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CARTAN MEETS CHAPLYGIN

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Abstract. In a note at the 1928 International Congress of Mathematicians Cartan outlined how his "method of equivalence" can provide the invariants of nonholonomic systems on a manifold Q with kinetic lagrangians [29]. Cartan indicated which changes of the metric outside the constraint distribution $E \subset TQ$ preserve the *nonholonomic connection* $D_X Y = \mathrm{Proj}_E \nabla_X Y$, $X, Y \in E$, where $\nabla_X Y$ is the Levi-Civita connection on Q and Proj_E is the orthogonal projection over E. Here we discuss this equivalence problem of nonholonomic connections for Chaplygin systems [30, 31, 62]. We also discuss an example-a mathematical gem!-found by Oliva and Terra [76]. It implies that there is more freedom (thus more opportunities) using a *weaker* equivalence, just to preserve the straightest paths: $D_X X = 0$. However, finding examples that are weakly but not strongly equivalent leads to an over-determined system of equations indicating that such systems should be rare. We show that the two notions coincide in the following cases: i) Rank two distributions. This implies for instance that in Cartan's example of a sphere rolling on a plane without slipping or twisting, a (2,3,5) distribution, the two notions of equivalence coincide; ii) For a rank 3 or higher distribution, the corank of D in D+[D,D] must be at least 3 in order to find examples where the two notions of equivalence do not coincide. This rules out the possibility of finding examples on (3,5) distributions such as Chaplygin's marble sphere. Therefore the beautiful (3,6) example by Oliva and Terra is minimal.

1. Introduction: d'Alembert, Hertz, Cartan, Chaplygin

A constrained mechanical system consists of a Lagrangian $L: TQ \to \mathfrak{R}$ and a distribution $E \subset TQ$ of rank $s < n = \dim Q$. D'Alembert's principle gives

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = \lambda, \quad \text{with} \quad \lambda = \Lambda C(q), \quad \text{such that} \quad E : C(q) \cdot \dot q = 0. \tag{1.1}$$

Equations (1.1) are given in column form, one entry for each $\dot q_j$, with C an $r \times n$ matrix, and $\Lambda = (\lambda_1, \dots, \lambda_r)$ a row r-vector, $r = n - s$ is the number of contraints. When cast in Hamiltonian form, with $p = \partial L/\partial \dot q$, $H = p\dot q - L$ (see eg. [11]) then

$$X_{nh} = (\dot q, \dot p) = (\partial H/\partial p - \partial H/\partial q + \lambda) = X_H + (0, \lambda), \quad \lambda \cdot \partial H/\partial p = 0.$$

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The multipliers $\lambda_1, \cdots, \lambda_r$ can be eliminated by brute force, more or less as follows. Matrix C has an invertible $r \times r$ block. One uses the corresponding r equations to solve for all the λ's. Splitting $C(q) \cdot \dot{q} = 0$ in two blocks, one could solve for r of the $\dot{q}$'s in terms of $s = n - r$ others, the coefficients being functions of all the q's. Also, by differentiating, one eliminates r of the $\ddot{q}$'s.

Eventually one gets a system of first order ODEs for the n-coordinates q and for s of the velocities, totalizing a system of $n + s$ ordinary differential equations. This procedure is very awkward. This is why geometric approaches are so helpful.

This paper is also intended as a review for a non-expert, familiar with basic differential geometry, at the level of the first chapters of [52]. For a crash course on Geometric Mechanics, with emphasis on control, of interest to mechanical engineers working in Robotics, see [70]. A geometric mechanics approach to nonholonomic systems was anticipated by A. Bakša [8], see also [66]. For a short survey on nonholonomic systems, see [10]. For a comprehensive introduction, see [3]. Our goal is to discuss the equivalence problem of Cartan's *nonholonomic connection* $D_X Y$ under changes of metric. We pursue a recent observation by Terra and Oliva [76] that there is more freedom if one is happy by keeping the same straightest paths $D_X X = 0$. We show nonetheless that in many cases the notions coincide.

1.1. From d'Alembert to Hertz. Jean le Rond d'Alembert asserted in his *Traité de Dynamique* that, due to the constraints imposed on the system, a force appears in the right hand side of Euler–Lagrange equations. This force produces no resultant work in the system[1]. It was noticed by H. Hertz in his *Prinzipien der mechanik* ([51], 1894) that d'Alembert's principle yields *different* equations than those obtained by the calculus of variations with constraints (the so-called "vakonomic" paths), except when the distribution is integrable (holonomic); in this case both lead to geodesics of the induced metric on every leaf L of the foliation. Inspired by Gauss' least constraint principle, Hertz proposed the principle of least curvature: the path followed by an *inertial* mechanical system is as straight as possible, given that the velocity vector must satisfy the constraints. This gives the correct equations for nonholonomic systems (hereafter shortened as nh)[2].

1.2. Appell, Whittaker, Maggi, Hamel. We will skip, for brevity, the accomplishments of their generation at the beginning of the 20$^{\text{th}}$ century. In danger of being superficial, we take the nerve to say that their contribution was mainly on the practical side: 'quasivelocities'. Let $q \in \mathfrak{R}^n$ be local coordinates on Q^n, and consider a local moving frame, defined by an $n \times n$ invertible matrix $R(q)$,

$$e_J = \frac{\partial}{\partial \pi_J} = \sum_{i=1}^{n} r_{IJ} \frac{\partial}{\partial q_I}, \quad \sum \dot{\pi}_J e_J = \sum \dot{q}_I \frac{\partial}{\partial q_i}, \quad \dot{\pi} = (R(q))^{-1} \dot{q}.$$

[1] Part II, Chapter 1 (1743, 1758). For an appraisal of d'Alembert's work, see [33]. The validation of d'Alembert's principle has been experimentally established (see eg. [54, 55, 68], as the limit of appropriate damping forces.

[2] An inertial system is governed exclusively by its kinetic energy. Poincaré appreciated Hertz's view that Mechanics could be founded on the concept of constraints rather than forces: "il nous force a réflechir, à nous affranchir de vieilles associations d'idées" [78]. We prefer to call the solutions of (1.1) *straightest paths*. The terminology *nonholonomic geodesic* would be more appropriate for the vakonomic setting.

In books on mechanical engineering the keyword *quasicoordinates* refers to nonexisting entities π such that

$$\frac{\partial f}{\partial \pi_J} = \sum_I \frac{\partial f}{\partial q_I}\frac{\partial q_I}{\partial \pi_J} = \sum_I \frac{\partial f}{\partial q_I} r_{IJ} = e_J(f).$$

Using such quasicoordinates, the multipliers can be eliminated automatically, and the number of ODEs would be just $n + s$. Moreover, they realized that when symmetries were present, a proper choice of quasicoordinates would yield reduced equations, which in some cases could be explicitly integrated in closed form.

1.3. Nh connections: Cartan and contemporaries. In the late 1920's Cartan, Synge, Schouten, Vranceanu and Wagner provided a differential geometric description of nh systems. For historical information on that period we refer to [**35**]. They considered only purely inertial Lagrangians (no potential energy) and we will follow suit here.

Let g denote the Riemannian metric on Q and $E \subset TQ$ a distribution.

DEFINITION 1.1. The (*partial*) *connection* on $E \subset TQ$, encoding d'Alembert's principle, is the combination of two operations,

i) computing the Levi-Civita connection $\nabla_X Y$, of g for $X, Y \in E$
ii) projecting $\nabla_X Y$ orthogonally back to E.

This is analogous to the induced Levi-Civita connection on a submanifold of a Riemannian manifold. The nh connection is therefore

$$D_X Y = \mathrm{Proj}_E \nabla_X Y \tag{1.2}$$

and the straightest path equation is given by

$$D_X X = 0, \quad X \in E.$$

Affine connections on TQ are classical objects. According to [**35**], the study of affine connections on subbundles of TQ (*partial connections*) started with Schouten, and was further developed by Vranceanu, Synge and Wagner. Without this knowledge, nonholonomic connections were discussed in [**63**] in which one of the authors was a participant. See [**45**] for a recent work in which the notion is used.

1.4. Chaplygin and Hamiltonization. The idea of Hamiltonization of nh systems goes back to Chaplygin himself. In the first decade of the 20th century he developed the method of *reducing multipliers* [**31**], applied to systems with abelian symmetries. In some cases, under a time reparametrization depending on the base variables $s \in S$, the reduced system can be cast in Hamiltonian form, and in several examples the reduced system is integrable (more information in section 6).

The jurisprudence was established by Chaplygin's sphere. It is a dynamically unbalanced sphere, but with the center of mass at the geometric center. The configuration space is $Q = SO(3) \times \mathfrak{R}^2$. The sphere rolls without slipping on a horizontal plane, twisting about the vertical axis allowed. The problem can be viewed as an abelian ($G = \mathfrak{R}^2$) Chaplygin nh system [**30**, **31**]. See also [**36**]. The no slip contraints define a distribution E of rank 3 that is strongly nonholonomic (i.e. 1-step, 3-5). It was shown to be Hamiltonizable when reduced to T^*S^2 [**13**].

1.5. Organization of the paper. In sections 2 and 3 we discuss É. Cartan's note at the 1928 ICM. Cartan's method to obtain the dynamics, in the case of Lie groups, is equivalent to Arnold–Euler's [2]. Section 3 gives the *initial G-structure* for the equivalence method. Section 4 builds on recent work by Terra and Oliva [76]. They consider a weaker notion of equivalence, the preservation of the acceleration $D_X X$ instead of the nonholonomic connection $D_X Y$. We present cases where the weak and strong notions coincide. The special case of Chaplygin systems is outlined in section 5, and section 6 presents final remarks and directions for further research.

We finish the introduction with a historical note[3].

1.6. Chaplygin and Cartan. They were born 3 days apart, Cartan on April 9, and Chaplygin April 5, 1869. What more in common? Both came from poor families and were discovered and protected by their elementary school teachers. They endured hardships and tragedies in WW2.

Chaplygin was very much involved in the revolutionary movement in the 1910's. He protested persecutions in the 1930's, defending Luzin fiercely. Still, he was awarded in most prestigious medals due to his achievements, specially during the War: Hero of Socialist Labour, Order of Lenin (twice), and Order of the Red Banner.

Cartan's son Louis, a member of the Resistance, was murdered by the Nazis. Henri Cartan, the oldest, was a leader in the creation of the European Math Union after the war and was influential in bringing Germany back.

Cartan can be described as a pure, pure mathematician, and Chaplygin a very applied one[4], but this may be misleading; they shared a common interest in gravitation and cosmology. Chaplygin gas is still a basic ingredient for dark matter theory. As Chern and Chevalley [32] wrote about Cartan, and Lyusternik on Chaplygin, both were excellent teachers and were all for Women in Mathematics.

Lyusternik tells the following anecdote in [71]: 'I was examined by Sergei Alekseevich on particle mechanics [entering the Moscow State University]. My answer was in no way remarkable. I was surprised several months later when taking the [course on] Mechanics of a System that he remembered my name. 'There's nothing surprising about that', I was told, 'Chaplygin has a phenomenal memory'. When he was Rector of the Higher Courses for Women, he knew all the students by name'.

[3] Information from "The Early Years of the Moscow Mathematical School" (Lyusternik, [71]) and the websites http://www-history.mcs.st-andrews.ac.uk/Biographies/Chaplygin.html, http://www-history.mcs.st-andrews.ac.uk/Biographies/Cartan.html

[4] Chaplygin's career was mostly in aeronautical engineering. We quote from the MacTutor site: The Central Aerohydrodynamic Institute, or TsAGI, was founded in 1918 and Chaplygin helped organize the Institute from that time. In fact from the time that TsAGI was founded, he devoted almost all his energies to the project. On the death of Zhukovsky in 1921, Chaplygin became Chairman of the Board, a position he held until 1930. He was Executive Director of the Institute from 1928 to 1931, then he became head of the scientific work of the Institute (...) In the autumn of 1941, the Central Aerohydrodynamic Institute was evacuated from Moscow to Kazan and Novosibirsk. Chaplygin took charge of the Novosibirsk branch of the Institute and rapidly organised the building of a wind tunnel and research laboratories. However, the hard work and difficult circumstances told on his health and he died from a brain haemorrhage in October 1942.'

1.7. Hilbert: "Mathematics Knows No Races" [82]. There was a strong political aspect at the ICM in Bologna, held from September 3 to September 10 1938. Hilbert intended to deliver a speech (but it did not happen, not clear why) with that timely title. But Hilbert's dream is becoming true (and hopefully will remain so). At ICM 2014 and 2018 Fields medals were awarded to mathematicians born in Third World countries, one of them Maryam Mirzakhani. She died so early.

2. Cartan's equivalence for nh connections

It is not a surprise that Cartan used *adapted moving frames*[5] in his approach for nh mechanics in [29]. He considered only kinematical Lagrangians $L = T$ on a configuration space Q, where T is the kinetic energy of a Riemannian metric and the constraints defined by a distribution $E \subset TQ$ with rank $s < n$.

Cartan advocated his *equivalence method* to obtain invariants characterizing the nonholonomic connection (1.2). See [50, 75, 77] for modern description of the equivalence method with applications.

Cartan's note was revisited in [63]. As pointed out by Terra and Oliva [76], there is an inaccuracy. It was asserted that Cartan's equivalence criterion (see section 3.2) provided *necessary and sufficient conditions* for a change of metric in Q to keep the straightest paths. Cartan's criterion gives sufficient conditions, but they are not always necessary, and this brings more opportunities. We will show nonetheless in section 4 that the exceptions are rare.

The difference $A(X,Y) = \tilde{\nabla}_X Y - \nabla_X Y$ between two affine connections in Q is a 2-tensor (very easily seen to be bilinear). Clearly ∇ and $\tilde{\nabla}$ have the same geodesics if and only if A is skew-symmetric. This implies (obviously) that $\nabla_X X = \tilde{\nabla}_X X$. One can make the same observation about partial connections, the difference between $D_X Y$ and $\tilde{D}_X Y$ being a "partial" 2-tensor. Terra and Oliva propose a *broader* notion of equivalence: the preservation of the straightest path equation $D_X X = 0$.

We start with the *stronger* notion of equivalence, preservation of $D_X Y$. We will consider the weaker notion $D_X X$ in section 4. We will show that for rank 2 distributions, preservation of $D_X Y$ is equivalent to preservation of $D_X X$. We also show that the notions coincide when $\dim(E + [E,E]) - \dim E < 3$.

DEFINITION 2.1. $E \subset TM$ is *bracket generating* if any local frame e_i for E together with its iterated Lie brackets $[e_i, e_j]$, $[e_i, [e_j, e_k]], \dots$ spans TQ.

The only case Cartan discussed in some detail were the 1-step distributions, those satisfying $E + [E,E] = TQ$. We call those distributions *strongly nonholonomic* (Snh). In this case, Cartan proved that, preserving the connection $D_X Y$, one can

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[5] For a thorough account of moving frames in mechanics, see [73], where there is an interesting historical observation: at the same time as Cartan (in France), Schouten (in Holland) was also interested in moving frames for nh mechanics [80]. In fact, nonholonomic mechanics was trendy at that time! Vranceanu, then a student of Levi-Civita in Rome, also made a contribution to nh geometry at the 1928 ICM [87]. In Ireland, Synge was also studying nh systems [84]. A few years later, Wagner studied the geometry of nonholonomic systems in his doctoral thesis under V. F. Kagan at Lomonosov Moscow State University (see [35] for an account of Wagner curvature tensor).

modify the metric in the orthogonal complement F of E in TQ, as long as F remains orthogonal to E. This seems uneventful in view of d'Alembert's principle.

The simplest example of a strongly nonholonomic distribution is a contact distribution (2-3) in $\mathfrak{R}^3$. Their invariants are given in [37]. Chaplygin's ball (3-5) is also Snh. For a $k \geqslant 2$-step distribution, Cartan observed something unexpected. Denote by $E^{(1)} = E + [E, E]$. As in the Snh case, one can change the metric in the orthogonal complement F of E inside $E^{(1)}$. F remains orthogonal to E. Surprisingly, there is total flexibility in changing the metric outside $E^{(1)}$, as if the nh geometry becomes 'saturated' at the level of $E^{(1)}$. The proof is in section 3.

About the invariants, Cartan indicated that for distributions with $k \geqslant 2$-steps, implementing the equivalence method would become cumbersome. We have the impression that at this point Cartan lost interest.

"l'interét géométrique sévanouirait rapidement à mesure que les cas envisagés deviendraient plus compliqués."

In fact, as far as we know, he did not pursue this study later. We took the nerve to do the equivalence problem for nh connections on 2-3 [37], the 2-step 2-3-5 distribution [38, 65], and the Engel 2-3-4 in [64].

Cartan was right in his warning: it can be difficult to interpret geometrically the invariants. At any rate, in spite of Cartan's somewhat pessimistic statements, with the help of computer algebra further cases may be done nowadays. Cartan's method of equivalence is algorithmic and (almost) unsupervised. We wish to advertise this study for the special case of Chaplygin systems, both in the strong and weak senses. Some thoughts are presented in section 5.

2.1. Derived ideal. The bracket generating condition can be refined in terms of a filtration of ideals of differential forms. To do this let $\mathcal{I}$ be the ideal of all differential forms annihilating E. It is locally generated by $n - s$ one-forms.

Definition 2.2. The set of forms $\mathcal{I}' = \{\eta \in I \mid d\eta \equiv 0 \mod (I)\}$ is called the derived ideal of $\mathcal{I}$.

Set $\mathcal{I} = \mathcal{I}^0$ and define $\mathcal{I}^j$ inductively by $\mathcal{I}^j = (\mathcal{I}^{j-1})'$. The condition that E is bracket generating can be redefined in terms of the sequence of inclusions $I^0 \supset I^1 \supset I^2 \supset \cdots$ terminating with the 0 ideal. In the vectorfields side, we denote $E' = E + [E, E]$ and define inductively $E^{(0)} = E$, $E^j = (E^{j-1})'$. Then $\mathcal{I}^{(j)}$ is the annihilator of $E^{(j)}$. This comes from the following lemma, a simple consequence of Cartan's magic formula:

Lemma 2.1. *$\mathcal{I}^{(1)}$ is the annihilator of $E^{(1)} = E + [E, E]$.*

Proof. Just stare at

$$d\theta(V, W) = \pm V\theta(W) \pm W\theta(V) \pm \theta[V, W]$$

where the correct signs do not matter. □

Remark 2.1. The filtration of ideals $\mathcal{I}^{(j)}$ forms the annihilators of $E^{(j)}$ where $E^{j+1} = E^j + [E^j, E^j]$ which form the big' growth vector. If, instead, we take $E = E^1 = H = H^1$ and $H^{j+1} = E + [E, H^j]$ we get the 'small' growth vector. In general E^i contains H^i. In the examples discussed here the big and the small sequences are the same. As we will see shortly, nonholonomic geometry (meaning $D_X Y$) sees only the first derived ideal.

2.2. The Levi-Civita connection. Once a frame in Q is chosen, it defines a riemannian metric T in Q for which it is an orthonormal basis. Let (e_I) be an orthonormal frame and (ω^I) its dual coframe satisfying $\omega_I(e_J) = \delta_{IJ}$. The metric writes in terms of the coframe as

$$g = (\omega^1)^2 + \cdots + (\omega^n)^2.$$

Recall that the Levi-Civita connection of the metric is given by (see eg. [52])

$$\nabla_X e_J = \omega_{IJ}(X) e_I, \tag{2.1}$$

where the connection forms ω_{IJ} satisfy (uniquely) the structure equations

$$d\omega^I + \omega_{IJ} \wedge \omega^J = 0, \quad \omega_{IJ} = -\omega_{JI}.$$

In the traditional Christoffel symbols notation one writes

$$\nabla_{e_K} e_J = \Gamma^I_{KJ} e_I, \quad \Gamma^I_{KJ} = \omega_{IJ}(e_K), \quad \Gamma^I_{KJ} = -\Gamma^J_{KI}, \tag{2.2}$$

Proposition 2.1 (Cartan's moving frames approach for geodesics).

$$\nabla_{\dot c} \dot c = 0 \iff \dot v_K = -v_I v_J \Gamma^K_{IJ}(q), \qquad \dot c = v_L e_L, \quad \Gamma^K_{IJ} = \omega_{KJ}(e_I).$$

2.3. Euler-Arnold equations: Γ^I_{KJ} versus c^K_{IJ}. In control problems, it is very useful to prepare a table of the expansions of the commutators $[e_J, e_K]$ in its own moving frame $\{e_L\}$

$$[e_J, e_K] = c^L_{JK}(q) e_L.$$

Thus it is important to relate the Christoffel symbols $\Gamma^I_{KJ} = \omega_{IJ}(e_K)$ with the values of $c^K_{IJ} (= \omega_K[e_I, e_J])$.

Proposition 2.2. *We have the relations*

$$\omega_{IK}(e_J) - \omega_{IJ}(e_K) = \omega^I[e_J, e_K] \tag{2.3}$$

$$2\Gamma^J_{IK} = 2\omega_{JK}(e_I) = \omega^I[e_J, e_K] - \omega^J[e_K, e_I] - \omega^K[e_I, e_J]. \tag{2.4}$$

Proof. $d\omega^I(e_J, e_K) = -\omega^I[e_J, e_K] = -\sum_L \omega_{IL} \wedge \omega^L)(e_J, e_K) = \omega_{IJ}(e_K) - \omega_{IK}(e_J)$. Hence (2.3) follows. These relations can be inverted with the usual trick. If we write the cyclic permutations of (2.3), we get three equations relating six quantities on the right hand side, but in reality they are just three objects due to the anti-symmetry of $\omega_{IJ}(e_K)$ in the lower indices I, J. Solving this linear system yields (2.4). ☐

Proposition 2.3 (Euler–Arnold equations). *The geodesic equations can also be written as*

$$\nabla_{\dot c} \dot c = 0 \iff \dot v_K = -v_I v_J [(c^I_{KJ} + c^J_{KI})/2] = -v_I v_J c^I_{KJ}(q), \quad \dot c = v_L e_L. \tag{2.5}$$

Proof. Immediate from the previous Proposition. One of the three terms in (2.4) does not contribute due to the skew symmetry with respect to I, J. ☐

EXAMPLE 2.1. For a Lie group, with say, a left invariant frame and left invariant metric the c^I_{KJ} are constants. The expert reader will recognize that (2.5) becomes the Arnold–Euler equations in the Lie Algebra, in the case of an orthonormal frame. For a general non-orthonormal frame (see eg. [62])

$$\dot{\Omega}_I = g^{IK} c^J_{LK} g_{JM} \Omega_M \Omega_L, \tag{2.6}$$

where we changed to the usual notation $v_I = \Omega_I$ and we define a left invariant metric by $2T = (G\Omega, \Omega)$, with $G = (G_{IJ})$ is a symmetric positive definite matrix. The vector Ω represents a Lie algebra element expanded in the basis e_K. The presence of entries g^{IK} of the cometric matrix G^{-1} is morally a "nonlocal" effect.

2.4. Straightest paths (à la Cartan or à la Euler–Arnold). We will use the following conventions and un-conventions: capital roman letters $I, J, \ldots$ run from 1 to n. Lower case roman $i, j, \ldots$ run from 1 to s (The dimension of the constraint distribution). Greek characters α, β, γ, from $s+1$ to n. Summation over repeated indices is assumed unless otherwise stated. We apologize if we are sometimes careless in using upper or lower indices, and sometimes we put Σ to indicate a summation more explicitly. We will further subdivide the Greek characters into lower and upper case to account for forms belonging to the derived ideal $\mathcal{I}'$ of $\mathcal{I} = \text{span}\{\omega^\alpha\}$.

Cartan proposed studying nonholonomic systems using *adapted coframes*

$$\omega = \begin{bmatrix} \omega^i \\ \omega^\alpha \end{bmatrix}, \quad \text{where} \quad 1 \leqslant i \leqslant s, \quad s+1 \leqslant \alpha \leqslant n.$$

This coframe is dual to an *adapted orthonormal moving frame* $\{e_i, e_\alpha\}$, the first s vectors being tangent to E.

PROPOSITION 2.4. *The Nh connection of Definition 1.1 is obtained simply by running the indices only up to s:*

$$D_{e_i} e_j = \Gamma^k_{ij} e_k \quad (1 \leqslant i, j, k \leqslant s).$$

A curve $c(t)$ is a straightest path if $\dot{c} = \sum_{k=1}^s v_k e_k$ ($v_k = \omega_k(\dot{c})$ quasivelocities) *with (see (2.2) and (2.5))*

$$\dot{v}_k = -v_i v_j \Gamma^k_{ij} \quad \textit{or equivalently,} \quad \dot{v}_k = -v_i v_j c^i_{kj}. \tag{2.7}$$

In the sequel we will sometimes use the $\Gamma^j_{ik} = \omega_{ji}(e_k)$ (that we call the Cartan format) and sometimes the $c^i_{kj} = \omega_i[e_k, e_j]$ (that we call the Arnold–Euler format), whichever is more convenient in the situation being considered.

COROLLARY 2.1. *Given a left invariant distribution E, and a left invariant metric (it may be given by a matrix G), the nonholonomic equations are obtained by the* orthogonal projection *with respect to this metric, of the Euler equations (2.5), (2.6) for non-orthonormal frames for $\dot{\Omega}$ on the distribution E.*

NOTA BENE. We stress that although $\Gamma^k_{ij} \neq c^i_{kj}$, the difference is *skew symmetric* in the indices i, j. For the straightest path equations the difference will disappear after the summations over i, j of the corresponding $\square v_i v_j$, so we may take the c^i_{kj}. But for the nh connection we need to stick to the Γ^k_{ij}.

2.5. Example: maximally symmetric 2-3-5 distribution. Let

$$\begin{aligned}
e_1 &= \frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_3} - x_1x_2\frac{\partial}{\partial x_4} - x_2^2\frac{\partial}{\partial x_5}\\
e_2 &= \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3} + x_1^2\frac{\partial}{\partial x_4} + x_1x_2\frac{\partial}{\partial x_5}\\
e_3 &= [e_1, e_2] = 2\frac{\partial}{\partial x_3} + 3x_1\frac{\partial}{\partial x_4} + 3x_2\frac{\partial}{\partial x_5}\\
e_4 &= [e_1, e_3] = 3\frac{\partial}{\partial x_4},\\
e_5 &= [e_2, e_3] = 3\frac{\partial}{\partial x_5}
\end{aligned}$$

which form a basis for a 5-dimensional nilpotent Lie algebra on $\mathfrak{R}^5$. The exponential map gives a diffeomorphism between the nilpotent algebra and the group. Using the Baker–Campbell–Hausdorf formula one determines the cubic polynomial group law which is:

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}\cdot\begin{pmatrix}y_1\\y_2\\y_3\\y_4\\y_5\end{pmatrix} = \begin{pmatrix} x_1+y_1\\ x_2+y_2\\ x_3+y_3+\frac{1}{2}(x_1y_2-x_2y_1)\\ x_4+y_4+\frac{1}{2}(x_1y_3-x_3y_1)+\frac{1}{12}(x_2y_1^2-x_1y_1y_2+x_1^2y_2-x_1x_2y_1)\\ x_5+y_5+\frac{1}{2}(x_2y_3-x_3y_2)+\frac{1}{12}(x_2y_1y_2-x_1y_2^2+x_1x_2y_2-x_2^2y_1)\end{pmatrix}.$$

By construction the vector fields X_i provide a left invariant frame on G. The dual coframe is

$$\begin{pmatrix}\eta^1\\\eta^2\\\eta^3\\\eta^4\\\eta^5\end{pmatrix} = \begin{pmatrix} dx_1\\ dx_2\\ \frac{1}{2}(-x_2dx_1+x_1dx_2-dx_3)\\ \frac{1}{6}(-x_1x_2dx_1+x_1^2dx_2-3x_1dx_3+2dx_4)\\ \frac{1}{6}(-x_2^2dx_1+x_1x_2dx_2-3x_2dx_3+2dx_5)\end{pmatrix}.$$

PROPOSITION 2.5. *Let $E = \mathrm{span}(e_1, e_2)$ be a rank two distribution on an n-dimensional Riemannian manifold Q. In terms of an orthonormal coframe ω,*

$$\dot v_1 = -\omega_{12}^1 v_1v_2 - \omega_{12}^2(v_2)^2, \quad \dot v_2 = \omega_{12}^1(v_1)^2 + \omega_{12}^2 v_1v_2$$

where

$$\omega_{12}^2 = -\frac{d\omega^1\wedge\omega^3\wedge\cdots\wedge\omega^n}{\omega^1\wedge\cdots\wedge\omega^n} \quad \text{and} \quad \omega_{12}^1 = -\frac{d\omega^2\wedge\omega^3\wedge\cdots\wedge\omega^n}{\omega^1\wedge\cdots\wedge\omega^n}. \tag{2.8}$$

PROOF. Since ω provides a basis for the cotangent space, we can express each component of the Levi-Civita connection one-form ω_{ij} as $\omega_{ij}^1\omega^1+\cdots+\omega_{ij}^n\omega^n$. Because $s = 2$ we only need ω_{12} and since the path c is tangent to E we only need $\omega_{12}^1\omega^1$ and $\omega_{12}^2\omega^2$. Using the structure equations we compute $d\omega^1\wedge\omega^3\wedge\cdots\wedge\omega^n = -\omega_{12}^1\omega^1\wedge\cdots\wedge\omega^n$ and $d\omega^2\wedge\omega^3\cdots\wedge\omega^n = \omega_{12}^2\omega^1\wedge\cdots\wedge\omega^n$. ☐

[This is easily modified for rank s distributions with $s > 2$. One obtains a linear system of equations for the required coefficients of the Levi-Civita connection.]

We take $g = (\eta^1)^2 + \cdots + (\eta^5)^2$ making the given frame orthonormal. Since $d\eta^1 = d\eta^2 = 0$, the previous proposition implies that the nonholonomic geodesic equations are $\dot v^1 = \dot v^2 = 0$. Hence

$$(\dot x_1, \dot x_2, \dot x_3, \dot x_4, \dot x_5) = Ae_1 + Be_2$$

which are equivalent to the system of equations

$$\begin{pmatrix} \dot x_1 \\ \dot x_2 \\ \dot x_3 \\ \dot x_4 \\ \dot x_5 \end{pmatrix} = \begin{pmatrix} A \\ B \\ x_1\dot x_2 - x_2\dot x_1 \\ x_1^2\dot x_2 - x_1x_2\dot x_1 \\ -x_2^2\dot x_1 + x_1x_2\dot x_2 \end{pmatrix}$$

whose solutions give the nonhonolomic geodesics in terms of polynomials of up to third degree (for x_4, x_5).

REMARK 2.2. Cartan introduced in his thesis [26], and developed in two very famous papers [27,28], the simplest exceptional complex Lie algebra $\mathcal{G}_2$. It turns out that E is the maximally symmetric 2-3-5 distribution and that its Lie algebra of symmetries is precisely the 14-dimensional, non-compact real form of $\mathcal{G}_2$.

Cartan's work has been discussed in the 1990's by Bryant and Hsu [22,23] and more recently in Agrachev, [1], Bor and Montgomey [12], and Baez [4,5]. Two ways have been found to realize "concretely" a Lie group for $\mathcal{G}_2$. One is via octonions and the other as the symmetries of the rolling distribution, without slip or twist, of two spheres in the ratio 1:3. It seems that Cartan never explicitly discussed rolling, but it is implicit in his work. Bryant pointed out that for the 1:3 (or 3:1, it does not matter) rolling distribution, Cartan's tensor F (the lowest order invariant for 2-plane fields in $\mathfrak{R}^5$) vanishes, thus it is locally equivalent to 'flat' distribution from Cartan's 1910 paper-the nilpotent 2-3-5 as we presented above.

3. G-structures in nonholonomic geometry

At the 1928 ICM Élie Cartan described the transformations of coframes on a manifold leading to the *same nh connection*. In modern language, a family of coframes together with a matrix group G of transformation is called a G-structure. It is a sub-bundle of the coframe bundle $\pi\colon F^*(Q) \to Q$, as a $Gl(n)$ principal bundle [50,75]. Both for subriemannian (SR) or nonholonomic (NH) geometry, the story begins with the change of coframes

$$\begin{pmatrix} \bar\omega^i \\ \bar\omega^\alpha \end{pmatrix} = \begin{pmatrix} C & B \\ 0 & A \end{pmatrix} \begin{pmatrix} \omega^i \\ \omega^\alpha \end{pmatrix} \tag{3.1}$$

with

$$C \in O(s), \quad A \in Gl(n-s), \quad B \in M(s, n-s).$$

3.1. "Master packages". We now derive the conditions for strong and weak equivalence. Dually to (3.1), the frames change via

$$(e_i, e_\alpha) = (\bar e_i, \bar e_\alpha) \begin{pmatrix} C & B \\ 0 & A \end{pmatrix}.$$

We can take C = identity without loss of generality. They correspond to changing the metric in Q but keeping it unaltered inside E. In the case of SR geometry, there are no restrictions in these matrices, this is the initial G-structure. For nh systems things are more subtle. Proposition 2.4 implies

PROPOSITION 3.1. i) *Strong equivalence: the nh connection is preserved under the change of metric if and only if for all* $1 \leqslant i, j, k \leqslant s$.

$$\bar{\Gamma}^k_{ij} = \Gamma^k_{ij}.$$

ii) *Weak equivalence: the trajectories are preserved under the change of metric if and only if*

$$\bar{c}^i_{kj} = c^i_{kj} + \textit{skew symmetric stuff in } i, j.$$

We thank Prof. Waldyr Oliva for calling our attention to the additional flexibility in the weak notion. It is due to the "nota bene" in section 2.4.

To find the transformation rules we take C = identity, so $e_i = \bar{e}_i$, $1 \leqslant i \leqslant s$ and re-compute the structure constants:

$$[\bar{e}_\ell, \bar{e}_i] = [e_\ell, e_i] = c^j_{\ell i} e_j + c^\alpha_{\ell i}(B_{j\alpha}\bar{e}_j + \text{stuff}\, \bar{e}_\lambda) = (c^j_{\ell i} + c^\alpha_{\ell i} B_{j\alpha})\bar{e}_j + \text{stuff}\, \bar{e}_\lambda.$$

We get immediately that for $1 \leqslant i, j, \ell \leqslant s$, and sum over $s+1 \leqslant \alpha \leqslant n$:

PROPOSITION 3.2.

$$c^j_{\ell i} \to \bar{c}^j_{\ell i} = c^j_{\ell i} + c^\alpha_{\ell i} B_{j\alpha}$$

(*which clearly is still skew symmetric with respect to indices* ℓ, i.)

$$\Gamma^j_{ik} \to \bar{\Gamma}^j_{ik} = \Gamma^j_{ik} + \frac{1}{2}\left(c^\alpha_{jk} B_{i\alpha} - c^\alpha_{ki} B_{j\alpha} + c^\alpha_{ji} B_{k\alpha}\right)$$

(*which clearly is still skew symmetric with respect to indices* j, k.)

PROPOSITION 3.3 (Master equations). *The "package of conditions" for nh connection equivalence is*

$$c^\alpha_{jk} B_{i\alpha} - c^\alpha_{ki} B_{j\alpha} + c^\alpha_{ji} B_{k\alpha} = 0. \tag{3.2}$$

For the straightest paths equivalence the conditions are

$$c^\alpha_{\ell i} B_{j\alpha} + c^\alpha_{\ell j} B_{i\alpha} = 0. \tag{3.3}$$

For a Chaplygin system where $Q \to S$ is a principal bundle, B and the $c^\alpha_{\ell j}$ will be functions of the base variable.

Since the preservation of the nh connection implies that the trajectories are maintained, (3.2) implies (3.3).

A question then arises: in what circumstances does (3.3) *in turn imply* (3.2)*, i.e. the two equivalence notions coincide?*

We present situations where the notions coincide in the next section 4. The counter example by Terra an Oliva [76] is reviewed in section 4.6.

3.2. Change of metric preserving the nh connection (Cartan's way). Both sets of conditions as given by the above proposition 3.3 seem difficult to analyze. The package (3.2) for nh connection equivalence looks specially diabolical. Probably, this is why Cartan chose to work with coframes structure equations. He focused only on the nh connection equivalence. Let us now review his approach.

In the previous section 2.1 we defined the *first derived ideal* $\mathcal{I}^{(1)}$. Recall that it is formed by combinations $\theta = \sum a_\alpha \omega^\alpha$ (coefficients are functions in Q) of the ω^α such that $d\theta \in \mathcal{I}$. This means that the 2-form $d\theta$, as a 2-form, can be written in terms of wedge products where every term contains *at least one* of the ω^α's; in other words, there are no terms of the form $\omega^i \wedge \omega^j$. Equivalently, $d\theta$ vanishes when applied to a pair of vectors in E. This is clear from Cartan's magic formula.

We mentioned in the introduction that Cartan observed that for nh systems, there must be restrictions on B in order to preserve the nh connection. He proved that for strongly nonholonomic (Snh) distributions, $B \equiv 0$. Thus, in that case, F must remain perpendicular to E in the new metric, although one can change the metric at will in F. *Probably the reason that Cartan was only interested in the nonholonomic connection (and not the trajectories) is because he showed that there is a unique (intrinsically defined!) special metric in F.* A more general result was presented in [63]. Let $r = \dim(\mathcal{I}^{(1)})$, so

$$\dim(E + [E, E]) = n - r.$$

As we mentioned before, we subdivide the $n - s$ Greek indices into r capital Greek indices Φ and $t = n - s - r$ lower case Greek (those listed as the last batch of objects). In the sequel, when we say "all the Greek", it means both lower and upper case.

Definition 3.1. The *specially adapted* frames for TQ

$$(e_i, e_\Phi, e_\phi), \quad 1 \leqslant i \leqslant s, \quad s + 1 \leqslant \Phi \leqslant s + r, \quad s + r + 1 \leqslant \phi \leqslant n \tag{3.4}$$

are those that satisfy the following requirements:

i) The e_i span E.
ii) The e_ϕ generate a complement F of E in $E + [E, E]$ (thus (e_i, e_ϕ) generate $E + [E, E]$, which is annihilated by $\mathcal{I}^{(1)}$).
iii) The e_Φ are chosen to complete the full frame for TQ (we placed them in the middle sector for convenience).

The dual coframe of the frame just constructed is denoted

$$(\omega^i, \omega^\Phi, \omega^\phi)^\dagger, \quad 1 \leqslant i \leqslant s, \quad s + 1 \leqslant \Phi \leqslant s + r, \quad s + r + 1 \leqslant \phi \leqslant n.$$

Notice that by duality, the $\omega^\Phi \in \mathcal{I}^{(1)}$. The last forms, in lower case Greek, are in $\mathcal{I}$ but not in $\mathcal{I}^{(1)}$. The most general change of frames among specially adapted ones is of the form

$$(e_i, e_\Phi, e_\phi) = (\bar{e}_i, \bar{e}_\Phi, \bar{e}_\phi) \begin{pmatrix} C & B_1 & B_2 \\ 0 & A_o & 0_{2,3} \\ 0 & A_1 & A_2 \end{pmatrix} \tag{3.5}$$

where the 0_{23} block is zero by construction, that is: the first and last batches still generate $E + [E, E]$. Also $C \in O(s)$, $A_o \in Gl(r)$, $A_2 \in Gl(n - s - r)$.

It is not hard to check that all possible metrics in Q are contemplated by this construction: given any frame $\{f_I\}_{I=1,\dots n}$ in Q, there is an orthogonal matrix $R \in O(n)$ such that the rotated frame $\{f_I\} \cdot R$ has the first s vectors in E and the first $s + r$ vectors in $E + [E, E]$.

The change of coframes writes as

$$\begin{pmatrix} \bar{\omega}^i \\ \bar{\omega}^\Phi \\ \bar{\omega}^\alpha \end{pmatrix} = \begin{pmatrix} C & B_1 & B_2 \\ 0 & A_o & 0_{2,3} \\ 0 & A_1 & A_2 \end{pmatrix} \begin{pmatrix} \omega^i \\ \omega^\Phi \\ \omega^\alpha \end{pmatrix}. \tag{3.6}$$

THEOREM 3.1. [63] *In the notation of* (3.5) *and* (3.6)*, make* C = identity *without loss in generality. The condition for the nh connection to be the same for the two metrics, determined by the corresponding coframes, is*

$$B_2 = 0.$$

For the proof, we must compare the structure equations

$$d\omega^i = -\omega_{ij} \wedge \omega^j - \omega_{i\alpha} \wedge \omega^\alpha, \quad d\bar{\omega}^i = -\bar{\omega}_{ij} \wedge \bar{\omega}^j - \bar{\omega}_{i\alpha} \wedge \bar{\omega}^\alpha. \tag{3.7}$$

DEFINITION 3.2. We say that forms (on any degree) are E-equivalent if their difference is in $\mathcal{I}$ and denote $\sim_E$. So anything containing one of the Greeks (the ω^α's or the ω^Φ's) can be dropped in terms of the equivalence relation.

LEMMA 3.1. *The nh connection is preserved if and only if* $d\bar{\omega}^i \sim_E d\omega^i$.

PROOF. By (3.6) with C = identity, we have $e_i = \bar{e}_i$. In view of (2.1) and (2.7) the connection is preserved by the change of metrics if and only if

$$\omega_{ij} \sim_E \bar{\omega}_{ij}.$$

So in view of (3.7) we get $d\bar{\omega}^i \sim_E d\omega^i$. □

PROOF OF THEOREM 3.1. Since $\bar{\omega}^i = \omega^i + (B_1)_{i\Phi}\omega^\Phi + (B_2)_{i\alpha}\omega^\alpha$ it follows from the lemma that $d[(B_1)_{i\Phi}\omega^\Phi + (B_2)_{i\alpha}\omega^\alpha] \sim_E 0$. Now,

$$d[(B_1)_{i\Phi}\omega^\Phi] = d(B_1)_{i\Phi} \wedge \omega^\Phi + (B_1)_{i\Phi} d\omega^\Phi \sim_E 0$$

as $d\omega^\Phi \sim_E 0$ (because by choice $\omega^\Phi \in \mathcal{I}^{(1)}$). It follows that

$$d[(B_2)_{i\alpha} \wedge \omega^\alpha] = (B_2)_{i\alpha} \wedge \omega^\alpha \sim_E 0.$$

This means that the 1-form given by the combination $(B_2)_{i\alpha} d\omega^\alpha$ is in $\mathcal{I}^{(1)}$. But unless all the coefficients vanish this is impossible. We exhausted the generators of $\mathcal{I}^{(1)}$ with the upper case Greek indices. Hence $B_2 = 0$. It was once said that when basic theorems are interesting they are trivial to prove. □

SUMMARY. In subriemannian geometry one can add *any* combination of forms in $\mathcal{I}$ to the ω_i. In nonholonomic geometry we can still add forms in $\mathcal{I}^{(1)}$ and preserve the connection. Thus nh geometry has some degree of "near-sightedness".

3.3. Initial G-structure for Nh connection equivalence. The condition $B_2 = 0$ means geometrically that the orthogonal complement F of E in $E^{(1)}$ cannot be changed. But the metric in F can be altered. Any complement H of $E^{(1)}$ to TQ can be declared orthogonal to $E^{(1)}$ without affecting the connection. From a different perspective, look back at the Levi-Civita connection of Q, restricted to the latin indices

$$\nabla_{e_k} e_j = \omega_{ij}(e_k)e_i + \omega_{\Phi j}(e_k)e_\Phi + \omega_{\alpha j}(e_k)e_\alpha. \tag{3.8}$$

In general the $\omega_{\Phi j}(e_k)e_\Phi \in H$ do not vanish, and they are outside $E + [E, E]$. This is not harmful, since the nh connections are obtained by *dropping out all the Greeks* e_Φ and e_α terms in (3.8). The first terms, defining the nh connections are the same in both metrics.

THEOREM 3.2. *The initial G-structure for the equivalence problem for Nh connections $D_X Y$ is the sub-bundle on the frame bundle consisting of coframes*

$$\omega = (\omega_i, \omega^\Phi, \omega^\phi)^\dagger, \qquad 1 \leqslant i \leqslant s, \quad s+1 \leqslant \Phi \leqslant s+r, \quad s+r+1 \leqslant \phi \leqslant n$$

where ω^Φ, ω^ϕ annihilate E and ω^Φ annihilate $E^1 = E + [E, E]$. The initial G-group acting on ω consists of matrices of the form

$$\begin{pmatrix} C & B_1 & 0 \\ 0 & A_o & 0 \\ 0 & A_1 & A_2 \end{pmatrix}$$

with $C \in O(s)$, $B_1 \in M(s, r)$, $A_o \in Gl(r, r)$, $A_1 \in M(n - s - r, r)$, $A_2 \in Gl(n - s - r)$.

Dimension count. Let $\dim E = s$, $\dim F = t$, $\dim H = r$, so that $n = \dim Q = r + s + t$. The number of independent functions is

$$s(s+1)/2 + t(t+1)/2 + r(s+t).$$

Revisiting example 2.5. Here $\mathcal{I} = \mathcal{I}^0 = \text{span}\{\eta^3, \eta^4, \eta^5\}$ and we can compute

$$d\eta^1 = dx_1 \wedge dx_2 \not\equiv 0, \quad d\eta^4 = dx_1 \wedge \eta^3 \equiv 0, \quad d\eta^5 = dx_2 \wedge \eta^3 \equiv 0 \quad (\text{all} \mod \mathcal{I}),$$

hence the uppercase Greeks are $\mathcal{I}^1 = \text{span}\{\eta^4, \eta^5\}$. Furthermore we can check that $d\eta^4$ and $d\eta^5 \not\equiv 0 \bmod I^1$ so that $\mathcal{I}^2 = 0$.

The column vector $(\eta^1, \eta^2, \eta^3, \eta^4, \eta^5)^{tr}$ is thus an adapted coframe, where for convenience we kept the uppercase Greeks η^4, η^5 at the end of the list. The initial $G_0 \subset Gl(5)$ consists of all matrices of the form

$$\begin{pmatrix} C_{11} & C_{12} & 0 & b_{14} & b_{15} \\ C_{21} & C_{22} & 0 & b_{24} & b_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{54} & a_{55} \end{pmatrix}.$$

In light of proposition 2.8, Cartan's condition is easy to appreciate. If we let C be the identity matrix and replace η_1 with $\tilde{\eta}_1 = \eta_1 + b_{14}\eta_4 + b_{15}\eta_5$ and η_2 with

$\tilde{\eta}_2 = \eta_2 + b_{24}\eta_4 + b_{25}\eta_5$, then the coefficients α^2_{12} and α^1_{12} in the motion equations (2.5) remain unchanged since

$$d\eta^4 \wedge \eta^3 \wedge \eta^4 \wedge \eta^5 = d\eta^5 \wedge \eta^3 \wedge \eta^4 \wedge \eta^5 = 0.$$

4. Remarks on weak equivalence-preserving the acceleration $D_X X$

Cartan's equivalence criterion-to preserve the nh connection, obviously implies that their nonholonomic paths are the same. Much to our surprise, Terra and Oliva have recently found an example of a change of metric that does not preserve the nh connection but still preserves the paths. We will discuss their example at the end of this section. First, we will make some considerations indicating that this is an "extreme event'. We start by showing that when $\dim E = 2$, preservation of trajectories implies, conversely, preservation of the connection. In [37, 38, 64], the cases of, respectively, 2-3, 2-3-5, and 2-3-4 distribution growth were studied.

4.1. Rank two distributions.

Proposition 4.1. *For any nonintegrable distribution E of rank 2 on a manifold Q of dimension n endowed with an arbitrary Riemannian metric, the straightest path equations are preserved if and only if $B_2 = 0$.*

Proof. Theorem 3.1 yields sufficiency. To show necessity, let $\omega = (\omega^1, \dots, \omega^n)^\dagger$, where † indicates transpose, be an specially adapted coframe (3.4) in Definition 3.1, with $\mathcal{I} = \mathrm{span}\{\omega^3, \dots, \omega^n\}$, $\mathcal{I}^{(1)} = \mathrm{span}\{\omega^3, \dots, \omega^{n-1}\}$, and $g = (\omega^1)^2 + \dots + (\omega^n)^2$. Here $\mathcal{I}^{(1)}$ is the annihilator of the three-dimensional distribution $E^{(1)} = E + [E, E]$, and $\omega^n \notin \mathcal{I}^{(1)}$. Note that in the special case $n = 3$, $\mathcal{I}^{(1)} = 0$ and $\omega^3 \notin I^{(1)}$.

The structure equations for ω are

$$d\begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \dots \\ \omega^{n-1} \\ \omega^n \end{pmatrix} \equiv \begin{pmatrix} f_1\omega^1 \wedge \omega^2 \\ f_2\omega^1 \wedge \omega^2 \\ 0 \\ \dots \\ 0 \\ f_n\omega^1 \wedge \omega^2 \end{pmatrix} \quad (\mathrm{mod}\, I).$$

Note that $f_n \neq 0$ since $\omega^n \notin I^{(1)}$. Applying Proposition 2.3, the straightest paths equations for the quasivelocities v_1 and v_2 are

$$\begin{aligned} \dot{v}_1 &= f_2 v_1 v_2 + f_1 (v_2)^2 \\ \dot{v}_2 &= -f_2 (v_1)^2 + f_1 v_1 v_2. \end{aligned} \tag{4.1}$$

Now suppose that in equation (3.6) $B_2 = (b_1, b_2)^\dagger$ and, without loss of generality, C, A_0, and A_2 are identity matrices, and that $A_1 = 0$. The modified coframe is thus $\tilde{\omega} = (\tilde{\omega^1}, \dots, \tilde{\omega}^n)^\dagger = (\omega^1 + b_1\omega^n, \omega^2 + b_2\omega^n, \omega^3, \dots, \omega^n)^\dagger$ with associated modified metric $(\tilde{\omega}^1)^2 + \dots + (\tilde{\omega}^n)^2$. Note that the quasivelocities v_1 and v_2 are unchanged.

The structure equations for $\tilde{\omega}$ are

$$d\begin{pmatrix}\tilde\omega^1\\ \tilde\omega^2\\ \tilde\omega^3\\ \cdots\\ \tilde\omega^{n-1}\\ \tilde\omega^n\end{pmatrix} \equiv \begin{pmatrix}(f_1+b_1f_n)\tilde\omega^1\wedge\tilde\omega^2\\ (f_2+b_2f_n)\tilde\omega^1\wedge\tilde\omega^2\\ 0\\ \cdots\\ 0\\ f_n\tilde\omega^1\wedge\tilde\omega^2\end{pmatrix} \quad (\mathrm{mod}\, I).$$

The straightest path equations for (the same) quasivelocities are now

$$\dot v_1 = (f_2+b_2f_n)v_1v_2 + (f_1+b_1f_n)(v_2)^2$$
$$\dot v_2 = -(f_2+b_2f_n)(v_1)^2 + (f_1+b_1f_n)v_1v_2.$$

Comparing with the original equations (4.1) and recalling that $f_n \neq 0$, we can see that it is necessary for $b_1 = b_2 = 0$ to preserve the nonholonomic paths. □

4.2. "Master equation" for keeping the straightest paths, revisited. Recall that the nonholonomic dynamics in the Euler–Arnold format is given by

$$\dot v_i = c^j_{\ell i}(q)v_jv_\ell, \quad 1\leqslant i,j,\ell\leqslant s, \quad \dot q = v_ie_i, \tag{4.2}$$

where the $e_i \in E$ $(i = 1,\dots s)$ are part of a full orthonormal frame e_J $(J = 1,\dots n)$, with

$$[e_I, e_J] = c^K_{IJ}e_K.$$

Recall that we change metric as follows. An orthonormal basis $\{\bar e_i, \bar e_\alpha\}$ for the new metric is related to $\{e_i, e_\alpha\}$ by

$$e_j = \bar e_iC_{ij}, \quad e_\alpha = \bar e_jB_{j\alpha} + \bar e_\lambda A_{\lambda\alpha}. \tag{4.3}$$

Here B is $s\times r$ and A is $r\times r$, functions of $q\in Q$. We take C =identity, so $e_i = \bar e_i$.

Definition 4.1. For each $i = 1,\dots s$, denote by C^i the $(n-s)\times s$ matrix

$$(C^i)_{\alpha\ell} = c^\alpha_{\ell i}.$$

We call them "weights". Note that the last r rows of those matrices are zero since we are working with specially adapted frames.

We get immediately from the package of conditions (3.3):

Proposition 4.2 (Master Equation for Weak Equivalence). *The necessary and sufficient condition for a change of metrics (4.3) to preserve the straightest paths equation (4.2) is:*

All the $s\times s$ "weighted matrices" BC^i for $i = 1,\dots s$ are skew symmetric.

Recall that this means

$$\sum_\alpha (c^\alpha_{\ell i}B_{j\alpha} + c^\alpha_{ji}B_{\ell\alpha}) = 0, \quad \text{for all} \quad 1\leqslant j,\ell\leqslant s \quad \text{and} \quad 1\leqslant i\leqslant s. \tag{4.4}$$

As we mentioned before, the skew symmetry requirement is natural since any skew symmetric bilinear form $v^\dagger Aw$ vanishes when $v = w$. In hindsight this proposition results from the Euler-Arnold form for the nonholonomic dynamics, using the $c^k_{ij}(q)$ instead of the $\Gamma^k_{ij}(q)$.

4.3. There are more equations than unknowns. Let $r = \dim \mathcal{I}^{(1)}$, so $t = n-s-r$ is the dimension of the complement F of E in $E^{(1)}$. This is the number of the α indices that matter forming matrix B_2. Note that for all the remaining α's we have $c^{\alpha}_{ij} = 0$. We differentiated those latter in Definition 3.1 by using upper case Greek letters (and indeed there are r of them).

Each matrix C^i has two blocks. The upper, which we call $\bar{C}^i$, is the relevant one, of size $(n-s-r)\times s$. The lower block is of size $r\times s$ and is filled with zeros. Therefore, in computing the products BC^i, we should divide matrix B in two blocks. The left one, which we call B_2, is of size $s\times(n-s-r)$ and meets the upper blocks $\bar{C}^i$. The right one, which we call B_1, is of size $s\times r$.

We can see that there are no conditions for B_1 since it meets the lower part of the C^i which are zero. The skew symmetry conditions (4.4) are limited to the small Greek indices α's, and apply to block B_2, of size $s\times(n-s-r)$. The number of unknowns is $u = s(n-s-r)$. On the other hand, the number p of equations encoded in $\bar{C}^i B_2 + (\bar{C}^i B_2)^T = 0$ is

$$p = s(s(s+1)/2) = s^2(s+1)/2.$$

We could assert for sure that there are nonzero solutions for B_2 if there were more unknowns than equations, $u > p$, i.e., if $s\times(n-s-r) > s^2(s+1)/2$, or equivalently, if

$$n > s(s+1)/2 + s + r \quad (*).$$

However, this is never the case. Since $\dim E^{(1)} \leqslant s + s(s-1)/2 = s(s+1)/2$ (the maximum number of new vectorfields that one can form with the brackets) we get

$$n \leqslant r + s(s+1)/2$$

which precludes (*). So there are always more equations than unknowns.

4.4. Corank 1, strongly nh, distributions. Here $s = n-1, r = 0$. Is it possible to find an example of nh geodesics equivalence with $B \neq 0$? We indicate why not using the Master Equations. We will give a more general argument in the next subsection, using Cartan's viewpoint. To begin with, note that there are far more conditions ($p = (n-1)^2 n/2$) than unknowns ($u = n-1$).

Matrices $C^i, 1 \leqslant i \leqslant n-1$ are row-vectors of size $n-1$

$$C^i = (c^n_{1i}, c^n_{2i}, \ldots, c^n_{n-1,i})$$

and the matrix $B_2 = B$ is a column vector of size $n-1$

$$B = (b_1, \ldots b_{n-1})^{\dagger}.$$

The product BC^j is a $(n-1)\times(n-1)$ matrix whose i-th row ($1 \leqslant i \leqslant n-1$) is

$$b_i(c^n_{1j}, c^n_{2j}, \ldots, c^n_{n-1,j}).$$

For the nh connection equivalence we know already that all $b_i, i = 1, \ldots, s = n-1$ must vanish since the distribution is Snh. But it would be interesting to have a direct argument using the conditions (3.2). Since there is only one $\alpha = n$, the usual alternating trick shows that every term in (3.2) vanishes, i.e. all $c^n_{jk} b_i = 0$. Since some $c^n_{ik} \neq 0$, we conclude that all $b_i = 0$.

It is a little bit more difficult to analyze the conditions for keeping the straightest paths (i.e., weak equivalence)

$$b_i c^n_{kj} + b_k c^n_{ij} = 0 \tag{4.5}$$

for all $1 \leqslant i, j, k \leqslant n-1$. It is convenient to form the skew symmetric matrix

$$\hat{C} = (c^n_{ij}).$$

Making $k = j$ in (4.5) kills automatically the first term and we get $b_j c^n_{ij} = 0$ (not a sum over j, it is just a single product).

Proposition 4.3. *If $c^n_{ij} \neq 0$ then $b_j = b_i = 0$.*

If there is an example with a $b_j \neq 0$, then the whole j-th column (so also the j-th row) of $\hat{C}$ must vanish. Conversely, suppose the j-th colum of $\hat{C}$ is formed by zeros (so also the j-th row). Then the equation (4.5) for that value j is void. It seems therefore that the most favorable case for the possibility of having a nonzero B is when *only one* structure coefficient is nonzero.

At first sight it seems that that pair of $b's$ vanishes and all the other could be arbitrary. We now show that (4.5) forces a lot of others (we think that this argument gives all) to be zero too.

When one structure coefficient is nonzero we have the group $\mathfrak{R}^m \times H_3$, where H_3 is the smallest Heisenberg group. Denote $(y_1, \ldots, y_m, x, y, z)$ the coordinates. The corank 1, strongly nh distribution E is generated by the $\partial/\partial y_i$, together with

$$X_1 = \partial/\partial x, \quad X_2 = \partial/\partial y + x\partial/\partial z.$$

The only nonzero commutator is $[X_1, X_2] = \partial/\partial z$.

B is a column matrix with $m+2$ entries. Let us look at the first m entries $b_i, i = 1, \ldots m$. The second term in (4.5) vanishes because all the possible $c^*_{i*} = 0$. But we may take $k = m+1$, $j = m+2$ and since $c^{m+3}_{m+1,m+2} = 1$ we conclude that $b_i = 0, i = 1, \ldots m$. In order to show that b_{m+1} and b_{m+2} also vanish in the nh geodesic equivalence we now look directly at H_3. More generally, the reasoning is valid for all Heisenberg algebras $H_n, n = 2m+1$.

The last coordinate is called z. The distribution of rank $n-1 = 2m$ (even) is generated by vectorfieds P_i's and Q_j's with $[P_i, Q_j] = \delta_{ij}\partial/\partial z$, $1 \leqslant i, j \leqslant m$. Conditions (4.5) lead to $B = 0$. This is because when we make $i = k$, we get $2b_k c^n_{kj} = 0$ (again, a plain product). But for every k there exists one (unique) j with $c^n_{kj} \neq 0$.

Therefore for the standard metric in the Heisenberg distribution, there is no difference between preserving the connection and preserving the straightest paths.

4.5. Corank of E in $E^{(1)} = E + [E, E] < 3$. The reasoning we now present takes care of the previous case $s = n-1$ and many others. It implies for instance that to find examples where the straightest paths are the same in non-equivalent nH connections, the corank of E in $E + [E, E]$ must be at least 3, no matter what the rank of E is. So such examples can happen in (3,6), (4,7,...),(4,8,...), etc. but not in (2,3), (2,3,4), (2,3,5), (3,5), etc. The minimal example is in $(3, 6)$, precisely the situation in [76].

We present the argument for a rank 3 distribution just to write things out explicitly. We show that to preserve the straightest paths, covectors that are not in $\mathcal{I}^1$ cannot be added to the $\omega_i, i = 1,2,3$ if $E + [E,E]$ has dimension 4 or 5.

Proposition 4.4. *Suppose E is a totally nonholonomic, rank 3 distribution on a manifold Q of dimension $n > 3$. If the corank of $E \subset E + [E,E]$ is one or two, then the block B_2 in (3.6) is zero.*

Proof. We slightly change the notational conventions as follows. Suppose ω is a specially adapted coframe for this distribution for which ω_ϕ, $4 \leqslant \phi \leqslant r$, span $\mathcal{I}^{(1)}$. When can we add multiples of ω_Φ, $r+1 \leqslant \Phi \leqslant n$ to the ω_i, $1 \leqslant i \leqslant 3$, while preserving the straightest path equations?

Let $[\omega_{ij}]$ be the Levi-Civita connection forms matrix relative to the coframe ω. Expand the components of ω as $\omega_{ij} = \sum_{k=1}^n \omega_{ij}^k \eta^k$. The nh trajectories are then

$$
\begin{aligned}
\dot v_1 &= -v_2(\omega_{12}^1 v_1 + \omega_{12}^2 v_2 + \omega_{12}^3 v_3) - v_3(\omega_{13}^1 v_1 + \omega_{13}^2 v_2 + \omega_{13}^3 v_3)\\
\dot v_2 &= v_1(\omega_{12}^1 v_1 + \omega_{12}^2 v_2 + \omega_{12}^3 v_3) - v_3(\omega_{23}^1 v_1 + \omega_{23}^2 v_2 + \omega_{23}^3 v_3)\\
\dot v_3 &= v_1(\omega_{13}^1 v_1 + \omega_{13}^2 v_2 + \omega_{13}^3 v_3) - v_2(\omega_{23}^1 v_1 + \omega_{23}^2 v_2 + \omega_{23}^3 v_3).
\end{aligned}
\tag{4.6}
$$

Now suppose $B_2 \neq 0$ and we modify ω by adding linear combinations $M_i = \sum a_\Phi \omega_\Phi$ to the ω_i, $1 \leqslant i \leqslant 3$, to obtain a new coframe $\tilde\omega$. The distribution E is preserved, but the metric has changed. Denote the Levi-Civita connection form relative to the modified coframe by $\tilde\omega = [\tilde\omega_{ij}]$ with components $\tilde\omega_{ij} = \sum_{k=1}^n \tilde\omega_{ij}^k \eta^k$.

The equations for the quasivelocities (the same as above) are now

$$
\begin{aligned}
\dot v_1 &= -v_2(\tilde\omega_{12}^1 v_1 + \tilde\omega_{12}^2 v_2 + \tilde\omega_{12}^3 v_3) - v_3(\tilde\omega_{13}^1 v_1 + \tilde\omega_{13}^2 v_2 + \tilde\omega_{13}^3 v_3)\\
\dot v_2 &= v_1(\tilde\omega_{12}^1 v_1 + \tilde\omega_{12}^2 v_2 + \tilde\omega_{12}^3 v_3) - v_3(\tilde\omega_{23}^1 v_1 + \tilde\omega_{23}^2 v_2 + \tilde\omega_{23}^3 v_3)\\
\dot v_3 &= v_1(\tilde\omega_{13}^1 v_1 + \tilde\omega_{13}^2 v_2 + \tilde\omega_{13}^3 v_3) - v_2(\tilde\omega_{23}^1 v_1 + \tilde\omega_{23}^2 v_2 + \tilde\omega_{23}^3 v_3).
\end{aligned}
\tag{4.7}
$$

Subtracting the expressions for $\dot v_1$ in (4.6) and (4.7) and equating the coefficients in $v_i v_j$ to zero we obtain

$$\omega_{12}^1 - \tilde\omega_{12}^1 = \omega_{12}^2 - \tilde\omega_{12}^2 = \omega_{13}^1 - \tilde\omega_{13}^1 = \omega_{13}^3 - \tilde\omega_{13}^3 = 0$$

and

$$\omega_{12}^3 - \tilde\omega_{12}^3 = -(\omega_{13}^2 - \tilde\omega_{13}^2). \tag{4.8}$$

Doing the same for $\dot v_2$ and $\dot v_3$ we find that

$$\omega_{12}^3 - \tilde\omega_{12}^3 = (\omega_{23}^1 - \tilde\omega_{23}^1) \tag{4.9}$$

$$\omega_{13}^2 - \tilde\omega_{13}^2 = -(\omega_{23}^1 - \tilde\omega_{23}^1) \tag{4.10}$$

with the difference between all other pairs equal to zero.

Equations (4.8), (4.9), and (4.10) are not independent and we can write

$$\omega_{12}^3 - \tilde\omega_{12}^3 = -(\omega_{13}^2 - \tilde\omega_{13}^2) = \omega_{23}^1 - \tilde\omega_{23}^1 = Q. \tag{4.11}$$

We know that connection forms ω and $\tilde{\omega} = \omega + \Delta$ lead to the same nH trajectories if and only if

$$\Delta = \begin{pmatrix} 0 & Q\eta^3 & -Q\eta^2 & * & * \\ -Q\eta^3 & 0 & Q\eta^1 & * & * \\ Q\eta^2 & -Q\eta^1 & 0 & * & * \\ ** & * & * & 0 & * \end{pmatrix} \quad (\operatorname{mod} I).$$

Since $d\tilde{\omega} = -\tilde{\omega} \wedge \tilde{\omega} = -(\omega + \Delta) \wedge \tilde{\omega}$ we must have

$$d \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix} \equiv \begin{pmatrix} -2Q\eta^2 \wedge \eta^3 \\ 2Q\eta^1 \wedge \eta^3 \\ -2Q\eta^1 \wedge \eta^2 \end{pmatrix} \quad (\operatorname{mod} I). \tag{4.12}$$

Since the dM_i are linearly independent, the M_i must be linearly independent. This is only possible if the corank of E in $E + [E, E]$ is three or more. □

The conclusion of this proposition is also true for distributions of rank greater than three. In this case (4.11) becomes

$$Q = \omega_{ij}^K - \tilde{\omega}_{ij}^k = -(\omega_{ik}^j - \tilde{\omega}_{ik}^j) = \omega_{jk}^i - \tilde{\omega}_{jk}^i$$

and, modulo $\eta - \{\eta^i, \eta^j, \eta^k\}$, (4.12) becomes

$$d \begin{pmatrix} M_i \\ M_j \\ M_k \end{pmatrix} \equiv \begin{pmatrix} -2Q\eta^j \wedge \eta^k \\ 2Q\eta^i \wedge \eta^k \\ -2Q\eta^i \wedge \eta^j \end{pmatrix} \quad (\operatorname{mod} I)$$

and the conclusion follows as before.

4.6. The example in $Q = SO(4)$ by Terra and Oliva [76]. In this example, $n = 6, s = r = 3$. First we will discuss it using the Cartan approach, and afterwards by a modification of the "master equations", which may be helpful in concocting other families of examples.

4.6.1. *Via Cartan equations.* Let $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ be a basis for $so(4)$ with

$$x_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$x_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad x_5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad x_6 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $X = \{X_1, X_2, X_3, X_4, X_5, X_6\}$ be the corresponding left invariant frame for TQ. Consider the bi-invariant metric g on $SO(4)$ induced by the Cartan–Killing form:

$$g(X_i, X_j) = -\frac{1}{2} \operatorname{tr}(x_i, x_j).$$

It is easy to see that relative to g, X is an orthonormal frame. The distribution $E = \mathrm{span}\{X_1, X_2, X_3\}$ is 1-step bracket generating with

$$[X_2, X_3] = X_4, \quad [X_3, X_1] = X_5, \quad [X_1, X_2] = X_6.$$

Let $\omega = [\omega^i]$ be the dual coframe with $\omega^i(X_j) = \delta_{ij}$. The structure equations are

$$\begin{aligned}
d\omega^1 &= -(-\omega^3 \wedge \omega^5 + \omega^2 \wedge \omega^6)\\
d\omega^2 &= -(-\omega^1 \wedge \omega^6 + \omega^3 \wedge \omega^4)\\
d\omega^3 &= -(\omega^1 \wedge \omega^5 - \omega^2 \wedge \omega^4)\\
d\omega^4 &= -(\omega^2 \wedge \omega^3 + \omega^5 \wedge \omega^6)\\
d\omega^5 &= -(-\omega^1 \wedge \omega^3 - \omega^4 \wedge \omega^6)\\
d\omega^6 &= -(\omega^1 \wedge \omega^2 + \omega^4 \wedge \omega^5).
\end{aligned}$$

$\mathcal{I}$ is spanned by $\{\omega^4, \omega^5, \omega^6\}$ and $\mathcal{I}^1 = 0$. The connection forms are

$$[\omega_{ij}] = \frac{1}{2}\begin{pmatrix} 0 & -\omega^6 & \omega^5 & 0 & -\omega^3 & \omega^2\\ \omega^6 & 0 & -\omega^4 & \omega^3 & 0 & -\omega^1\\ -\omega^5 & \omega^4 & 0 & -\omega^2 & \omega^1 & 0\\ 0 & -\omega^3 & -\omega^2 & 0 & -\omega^6 & \omega^5\\ \omega^3 & 0 & -\omega^1 & \omega^6 & 0 & -\omega^4\\ -\omega^2 & \omega^1 & 0 & -\omega^5 & \omega^4 & 0 \end{pmatrix}.$$

If c is a straightest path with $\dot c = v_1 X_1 + v_2 X_2 + v_3 X_3$, the nonholonomic equations are simply

$$\dot v_1 = \dot v_2 = \dot v_3 = 0 \quad (\text{since } \omega_{ij}(\dot c) = 0,\ 1 \leqslant i, j \leqslant 3).$$

For $0 < a < 1$ let G be a deformation of g given by (relative to the ordered basis X)

$$\begin{pmatrix} 1 & 0 & 0 & a & 0 & 0\\ 0 & 1 & 0 & 0 & a & 0\\ 0 & 0 & 1 & 0 & 0 & a\\ a & 0 & 0 & 1 & 0 & 0\\ 0 & a & 0 & 0 & 1 & 0\\ 0 & 0 & a & 0 & 0 & 1 \end{pmatrix}.$$

Since the distribution F spanned by X_4, X_5, X_6 is not orthogonal to E when $a \neq 0$, the nh connections are different. Terra and Oliva observed in [76] *that one gets the same nonholonomic paths.*

In terms of the original coframe ω, the coframe $\tilde\omega$ for this metric is

$$\begin{pmatrix} \tilde\omega^1\\ \tilde\omega^2\\ \tilde\omega^3\\ \tilde\omega^4\\ \tilde\omega^5\\ \tilde\omega^6 \end{pmatrix} = \begin{pmatrix} \omega^1 + a\omega^4\\ \omega^2 + a\omega^5\\ \omega^3 + a\omega^6\\ \sqrt{1-a^2}\,\omega^4\\ \sqrt{1-a^2}\,\omega^5\\ \sqrt{1-a^2}\,\omega^6 \end{pmatrix}.$$

Note that we have added to ω^1, ω^2, and ω^3 multiples of ω^4, ω^5, and ω^6, which are not in the first derived ideal. So the connection form restricted to E will change. True enough, the structure equations for $\tilde{\omega}$ are

$$\begin{aligned}
d\omega^1 &= -(-\omega^3 \wedge \omega^5 + \omega^2 \wedge \omega^6) - a(\omega^2 \wedge \omega^3 + \omega^5 \wedge \omega^6)\\
d\omega^2 &= -(-\omega^1 \wedge \omega^6 + \omega^3 \wedge \omega^4) - a(-\omega^1 \wedge \omega^3 - \omega^4 \wedge \omega^6)\\
d\omega^3 &= -(\omega^1 \wedge \omega^5 - \omega^2 \wedge \omega^4) - a(\omega^1 \wedge \omega^2 + \omega^4 \wedge \omega^5)\\
d\omega^4 &= -\sqrt{1-a^2}(\omega^2 \wedge \omega^3 + \omega^5 \wedge \omega^6)\\
d\omega^5 &= -\sqrt{1-a^2}(-\omega^1 \wedge \omega^3 - \omega^4 \wedge \omega^6)\\
d\omega^6 &= -\sqrt{1-a^2}(\omega^1 \wedge \omega^2 + \omega^4 \wedge \omega^5).
\end{aligned}$$

As expected the connection form, when restricted to E changes: mod $(\mathcal{I})$ we get

$$\tilde{\omega}_{12} = -\frac{a}{2}\tilde{\omega}^3, \quad \tilde{\omega}_{13} = \frac{a}{2}\tilde{\omega}^2, \quad \tilde{\omega}_{23} = -\frac{a}{2}\tilde{\omega}^1.$$

The nonholonomic equations for the deformed metric are then

$$\begin{aligned}
\dot{v}_1 &= -v_1\omega_{11}(\dot{c}) - v_2\omega_{12}(\dot{c}) - v_3\omega_{13}(\dot{c}) = -v_1(0) - v_2\Big(-\frac{a}{2}v_3\Big) - v_3\Big(\frac{a}{2}v_3\Big) = 0\\
\dot{v}_2 &= -v_1\omega_{21}(\dot{c}) - v_2\omega_{22}(\dot{c}) - v_3\omega_{23}(\dot{c}) = -v_1\Big(\frac{a}{2}v_3\Big) - v_2(0) - v_3\Big(-\frac{a}{2}v_1\Big) = 0\\
\dot{v}_3 &= -v_1\omega_{31}(\dot{c}) - v_2\omega_{32}(\dot{c}) - v_3\omega_{33}(\dot{c}) = -v_1\Big(-\frac{a}{2}v_2\Big) - v_2\Big(\frac{a}{2}v_1\Big) - v_3(0) = 0.
\end{aligned}$$

While the nonholonomic connection has changed, the changes cancel exactly in the straightest path equations. A mathematical gem indeed!

4.6.2. *Via Euler-Arnold equations* (2.6). With the expectation to concoct other examples, we invoke Proposition 2.1 on a Lie algebra of dimension n. We denote by G the matrix of the metric relative to a basis $e_I, i = 1, \ldots n$, and we denote by $\Omega_I, I = 1, \ldots, n$ the coordinates in this basis. Denote by P_G the orthogonal projection over E, spanned by the first vectors $i = 1, \ldots, s$. The equations for the nonholonomic system are obtained by applying P_G to the right hand side of $\dot{\Omega}$ in the Euler-Arnold equations for geodesics in the Lie group.

We now recall a Linear Algebra formula to project over a subspace generated by vectors $a_1, \ldots, a_s$. Let A be the $n \times s$ matrix whose columns are the a_i.

$$P_G(\Omega) = A(A^{\dagger}GA)^{-1}A^{\dagger}G\Omega.$$

This is often more convenient than a Gram–Schmidt procedure on the a_i.

When the $a_i = e_i$ are the first s vectors of the basis e_I, then

$$A = \begin{bmatrix} I_{s\times s} \\ 0_{r\times s} \end{bmatrix}, \quad r = n - s.$$

A simple computation yields that the projection has matrix

$$P_G = \begin{bmatrix} I & (G_{11})^{-1}G_{12} \\ 0 & 0 \end{bmatrix},$$

or simplifying the notation, by cutting the rows of zeros

$$P_G(\dot\Omega) = [I \mid (G_{11})^{-1}G_{12}]\dot\Omega$$

where G is partitioned as (G_{11} is $s\times s$ and G_{12} is $s\times r$)

$$G = \begin{bmatrix} G_{11} & G_{12} \\ * & * \end{bmatrix}.$$

We thus get a recipe to compute the nh equations for the $\dot\Omega_i, i = 1,\dots,s$: *Add to each of the first s equations $\dot\Omega_i$ a suitable combination of the remaining $\dot\Omega_\alpha$, $\alpha = s+1,\dots,n$, and afterwards cross out all terms containing one of the Ω_α's (since they vanish on E). The coefficients on the combination are $\tilde g^{ij}g_{j\alpha}$, where the tilde means that we are inverting the minor block G_{11} (that we denote $\tilde G$).* (Notice that the formulas for all the $\dot\Omega_I$ require the inverse of the full matrix G).

We now construct a family of metrics for which the projections become very simple. Consider an even dimensional vector space, of dimension $n = 2s$, where half of the basis is denoted A_i and the other half B_α. Let

$$G = \begin{bmatrix} I & S \\ S^\dagger & I \end{bmatrix}.$$

where K is a symmetric matrix so that G is still invertible. Actually, we only need K to commute with its transpose. It is immediate to verify that in this case

$$G^{-1} = \begin{bmatrix} (I-SS^\dagger)^{-1} & -S(I-SS^\dagger)^{-1} \\ -S^\dagger(I-SS^\dagger)^{-1} & (I-SS^\dagger)^{-1} \end{bmatrix}.$$

$$P_G = [I \mid S].$$

Taking into account that

$$g_{Jm} = \begin{cases} \delta_{Jm}, & 1 \leqslant J \leqslant s \\ S_{\alpha m}, & s+1 \leqslant J = s+\alpha \leqslant n \end{cases}$$

the resulting equations for the nh dynamics are

$$\dot\Omega_i = \sum_{1\leqslant m,\ell\leqslant s}\left(c_{\ell i}^m + \sum_{\alpha=1}^{s} c_{\ell i}^{s+\alpha}S_{\alpha m}\right)\Omega_m\Omega_\ell, \quad i = 1\dots s.$$

We reintroduced the summation symbols for better clarity. We obtain a result similar to the "master equations".

Proposition 4.5. *The nonholonomic dynamics is independent of the matrix S if and only if the expression*

$$A_{\ell,m}^i = \sum_{\alpha=1}^{s} c_{\ell i}^{s+\alpha}S_{\alpha m} \tag{4.13}$$

is skew symmetric in the indices ℓ, m, for all $i = 1,\dots s$.

We can now re-check Terra and Oliva's example. We shuffle the basis for $so(4)$. For e_4, e_5, e_6 we take the standard 3×3 skew symmetric matrices bordered by zeros in the fourth row and column,

$$(e_i)_{jk} = (A_i)_{jk} = -\epsilon_{ijk}, \quad i, j, k = 1, 2, 3$$

and for $e_1 = B_1$, $e_2 = B_2$, $e_3 = B_3$ the only nonzero entries will be in the last row and colum, as shown below:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The structure constants are

$$[A_i, A_j] = \epsilon_{ijk} A_k, \quad [B_i, B_j] = \epsilon_{ijk} A_k, \quad [A_i, B_j] = \epsilon_{ijk} B_k.$$

Relative to the basis $e_1 = B_1$, $e_2 = B_2$, $e_3 = B_3$, $e_4 = A_1$, $e_5 = A_2$, $e_6 = A_3$, as done in the previous subsection, the family of metrics is given by the 6×6 matrix (symmetric, positive definite) G with

$$S = aI_{3\times3}.$$

For $a = 0$ the metric is bi-invariant. The *unconstrained* dynamics system is trivial: $\dot{\Omega}_I = 0$, $I = 1, \ldots, 6$. In fact, since $g_{JM} = \delta_{JM}$, we get $\dot{\Omega}_K = c^J_{LK} \Omega_J \Omega_L$. Observe that all the non-vanishing structure constants satisfy $c^J_{LK} = -c^L_{JK}$ so the terms in the right hand side cancel in pairs[6].

We test the condition (4.13) with $0 < a < 1$.

Due to the symmetries between the indices 1,2,3 in G and in the Lie brackets, it is enough to test for $i = 1$ and $\ell, m = 2, 3$, with $S_{pq} = a\delta_{pq}$.

$$\begin{array}{llll} \ell = m = 2: & c^5_{21} S_{22} & \text{but} & c^5_{21} = \epsilon_{212} = 0. \\ \ell = m = 3: & c^6_{31} S_{22} & \text{but} & c^6_{31} = \epsilon_{313} = 0. \\ \ell = 2, m = 3: & c^6_{21} S_{33} & \text{but} & c^6_{21} = \epsilon_{213} = -1. \\ \ell = 3, m = 2: & c^5_{31} S_{22} & \text{but} & c^5_{31} = \epsilon_{312} = +1. \end{array}$$

The skew symmetry requirement of Proposition 4.5 is fulfilled. Although the unconstrained Euler–Arnold system does not vanish, it projects orthogonally over E, so the nonholonomic dynamics is trivial.

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[6] This is true for $SO(n)$. The standard bi-invariant metric in $sO(n)$ is given by the Killing form, $\langle \Omega_1, \Omega_2 \rangle = -(1/2)\,\mathrm{tr}(\Omega_1 \Omega_2)$ which defines an isomorphim $sO(n) \equiv sO(n)^*$. Euler's equations then write as $\dot{M} = [M, \Omega]$, $M = G(\Omega) \in sO(n)$. The positive definite bilinear operator giving the total energy of the n-dimensional rigid body is defined by $H = \langle G(\Omega_1), \Omega_2 \rangle$. Obviously for $G = id$, $\dot{M} = [\Omega, \Omega] \equiv 0$. $SO(4)$ is special: it has a two parameter family of bi-invariant metrics.

5. Cartan meets Chaplygin

We outline a research proposal: equivalence problems for Chaplygin systems. There is a new twist. One can focus on the dynamics on the base. The horizontal lift to the total space will be considered "just" a quadrature (this makes no justice to linear ODEs with time varying coefficients).

Given two purely inertial G-Chaplygin systems (symmetry group G),

$$(G^r \hookrightarrow Q^n \to S^s, T, \omega : TQ \to \mathcal{G}), \quad (G \hookrightarrow \tilde{Q}^n \to \tilde{S}^s, \tilde{T}, \tilde{\omega} : T\tilde{Q} \to \mathcal{G}), \tag{5.1}$$

with constraint distributions $E = \ker\omega$, $\tilde{E} = \ker\tilde{\omega}$, respectively, find what are the invariants that imply the existence of a local diffeomorphism $\phi : Q \to \tilde{Q}$ satisfying a required criterion, which could be, if $E = \tilde{E}$,

i) Preserving the nh connections, as discussed in sections 3 and 4:

$$\tilde{D}_{\tilde{X}}\tilde{Y} = \phi_*(D_X Y), \quad \tilde{X} = \phi_*(X), \tilde{Y} = \phi_* Y.$$

ii) Sending the straightest paths in Q to the straightest paths in $\tilde{Q}$.

$$\tilde{D}_{\tilde{X}}\tilde{X} = \phi_*(D_X X), \quad \tilde{X} = \phi_*(X).$$

If $E \neq \tilde{E}$ (we change the connection in addition to the metric):

iii) Preserving the projected connections in S.

iv) Sending the reduced paths in S to the reduced paths in $\tilde{S}$.

In equivalence problems it is traditional to work locally, on neighborhoods of points $q \in Q$ and $\tilde{q} \in \tilde{Q}$. So there is no loss in generality to assume

$$Q = U \times G, \quad \tilde{Q} = \tilde{U} \times G,$$

with $U = \tilde{U}$ an open set in $\mathfrak{R}^s$. To start, one needs to describe the initial subbundles inside the full coframe bundle $F^*(U \times G)$ and the corresponding matrix group.

5.1. The reduced affine connection. Let T be a G-invariant riemannian metric in Q. Given X, Y vectorfields in S, one defines an affine connection in S as follows. Lift them to horizontal vectorfields $h(X), h(Y)$ in Q. Going back to definition 1.1, compute

$$D_{h(X)}h(Y) = \mathrm{Proj}_E \nabla_{h(X)}h(Y)$$

where ∇ is the Levi-Civita connection in Q associated to the metric T. It is easy to see that it is also G-invariant.

Definition 5.1. The *reduced affine connection* is

$$D^S_X Y = \pi_*(D_{h(X)}h(Y)). \tag{5.2}$$

Query 5.1. D^S is an affine connection in S in the usual sense. One should be able to compute its structure equations, plus the torsion and curvature tensors in terms of moving frames in S. For this task, one can use results by A. Lewis in [69].

Definition 5.2. The reduced Chaplygin dynamics is governed by

$$D^S_X X = 0$$

the geodesic (i.e. straightest path) equation of the affine connection D^S. It is a *metric* connection with respect to the projected metric T_S.

Problem. In (5.1), fix a principal bundle $G^r \hookrightarrow Q^n \to S^s$, with vertical spaces V. We allow T and ω to change. When do the reduced connections $D = D_S$ and $\tilde{D} = \tilde{D}^S$ given by the definition (5.2) have the same geodesics (criterion iv)?

One considers the *difference tensor*

$$B(X,Y) = \tilde{D}^S_X Y - D^S_X Y$$

which is easily seen to be bilinear. As it is well known (see [52, section 5.4.]), D and $\tilde{D}$ have the same geodesics if and only if B is skew-symmetric, which is equivalent to the (obvious) condition

$$D^S_X X = \tilde{D}^S_X X.$$

At the Q level, this amounts to the requirement that

$$\mathrm{Proj}_E(\nabla_{h(X)} h(X)) - \mathrm{Proj}_{\tilde{E}}(\tilde{\nabla}_{\tilde{h}(X)} \tilde{h}(X)) \in V \tag{5.3}$$

for all vectorfields X in S. V is the (same) vertical distribution of the two principal connections, the first orthogonal projection is relative to metric T and the second to $\tilde{T}$, and likewise one takes the two distinct Levi-Civita covariant derivatives.

6. Final comments and other research directions

6.1. Cartan equivalence of Chaplygin systems. The basic task is how to use (5.3) in section 5 to obtain the initial structure for Cartan equivalence in terms of frames in S and the Lie algebra of the group G. For integrable distributions the solution is very simple. Both reduced dynamics are hamiltonian, with $H = \tilde{H}$ being the respective kinetic energies in Q restricted to the corresponding distributions and then projected to S. Since for metrics the unique invariant is the Riemannian curvature, the requirement is that $T|_E$ and $\tilde{T}|_{\tilde{E}}$ be the lifts (to their distributions) of a same metric g in S, the complements being arbitrary. The reduced connection is simply the Levi-Civita connection of the metric gS.

For nonintegrable distributions there is a complication: the projected system in S non-hamiltonian. The reduced equations are of the form $D^g_X X = F(X)$, where F is a "gyroscopic type" force, i.e. $g(F(X), X) \equiv 0$. This force comes from a "(J, K)" term combining the momentum map of the G-action in Q and the curvature of the connection (see eg [62]). The problem becomes more interesting and is related to the query 5.1. We also thank the referee for pointing out that a duality exists between a Levi-Civita type connection and an almost-Poisson bracket, in the more abstract context of mechanics in Lie algebroids [9, 49, 67].

6.2. Non-equivalent connections with the same straightest paths. The example of Terra and Olive generates interest in finding in what circumstances the weak equivalence does not imply the strong. We saw that there is a need for having "room" for at least three dimensions between E and $E + [E, E]$. In such cases, are there consequences for the equivalence problem? For hamiltonization?

6.3. Perspectives on Hamiltonization. Chaplygin's reducing multipliers method was geometrized in [21]. We mentioned in the introduction that it is tied intimately to hamiltonization and integrability. Currently, the *almost Poisson description* of nh systems is how one starts the quest for hamiltonization [19]. Hamiltonization is done usually after reduction of symmetries and a change of time scale depending on reduced variables (in Chaplygin systems the base manifold). Time change is related to the existence of a smooth invariant measure.

Unfortunately, our impression is that Cartan's equivalence approach (changing the metric) is of no avail for these purposes (we hope to be wrong). At any rate, we are preparing a sequel paper, in which Cartan moving frames in Q are extended to moving frames in T^*Q and then used to produce a simple way to obtain the Maschke/Schaft bracket [24, 72, 79][7].

Necessary and sufficient conditions for the existence of invariant measures can be found in eg. [25, 43]. Compiling a list of the known nh systems with a smooth invariant measure is in order, together with information about which ones have been hamiltonized and their integrability. For a rolling rigid body (several versions of the problem) results on the existence of an invariant measure and Hamiltonization have been gathered in the form of tables [15,16,18]. In these papers, a conformally Hamiltonian representation is found for a reduced system on T^*S^2. See [46] for a negative case. Chaplygin sphere was shown to be Hamiltonizable in [13] and the methodology explained in [17]. A glimpse of the subject (we apologize for the many omissions since the theme is booming) can be found in references such as (in random order) [6,7,34,39–42,44,48,53,56–61,85,86]. There are many more and more will be coming! Some recent information follows, kindly given to us by Luis Naranjo and Ivan Mamaev.

6.3.1. *Possibility of hamiltonization in the first level ($Leg(E) = \mathcal{M} \subset T^*Q$).* Sufficient conditions in order for a Chaplygin system to be hamiltonizable at the first level $T^*(Q/G)$, similar to Stanchenko's [83], were found by Naranjo and Marrero (personal communication). For Chaplygin's sphere, such conditions are not satisfied. It is believed that, in general, one cannot achieve hamiltonization at the first level. However, this question should be formulated in a clearer manner[8].

6.3.2. *Obstructions to Hamiltonization.* The conformal Hamiltonization of a bivector B seems a very interesting problem. Given a function H, a mandatory requirement is the existence of an invariant measure for $X_H = B \cdot dH$. In the case of nh systems, some obstructions to Hamiltonization are discussed in [14, 19, 20].

6.3.3. *Turning integrals into Casimirs.* The underlining idea is to modify the Mashke van der Schaft bracket [72, 79] to make first integrals into Casimirs of the reduced system [47].

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[7] Adding semi-basic bivectors that do not affect the dynamics (à la Stanchenko [83]) will become evident in this approach. The experts will agree that such preliminary constructions, before reduction, can help hamiltonization.

[8] For instance, in the case of Chaplygin's sphere, if it is homogeneous, then the system on $\mathcal{M}$ can be understood as an invariant subsystem of a Hamiltonian system, namely, that of a homogeneous sphere moving in R^2 without constraints. In this case $Leg(E)$ is a level set of the first integrals of the Hamiltonian system on T^*Q.

6.4. Themes from Cartan's ICM1928 paper. One wonders why Cartan did not discuss the weak notion of equivalence. We believe that he was concerned mainly with $D^E_X Y$ in the strongly nonholonomic case (1 step, $E + [E, E] = TQ$). Recall, we saw that in this case, for the preservation of the partial nh-connection on E, the metric can be changed outside E as long as the orthogonal space $F = E^{\perp}$ with respect to the original metric in Q stays the same. Cartan was interested in a *complementary partial connection* on F. For this he shows in § 6 to 8 that there is an intrinsically defined metric in F such that the partial connection D^F (now projecting to F) plus a suitable 2-tensor has zero torsion.

Moreover, it seems to us that at their time Cartan and contemporaries anticipated a construction that is nowadays popular in robotics, control theory, computer vision and statistics on manifolds (see eg. [**74**, **81**]). Namely, obtaining the development of a curve in Q over a fixed tangent space using the connection. The 'true' mechanics at a point $q \in Q$ is transported to a mirror Euclidian setting. The inertial motion in Q becomes a motion with Euclidian metric but with an applied force (using parallel transport of an object at q). We think it will be rewarding to revisit those aspects of Cartan's paper, not discussed in this review, and just touched upon in [**63**].

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КАРТАН СРЕЋЕ ЧАПЛИГИНА

Резиме. На Међународном конгресу математичара 1928, Картан је напоменуо да његов "метод еквиваленције" доводи до инваријанти нехолономног система на конфигурационом простору Q са Лагранжијаном одређеним кинетичком енергијом [29]. Картан је дао карактеристику промене метрике ван дистрибуције веза $E \subset TQ$ која очувава нехолономну повезаност $D_X Y = \mathrm{Proj}_E \nabla_X Y$, $X, Y \in E$, где је $\nabla_X Y$ Леви-Чивита повезаност на Q и Proj_E ортогонална пројекција на E. Овде разматрамо наведени проблем еквиваленције нехолономних повезаности за Чаплигинове системе [30, 31, 62]. Такође разматрамо и пример - математички драгуљ! - кога су пронашли Олива и Тера [76]. Ту имамо више слободе (дакле и више могућности) користећи слабију еквивалентност - чувају се само најправљи путеви: $D_X X = 0$. Ипак, тражење примера који су слабо, али не строго еквивалентни доводи до преодређеног система једначина, што даје индицију да су такви системи ретки. Ми показујемо да се два приступа поклапају у следећим случајевима: i) Дистрибуцијама ранка 2. На пример, у Картановом примеру сфере која се котрља по равни без клизања и увијања када имамо (2,3,5) дистрибуцију; ii) Када је дистрибуција ранка 3 или више, неопходан услов да се приступи разликују је да је коранк од D у D+[D,D] барем 3. Ово искључује могућност налажења примера у случају (3,5) дистрибуција, као код Чаплигинове сфере. Дакле предиван пример (3,6) дистрибуције који су добили Олива и Тера је минималан.

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