl}$, it is straightforward to obtain (recall that we assume that $i, j, k \in \{1, \dots, n-1\}$ and $i \neq j$):

$$\gamma \wedge E_i[R_{kj}] = -\gamma_j R_{ki} = \gamma \wedge E_i\left[\frac{\gamma_j}{\gamma_n} R_{kn}\right], \quad \gamma \wedge E_n[R_{kj}] = -\gamma_j R_{kn}. \tag{A.13}$$

With a little bit more work, and using $\sum_{l=1}^{n} R_{kl}\gamma_l = \sum_{l=1}^{n} R_{kl}R_{nl} = \delta_{kn} = 0$, one obtains

$$\gamma \wedge E_n\left[\frac{\gamma_j}{\gamma_n} R_{kn}\right] = -\frac{\gamma_j}{\gamma_n^2} R_{kn} - \gamma_j R_{kn}. \tag{A.14}$$

Identities (A.13) and (A.14) imply that (A.11) holds.

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ХАМИЛТОНИЗАЦИЈА, ОЧУВАЊЕ МЕРЕ И ПРВИ ИНТЕГРАЛИ ВИШЕДИМЕНЗИОНЕ ГУМЕНЕ РУТОВЕ СФЕРЕ

Резиме. Разматрамо вишедимензионо уопштење проблема котрљања сфере, са основосиметричном расподелом масе, без клизања и ротације по равни. Користећи најновије резултате из García-Naranjo [21] и García-Naranjo and Marrero [22], показали смо да редуковане једначине кретања имају инваријантну меру и да се могу представити у Хамилтоновом облику помоћу Чаплигинове методе редукционог множитеља. Такође смо доказали општи резултат о постојању првих интеграла за одређене Хамилтонизабилне Чаплигинове системе са унутрашњим симетријама који се користи за одређивање очуваних величина проблема.

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