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SINGULARITIES OF INTEGRABLE LIOUVILLE SYSTEMS, REDUCTION OF INTEGRALS TO LOWER DEGREE AND TOPOLOGICAL BILLIARDS: RECENT RESULTS

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Abstract. In the paper we present the new results in the theory of integrable Hamiltonian systems with two degrees of freedom and topological billiards. The results are obtained by the authors, their students, and participants of scientific seminars of the Department of Differential Geometry and Applications, Faculty of Mathematics and Mechanics at Lomonosov Moscow State University.

1. Reduction of the degree of integrals for Hamiltonian systems with two degrees of freedom with the help of billiards

The integrability of a billiard in a domain bounded by an ellipse was noted by Birkhoff [27]. The integrability of the billiard is preserved if we consider the flat domain bounded by arcs of confocal ellipses and hyperbolas whose boundary does not contain angles equal to $\frac{3\pi}{2}$. In this case, all the angles of the boundary are equal to $\frac{\pi}{2}$, since confocal quadrics always intersect at a right angle. In the book by Kozlov, Treschev [8] it is noted that these dynamical systems are completely Liouville integrable. For the flat billiard in an ellipse, there are coordinates such that the motion is represented as a periodic motion along tori. Up to Liouville equivalence, such systems were studied in detail in [20, 21, 23] by Dragović, Radnović, and in [28] by Fokicheva. In an interesting paper [22] Dragović and Radnović studied the Liouville foliation for the flat billiard in an ellipse, as well as geodesic flows on the ellipsoid in the Minkowski space, giving an answer in terms of the Fomenko–Zieschang invariants. See also the important papers [24–26] by Dragović and Radnović devoted to the analysis of pseudo-integrable billiards.

Fokicheva classified all topological billiards bounded by the arcs of confocal conics (the families of confocal ellipses and hyperbolas and the confocal parabolas) [29, 30]. Further, Fokicheva investigated the topology of Liouville foliations on

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isoenergy surfaces for such billiards by calculating the Fomenko–Zieschang invariants of these systems.

Let us recall that the two smooth integrable systems are called Liouville equivalent if and only if there exists a diffeomorphism sending the Liouville foliation of the first system to the Liouville foliation of the second one.

The topological type of the Liouville foliation is completely determined by the Fomenko–Zieschang invariant, which is the graph with numerical marks, as described in [**1**, **4**, **16**]. For many integrable cases of the rigid body dynamics the calculation of the Fomenko–Zieschang invariants made it possible to detect the Liouville equivalence of these systems to some topological billiards by comparing marked molecules (see Veduyshkina and Fomenko [**7**]). Thus, roughly speaking, locally flat topological integrable billiards "visually model" many fairly complicated integrable cases in the dynamics of the rigid body and other physical systems.

In the paper by Fomenko and Vedyushkina [**42**] the investigation of integrable billiards was continued.

In the theory of Hamiltonian systems with two degrees of freedom, systems that are integrable by means of integrals of large degrees, for example, 3 and 4, are very well known (see, for example [**1**]). Such systems include, for example, the famous Kovalevskaya system, and its generalizations — the Kovalevskaya-Yahya system and an analogue of the Kovalevskaya system on the Lie algebra $so(4)$ (in this case the additional integral has degree 4), then the Goryachev system, Chaplygin-Sretensky system (here the degree of the additional integral is equal to 3), Dullin-Matveev system (degree 3), and also the integrable system by Sokolov (integral of degree 4). Note that in many classical integrable cases the additional integral is quadratic (Euler, Jacobi, Lagrange, Zhukovsky, Clebsch systems, etc.). Studying systems with integrals of degrees 3, 4 and higher is usually much more difficult. Therefore, the problem of a possible reduction of the degrees of integrals 3 and 4 to the integrals of a lower degree is very well known. It turns out that, in the general case, there are topological obstacles. Based on the Maupertuis principle, A. V. Bolsinov and A. T. Fomenko proved that, for example, the integral of degree 4 in the Kovalevskaya case and the integral of degree 3 in the Goryachev–Chaplygin case cannot be reduced to linear and quadratic integrals.

THEOREM 1.1 (A. V. Bolsinov and A. T. Fomenko [**1**–**3**]). a) *The Kovalevskaya integrable case generates* (*by the Maupertuis principle*) *on a two-dimensional sphere a Riemannian metric, the geodesic flow of which is integrable with the help of the integral of degree* 4. *This integral cannot be reduced to a linear or quadratic one.*

b) *The Goryachev–Chaplygin integrable case generates* (*according to the Maupertuis principle*) *on a two-dimensional sphere a Riemannian metric, the geodesic flow of which is integrable with the help of the integral of degree* 3. *This integral cannot be reduced to a linear or quadratic one.*

This fact was proved on the basis of the theory by Fomenko–Zieschang [**4**]. Let us present here (as an example) a brief scheme of the proof for the Goryachev–Chaplygin system. Let us consider the so-called coarse molecule W for the geodesic flow of the metric on a sphere generated by this system according to the Maupertuis principle. This flow is trajectory equivalent to the Goryachev–Chaplygin case and, therefore, has the same Liouville foliation on the isoenergy 3-manifold. Therefore, this molecule W coincides with the molecule for the Goryachev–Chaplygin case, calculated by A. O. Oshemkov, see [5]. Suppose that the Goryachev–Chaplygin integral is reduced to a quadratic one. In this case, we can use the results from [1, Vol. 2, Chapter 3]. In this book there is a list of the so-called marked molecules W^* for all geodesic flows on a sphere, which are integrable with quadratic and linear integrals. We can see that the molecule for the Goryachev–Chaplygin flow on the sphere does not coincide with any of the molecules of this classification (for integrals of degrees 1 and 2). Since the graph W^* is a Liouville invariant of the integrable system, we have a contradiction.

The theory of Fomenko–Zieschang originally dealt with smooth Liouville foliations and their fiber-wise smooth diffeomorphisms, that is, sending the leaves of the first foliation to the leaves of the second one. Recently, a new class of integrable topological billiards has been discovered. They are realized as the dynamics of a material point on the two-dimensional locally-Euclidean cell complexes, the edges of which are arcs of confocal quadrics (see [6]). The corresponding Hamiltonian system is realized on a four-dimensional piecewise-smooth manifold and (after reduction) on a three-dimensional piecewise-smooth isoenergy surface. The corresponding Liouville foliation consists of the regular piecewise smooth two-dimensional tori and some singular fibers. The foliated small neighborhood of the singular fiber is called "3-atoms". Here, the Liouville equivalence of the billiards is given by piecewise smooth fiberwise diffeomorphisms of the Liouville foliations. As was found by the authors in [7], for many cases, the topological billiards model (that is, are piecewise smoothly Liouville equivalent) important integrable systems with two degrees of freedom. Unexpectedly, with the help of integrable billiards, it is possible to reduce degrees 3 and 4 of the integrals to the integrals lower degree. This reduction is possible for some isoenergy 3-surfaces. Moreover, the integrals of degrees 3 and 4 are reduced to the one and the same canonical quadratic integral on corresponding billiards. Such canonical reduction (to the canonical quadratic integral) became possible because of considering piecewise smooth Liouville equivalences.

The confocal quadrics are given by the following equation

$$(b-\lambda)x^2 + (a-\lambda)y^2 = (a-\lambda)(b-\lambda), \quad 0 \leqslant \lambda \leqslant a. \tag{1.1}$$

In further considerations, the quadrics are assumed to be confocal, and $\infty > a > b$.

Let the boundary of the domain Ω on the plane $\mathbb{R}^2$ be the union of piecewise smooth curves consisting of the arcs of confocal quadrics, with angles in the vertices of the curves equal to $\frac{\pi}{2}$. Then the billiard Ω (which is called an elementary billiard) is integrable. Namely, in addition to the constant length of the velocity vector along the trajectory, the following quadratic function is preserved (see [8]):

$$\Lambda = \frac{(x_1v_2 - x_2v_1)^2 + (v_1)^2b + (v_2)^2a}{(v_1)^2 + (v_2)^2},$$

where (x_1, x_2) are the coordinates of the billiard particle, and (v_1, v_2) are the coordinates of the velocity vector.

Here Λ is the parameter of the ellipse or hyperbola which are tangent to the lines containing the trajectory of the billiard particle.

Let us consider the 2-dimensional cell complex obtained by gluing the elementary billiards along common segments of their boundaries. In case when along each edge no more than two billiards-sheets are glued together, the resulting manifold is called a topological billiard (see [6]). In the case when along some edge more than two billiards are glued, this edge must be endowed by a permutation $\sigma \in S_n$, where n is the number of billiard sheets glued along this edge. This cell complex with such permutations on the edges is called the billiard book (see [10]). The motion of the material point on it is determined as follows: the trajectory, when it hits the edge of the billiard, moves from one elementary domain to another one, according to the assigned permutation on this edge.

It is a remarkable and important fact that the resulting dynamical system is integrable. The resulting system turns out to be integrable with the same pair of integrals which are determined on the locally plane elementary billiard sheets.

The complete classification of the topological billiards up to Liouville equivalence was done by Vedyushkina in [6]. In the papers [9, 10] Fomenko, Kharcheva, Vedyushkina investigated the topology of the Liouville foliations for the billiard books. It was shown that an arbitrary 3-bifurcation of 2-dimension Liouville tori for any nondegenerate integrable Hamiltonian system with two degrees of freedom can be simulated (modeled) by a suitable billiard book. In other words, an arbitrary 3-atom can be realized by some billiard book.

Let us describe the decreasing of the degree of some integrable systems with the help of billiards.

THEOREM 1.2 (A. T. Fomenko, V. V. Vedyushkina). *The integrable systems by Kovalevskaya* [1]*, Kovalevskaya-Yahya* [11]*, Kovalevskaya on the Lie algebra* $so(4)$ [12]*, Goryachev–Chaplygin–Sretensky* [5]*, Sokolov* [13]*, Dullin–Matveev* [14] *with the integrals of degrees* 3 *and* 4 *are modeled* (*that is, are piecewise smooth Liouville equivalent*) *in the suitable energy zones* (*that is, on the suitable constant-energy* 3*-manifolds*) *by the integrable topological billiards with the canonical integral of degree* 2*. In other words, the integrals of higher degrees are reduced to the one and the same quadratic integral*

$$\Lambda = \frac{(x_1v_2 - x_2v_1)^2 + (v_1)^2b + (v_2)^2a}{(v_1)^2 + (v_2)^2}$$

on the corresponding billiards (*let us note that on the isoenergy* 3*-surface we have* $(v_1)^2 + (v_2)^2 = 1$).

REMARK 1.1. These results are presented in Figure 1. In the first column, there are corresponding billiards, in the second column there are corresponding Fomenko–Zieschang invariants for these integrable systems, in the third column the corresponding cases of integrability are indicated. Here the numbering of the invariants and the isoenergy zones which are indicated in brackets are taken from the papers [1, 5, 11–14]. In the fourth column, the topological type of the corresponding isoenergy 3-manifold is presented.

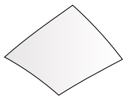
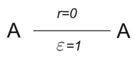
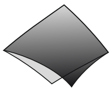
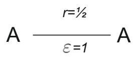
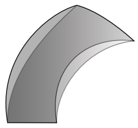
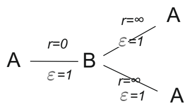
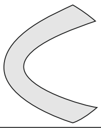
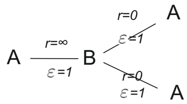
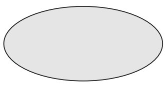
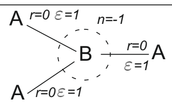
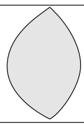
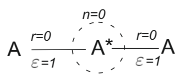
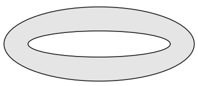
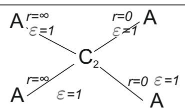
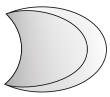
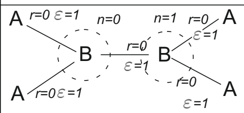
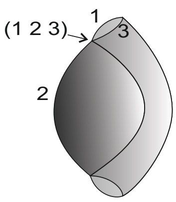
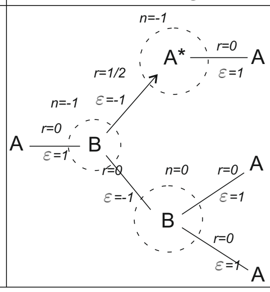
Integrable billiard	Fomenko-Zieshang invariant	Famous cases of integrability	Type of Q^3
		Kovalevskaya (1), Kovalevskaya-Yahya (h1), Kovalevskaya on so (4) (1,7,11), Dullin-Matveyev (1), Goryach-Chaplygin-Sretensky (1), Sokolov (A)	S^3
		Dullin-Matveyev (2)	RP^3
		Kovalevskaya (5), Kovalevskaya-Yahya (h16, h28), Kovalevskaya on so(4) (32), Goryachev-Chaplygin-Sretensky (4)	$S^1 \times S^2$
		Kovalevskaya on so(4) (10)	S^3
		Kovalevskaya-Yahya (h18), Kovalevskaya on so(4) (2.9), Sokolov (B)	S^3
		Kovalevskaya on so(4) (6), Goryachev-Chaplygin-Sretensky (2)	S^3
		Sokolov(I)	$S^1 \times S^2$
		Kovalevskaya on so(4) (8)	S^3
		Goryachev-Chaplygin-Sretensky (6)	$S^1 \times S^2$

FIGURE 1.

It should be noted that in some zones of energy, the systems listed above are sometimes smoothly Liouville equivalent to other systems with quadratic integrals. However, these smooth equivalences reduce high degree integrals to, generally speaking, different integrals of degree 2. An important advance of Theorem 1.2 is that piecewise smooth equivalences reduce the degree of integrals and reduce them to the one and the same canonical quadratic integral on the billiard. Namely, this remarkable integral is the parameter of confocal quadrics which form the boundary of the corresponding billiard table. In other words, the difference between the discovered piecewise smooth reduction and smooth reduction is that instead of the collection of different additional integrals of lower degree, we get the collection of the billiards with one and the same additional integral of degree two.

2. Integrable billiards in the Minkowski space: new results

In this section the results of the interesting paper by V. Dragović and M. Radnović [22] are developed.

Let us suppose that the plane $\mathbb{R}^2$ is endowed with Minkowski metric with a scalar product $\langle x, y\rangle = x_1y_1 - x_2y_2$.

Since the scalar product can be negative, all vectors are divided into 3 non-intersecting subsets. The vector v is called

- *space-like*, if $\langle v, v\rangle > 0$;
- *time-like*, if $\langle v, v\rangle < 0$;
- *light-like*, if $\langle v, v\rangle = 0$.

The family of confocal quadrics on the Minkowski plane is given by

$$\frac{x^2}{a-\lambda} + \frac{y^2}{b+\lambda} = 1, \ \lambda \leqslant a. \tag{2.1}$$

Here $a > b > 0$ are the real numbers.

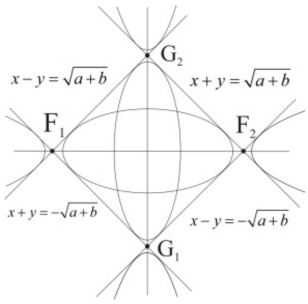


FIGURE 2. A family of confocal quadrics on the Minkowski plane. The bold dots indicate the four foci of the family of quadrics.

Among this family of quadrics we distinguish 3 subfamilies

- by $\lambda \in (-\infty, -b)$ the quadric is a hyperbola with the real axis x,
- by $\lambda \in (a, \infty)$ the quadric is a hyperbola with the real axis y,
- by $\lambda \in (-b, a)$ the quadric is an ellipse.

In addition, the values $\lambda = a, -b, \infty$ correspond to degenerate quadrics that are the axis y, the axis x and the line at infinity, respectively.

Let us define a billiard reflection in the Minkowski plane. Let v be the vector and ℓ is be the line. We represent a vector in the form $v = v_n + v_\ell$, where v_n is the normal component to the line ℓ (in the sense of Minkowski metric) containing the velocity vector, and v_ℓ belongs to ℓ. Let us call *the billiard reflection* v' of the vector v with respect to the line ℓ the following vector $v' = -v_n + v_\ell$. Vector v is a billiard reflection of the vector v'. Note that if the vector v_n is light-like, the reflection is undefined.

Note that the reflection does not change the type of the vector. Indeed, it is not difficult to verify that $\langle v, v\rangle = \langle v', v'\rangle$, where v' and v are the billiard reflections of each other.

It turns out that the flat billiard bounded by the arcs of confocal quadrics in the Minkowski space is also integrable. Along the trajectories of the billiard, the parameter of the confocal quadric is preserved (here we mean the billiard reflection in the Minkowski plane, see above).

Let us consider in detail the billiard in the flat domain bounded by the ellipse. At four points, where the normal to the tangent is light-like (namely, the points of tangency of the lines $x \pm y = \pm\sqrt{a+b}$ and the ellipse), we can extend the billiard reflection by continuity. At these points the vector v', which is the billiard reflection of the vector v, can be defined by the equation: $v' = -v$.

Let us demonstrate the evolution of trajectories induced by the change of the parameter Λ of the caustic.

For $\Lambda \in (-\infty, -b)$, the straight lines containing the trajectories of the billiard are tangent to the hyperbolas with the real axis x to the family (2.1). Then all trajectories are time-like and completely fill the entire interior of the ellipse. Similarly, for $\Lambda \in (a, \infty)$, the caustics are the hyperbolas with the real axis y, the trajectories are time-like and also fill the entire interior of the ellipse.

For $\Lambda = \infty$ the caustic is a line at infinity. Let us note that when Λ tends to $+\infty$ and $-\infty$ the pseudo-Euclidean length of the velocity vector can only decrease and its tilt angle tends to the tilt angle of the common tangents lines. Therefore, in the limit, we obtain light-like trajectories.

For $\Lambda \in (-b, 0)$ the trajectories are tangent to the confocal ellipse, and fill the two marked parts of the ellipse (see highlighted regions in Fig. 3a). These trajectories are time-like.

For $\Lambda \in (0, a)$ the trajectories are also tangent to the confocal ellipse, but fill the other two marked parts of the ellipse (see Fig. 3b). These trajectories are time-like.

For $\Lambda = a$, the tangent line to the trajectory is the y axis. The set of trajectories, which are time-like, consists of one periodic trajectory located along the axis y and two disjoint sets of homoclinic trajectories with tangent lines passing through pairs of foci located on the axis y.

For $\Lambda = -b$, the tangent line to the trajectory is the x axis. The set of trajectories, which are space-like, consists of one periodic trajectory located along the axis x and two disjoint sets of homoclinic trajectories with tangent lines passing through a pair of foci lying on the axis x.

For $\Lambda = -0, +0$ the caustic is the boundary ellipse. In the first case, the trajectories are space-like and the motion occurs along the segments marked in Fig. 3c. In the second case the trajectories are time-like and the motion occurs along the segments marked in Fig. 3d. There are no other trajectories for such Λ, since the segments are convex with respect to the interior of the ellipse. Hence, the material point cannot be reflected from the boundary in any other way.

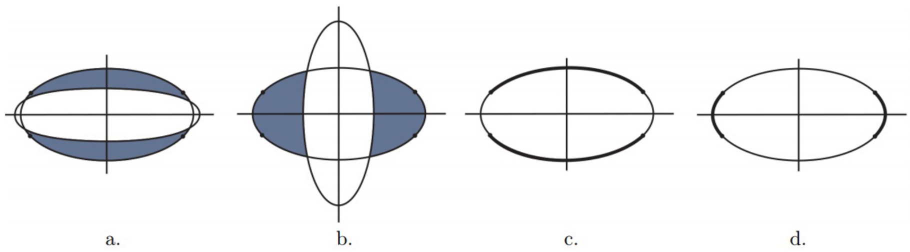


FIGURE 3. Areas of possible motion for different values λ

DEFINITION 2.1. A simple billiard in a Minkowski metric is a compact, connected subset of the Minkowski plane whose boundary consists of arcs of confocal quadrics for the family (2.1) and does not contain angles greater than π.

Classification of simple billiards on the Minkowski plane. Common tangent families of confocal quadrics divide the plane into several parts, and confocal ellipses or hyperbolas can be located only in five of this regions. Let us call them I, II, III, IV and V, as shown in Fig. 4

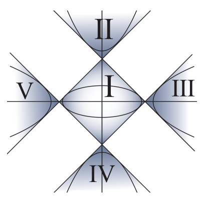


FIGURE 4. Areas I, II, III, IV and V on the plane.

DEFINITION 2.2. We call a simple billiard *elliptic* if its boundary consists of arcs of confocal ellipses and, possibly, parts of coordinate axes, and *hyperbolic*, if its boundary consists of arcs of confocal hyperbolas and, possibly, parts of coordinate axes.

Note that in this situation there is no simple billiard, with the boundary consisting of arcs of ellipses and hyperbolas simultaneously, since they are located in different parts of the plane (ellipses are located in I only, and hyperbolas are located in areas II, III, IV, V).

On the boundary of each simple domain there are singular points, where the material point after hitting continues its motion in the strictly opposite direction. This is either the nonregular point of the boundary (the intersection of the arcs of boundary quadrics), or the point where the common tangent lines are tangent also to the boundary.

DEFINITION 2.3. These points divide the boundary into different parts, which will be called *segments*.

DEFINITION 2.4. Two simple billiards Ω and Ω' are called *equivalent* if:

- they are obtained from each other by reflection relative to the x axis, y axis or by rotation on the angle $\pi/2$ (the reflection and rotation in this definition are Euclidean);
- changing the λ parameter of the boundary segment (excluding the λ corresponding to the values of a and $-b$) so that the parameter during this transformation λ does not take the values of a, $-b$ and ∞.

THEOREM 2.1. *Any simple elliptic billiard in the Minkowski plane is equivalent to one of the billiards shown in Fig.* 5

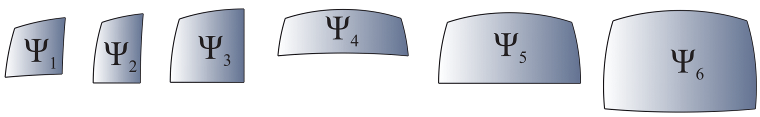


FIGURE 5. Classification of simple elliptic billiards in the Minkowski plane.

Let us describe the topological billiards on the Minkowski plane.

DEFINITION 2.5. Let l_1 and l_2 be the convex or straight congruent boundary segments of simple Ψ_1 and Ψ_2 billiards. That is, l_1 and l_2 belong to the same quadric of the same family of confocal quadrics. Then we define the *gluing* of such simple billiards along the segments l_1 and l_2. Their images after this gluing will be called the *gluing edge*. Here we determine the gluing along l_1 and l_2 by identical isometry. The end points of the gluing edge are called the *vertices of gluing*. The edges which are not glued are called *free edges* of the billiard. The segments having the same vertex are called *adjacent* segments.

DEFINITION 2.6. *The topological billiard* Δ is a two-dimensional piecewise smooth orientable manifold obtained by gluing several simple billiards. In this case, it is required that in each vertex of the gluing we have one of the following 3 situations: the gluing edge and two free edges, two gluing edges (such vertices are called *conical points*), or four gluing edges (such vertices are called *inner gluing vertices*).

Now we will describe the law of reflection in topological billiards.

After the gluing operation, the law of reflection for free edges remains the same as for the simple billiard. Then on the gluing edge the law is as follows: the point continues its motion on another sheet (domain) as after the regular (standard) reflection on this edge. Let us recall that the square of the Euclidean length of the velocity vector is preseved.

We should also note the case of conic points. The reflection in a conic point is determined by continuity, namely: upon hitting such a point, the material point continues its motion in the same domain in the opposite direction.

This reflection preserves the integral $v_E = v_1^2 + v_2^2$, i.e. the square of Euclidean velocity, and, as in the case of simple billiards, preserves the caustic parameter Λ. Let us repeat that all the segments of the boundaries belong to the same family of confocal quadrics.

Since these two integrals are functionally independent and are in involution with respect to the standard Poisson bracket, we can state that the topological billiard system is piecewise integrable in the sense of Liouville. For more detailed definitions, see V. V. Fokicheva's work [30].

We denote such topological billiards by $\Delta(2\Psi_i)_p^s$, where the index s denotes the number of glued edge segments, namely: $1, 2, 3$ or 4; the index p denotes the number of glued straight segments. The dashes in the superscript denote the number of intersections of the convex segments of the boundary with the coordinate axes.

Topological billiards are called equivalent if they are obtained from each other by replacing the components of their simple billiards Ψ_i with equivalent ones.

E. E. Karginova considered series of interesting examples of topological billiards obtained by gluing together two equivalent simple billiards. More precisely — the billiards $\Delta(2\Psi_3)_2^2$, $\Delta(2\Psi_5)^{3\prime}$, $\Delta(2\Psi_5)^2$, $\Delta(2\Psi_5)_1^3$ and $\Delta(2\Psi_3)_1^2$. For each of them, she calculates the Fomenko–Zieschang invariant, which classifies the Liouville foliations on the isoenergy surfaces.

Theorem 2.2 (E. E. Karginova). *The Fomenko–Zieschang invariants for topological billiards* $\Delta(2\Psi_3)_2^2$, $\Delta(2\Psi_5)^{3\prime}$, $\Delta(2\Psi_5)^2$, $\Delta(2\Psi_5)_1^3$ *and* $\Delta(2\Psi_3)_1^2$ *are shown in Fig. 6.*

The complete Liouville classification of topological billiards will be described in a future paper by E. E. Karginova.

3. Singularities of integrable system. Classification of gluing matrices on the edges of loop molecules for the center-center type points

These results are obtained by A. I. Zhila.

In the theory of topological Fomenko–Zieschang invariants, it is well known that for the points of center-center type, the r -mark on a loop molecule is always equal to 0. It is known that ε -mark depends on the orientation of the isoenergy manifold Q^3, then on the orientation of the critical circles (for the additional integral of the Liouville system), and on the orientation of the molecule's edges. In this section we consider the method of algorithmic definition of the orientation on the basic cycles on the Liouville tori. Then we will find gluing matrices on loop molecules

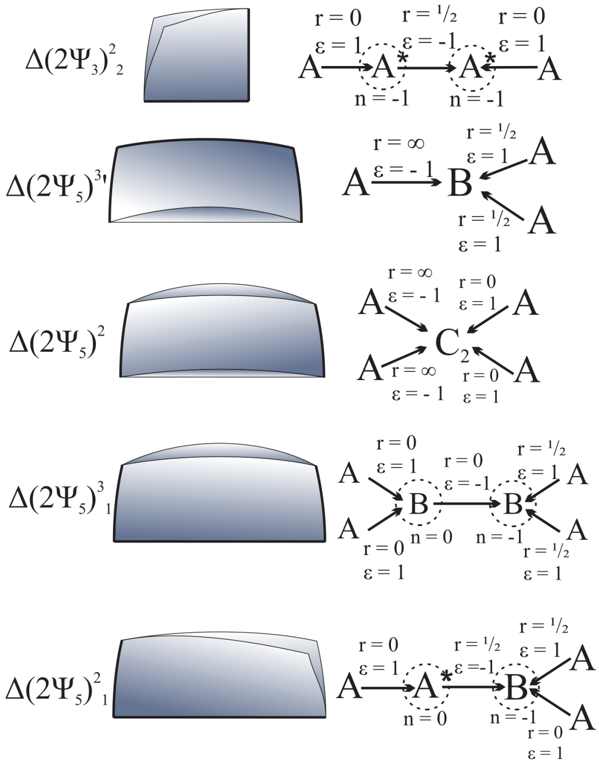


FIGURE 6. Fomenko–Zieschang invariants for the topological billiards. In the left column is the notation of the billiard, then — the billiard itself, and in the right column we can see the corresponding marked molecule.

for the center-center points depending on the location of the arcs of the bifurcation diagram.

Using topological invariants, we can recognize Liouville equivalence or non-equivalence for integrable systems. This investigation is based on the Fomenko–Zieschang theory for the integrable systems. In particular this classification is based on the Fomenko invariants (so-called coarse molecules) and Fomenko–Zieschang invariants (so-called marked molecules) and on the bifurcation complexes (for details, see [1, 4, 16, 17]).

Consider a smooth curve on the bifurcation diagram and the isoenergy manifold corresponding to it. We want to calculate the Fomenko–Zieschang invariant corresponding to the Liouville foliation on this manifold. We can, for example, extract information about marks and basic cycles from the loop molecules of the singular points (about loop molecules, see [18]). Singular points of the center-center type were found for many integrable system. Let us know, that r-mark on the loop molecule of any such singularity is equal to zero. In the work by V. A. Kibkalo [19] admissible bases (admissible coordinate systems) for two intersecting arcs of the bifurcation diagram are expressed via uniquely defined λ-cycles similarly to [18]. It has already been applied for the calculation of Fomenko–Zieschang invariants of Liouville foliation on isoenergy manifolds. In the work by A. I. Zhila, all possible gluing matrices are calculated for the case of center-center type singular points. After this it becomes possible to calculate the ε marks of arbitrary loop molecules.

Let us recall the rules of the basic cycles choice.

Let us consider a molecule of type $A - A$ for the center-center point. Let us consider the curve connecting the arcs of the bifurcation diagram corresponding to the bifurcations of type A. We want to construct basic cycles and calculate gluing matrices for the tori corresponding to the points of this admissible curve.

Let us consider two curves α_1 and α_2 of the bifurcation diagram, which are on the boundary of the image of momentum mapping (see figure 7). Then each curve corresponds to the bifurcation of type $A - A$. These curves intersect at a point which preimage contains a singular point of rank 0. Let us draw an admissible curve γ and consider the torus T in the preimage of a point on the curve γ. Now we can determine the coordinates on this torus, which are generated by the coordinates $(\lambda_{\alpha_1}, \mu_{\alpha_1})$ and $(\lambda_{\alpha_2}, \mu_{\alpha_2})$ on the tori T_1 and T_2 located near the α_1 and α_2 curves.

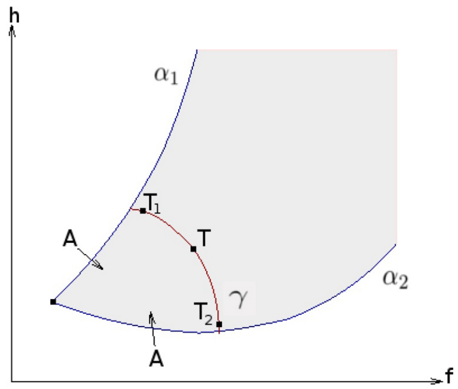


FIGURE 7. Admissible curve γ.

For the tori T_1 and T_2 near the curves α_1 and α_2 corresponding to the bifurcations of the type A, the following statements are held.

(1) The direction of the cycle μ_{α_i} coincides with the direction sgrad H on the critical circle.
(2) The cycle λ_{α_i} is contractible to a point when the torus approaches the corresponding curve α_i. Moreover, the pair $(\lambda_{\alpha_i}, \mu_{\alpha_i})$ should be positively oriented on the torus T_i, where $i = \overline{1, 2}$.

As the first basis cycle λ, we take the meridian of the solid torus, i.e. the contractible cycle. As the second cycle μ, we take an arbitrary cycle that complements λ up to a basis. The pair (λ, μ) should be positively oriented.

The coordinates generated by the coordinates $(\lambda_{\alpha_1}, \mu_{\alpha_1})$ and $(\lambda_{\alpha_2}, \mu_{\alpha_2})$ on the tori T_1 and T_2, respectively, are consequently defined on the torus T.

Considering these pairs of cycles as bases in the one-dimensional homology group of the torus, we obtain the gluing matrix

$$C = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \text{where } \begin{pmatrix} \lambda_{\alpha_1} \\ \mu_{\alpha_1} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \lambda_{\alpha_2} \\ \mu_{\alpha_2} \end{pmatrix}.$$

Let us recall the rule for the determining orientation on the admissible bases.

(1) The orientation on the symplectic manifold M^4 is given by the form $\omega\wedge\omega$.
(2) Let us consider the momentum mapping $\mathcal{F} = H \times F : M^4 \to \mathbb{R}^2(h, n)$. Then, on the surface $Q^3_\gamma = \{x \in M^4 |\ \mathcal{F}(x) \in \gamma\}$ we can determine the orientation by the normal vector n to the Q^3_γ in M^4. For example, a triple of the vectors e_1, e_2, e_3 will be positively oriented in Q^3_γ iif a quadruple of vectors e_1, e_2, e_3, n if a quadruple of vectors M^4. The normal vector to an admissible curve will be chosen in such a way that it is directed from the center-center point to a noncompact region in the image of momentum mapping. In the case when the curve γ is the line $H = const$, the manifold Q^3_γ is called an isoenergy surface and is denoted by Q^3_h (if $const = h$). Then as the normal n we take the vector $\operatorname{grad} H|_{M^4}$.
(3) On the torus $T^2 \subset Q^3_\gamma$ the orientation is given by the normal N to the torus in Q^3_γ. In other words, a pair of vectors e_1, e_2 will be positively oriented on T^2 if a triplet of the vectors e_1, e_2, N was positively oriented in Q^3_γ.
(4) As a result, we obtain that the positive orientation of a pair of the vectors e_1, e_2 is given by the condition

$$(\omega \wedge \omega)(e_1,\, e_2,\, N,\, n) > 0.$$

For example, based on the mentioned rules for defining orientation, we can see that in figure 7 for the basis cycles $(\lambda_{\alpha_i}, \mu_{\alpha_i})$ the following conditions must be held:

$$(\omega \wedge \omega)(\dot{\lambda_{\alpha_1}},\, \dot{\mu_{\alpha_1}},\, \operatorname{grad} F,\, \operatorname{grad} H) > 0$$

$$(\omega \wedge \omega)(\dot{\lambda_{\alpha_2}},\, \dot{\mu_{\alpha_2}},\, -\operatorname{grad} F,\, \operatorname{grad} H) > 0.$$

Here $\dot{\lambda_{\alpha_i}}$, $\dot{\mu_{\alpha_i}}$ are the velocity vectors of the cycles λ_{α_i}, μ_{α_i}. These vectors are tangent to the cycles of bases on the torus.

Now let us present a complete classification of the loop marked molecules for the center-center type points.

Let a center-center singular point correspond to the intersection of two curves α and δ on the plane (h, f) of the bifurcation diagram. Let us consider all possible different cases for the location of these curves with respect to each other. Let us fix the coordinates on the tori in the preimage of an admissible curve between α and δ. As a result we can now calculate ε-marks.

Theorem 3.1 (A. I. Zhila). *All the gluing matrices of the loop molecules of a center-center type point, depending on the relative position of the arcs of the* *bifurcation diagram on the plane (h, f) (in case when the positive orientation is determined by the condition $(\omega \wedge \omega)(\dot{\lambda},\, \dot{\mu},\, N,\, n) > 0$), are shown in figure 8.*

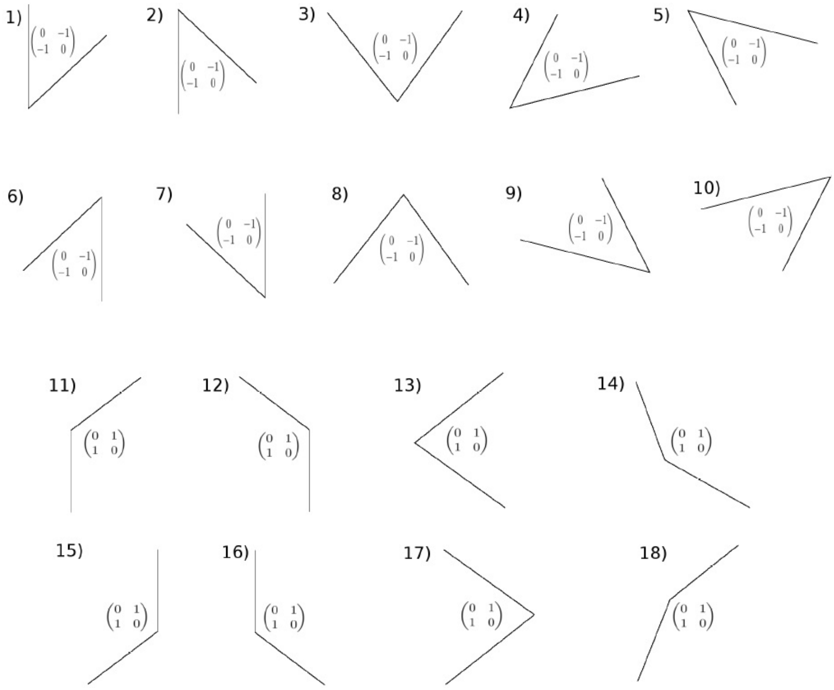


FIGURE 8. Gluing matrices for the center-center point. The complete list.

The problem of the calculation of gluing matrices for the points of center-center type was also investigated in the work by V. A. Kibkalo [19]. In this work, the orientation of the tangent basis (u, v) to the torus T^2 was defined by the condition $(\omega \wedge \omega)(\operatorname{grad} H, N, u, v) > 0$, where N is the outer normal vector to the 3-atom in M^4.

THEOREM 3.2 (V. A. Kibkalo). *Let us consider the singular point of type center-center on the bifurcation diagram. This point is the intersection of two arcs $\gamma_i, i = 1, 2$. Let $\varepsilon_i = \pm 1, i = 1, 2$ be the signs of the derivatives of the Hamiltonian H in the direction of intersecting arcs $\gamma_i, i = 1, 2$ respectively. Then the admissible coordinate systems (λ_i, μ_i) can be chosen such that*

$$\begin{pmatrix} \lambda_2 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_1 \\ \varepsilon_2 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \mu_1 \end{pmatrix}.$$

But the gluing matrix was not found explicitly in [19]. Nevertheless, theorem 3.2 determines the relations between admissible bases. These relations make it possible to reconstruct the gluing matrices.

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СИНУЛАРИТЕТИ ИНТЕГРАБИЛНИХ СИСТЕМА, РЕДУКЦИЈА ИНТЕГРАЛА НА МАЊЕ СТЕПЕНЕ И ТОПОЛОШКИ БИЛИЈАРИ: НЕДАВНИ РЕЗУЛТАТИ

Резиме. У раду су приказани нови резултати у теорији интеграбилних Хамилтонових система са два степена слободе и тополошким билијарима. Поред резултата које су добили аутори, представљени су и резултати њихових ученика, као и учесника научних семинара Катедре за диференцијалну геометрију и примене, Факултета за математику и механику Московског државног универзитета Ломоносов.

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