THEORETICAL AND APPLIED MECHANICS
Volume 46 (2019) Issue 1, 1–14 DOI: https://doi.org/10.2298/TAM190123005K

ON THE DYNAMICS OF SYSTEMS WITH ONE-SIDED NON-INTEGRABLE CONSTRAINTS

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Abstract. In the paper we take the first steps in studying the dynamics of systems with one-sided differential constraints defined by inequalities in the phase space. We give a general definition of motion for systems with such constraints. Within the framework of the classical non-holonomic model, and also for systems with servoconstraints (according to Béghin), we present the conditions under which the system leaves two-sided differential constraints. As an example, we consider the Chaplygin sleigh with a one-sided constraint, which is realized by means of an anisotropic force of viscous friction. Variational principles for the determination of motion of systems with one-sided differential constraints are presented.

1. Introduction

Let $x_1, \dots, x_n$ be generalized coordinates of a mechanical system and $M = \{x_1, \dots, x_n\}$ be the configuration space. The inertial properties of the system are defined by the kinetic energy

$$T = \frac{1}{2} \sum g_{ij} \dot{x}_i \dot{x}_j.$$

Here T is a positively defined quadratic form w.r.t. the velocities $\dot{x}_1, \dots, \dot{x}_n$ with smooth coefficients g_{ij}. Let

$$F = (F_1, \dots, F_n)$$

be generalized forces acting on the system; they depend on the state $x, \dot{x}$ of the system.

The phase space $\Gamma = TM$ is the tangent bundle of the configuration space. The dimension of Γ is $2n$; the local coordinates on Γ are coordinates x and velocities $\dot{x}$ of the system.

Let $f\colon \Gamma \to \mathbb{R}$ be a smooth function defined on the phase space. We assume that

$$\frac{\partial f}{\partial \dot{x}} \neq 0 \tag{1.1}$$

2010 *Mathematics Subject Classification:* 70F25; 70H45.
Key words and phrases: non-integrable constraints, servoconstraints, non-holonomic mechanics, vakonomic mechanics, one-sided constraint, unilateral constraint.

at all points of Γ. Usually, two-sided (bilateral) constraints are considered, i.e., for each moment of time, the state of our system satisfies the following condition

$$f(\dot{x}, x) = 0. \tag{1.2}$$

In the paper, we consider the dynamics of a mechanical system with a one-sided (unilateral) constraint for which, for each moment of time, we have

$$f(\dot{x}, x) \geqslant 0. \tag{1.3}$$

The boundary of this region is defined by equation (1.2).

We will call one-sided constraint (1.3) integrable (or, respectively, non-integrable) if boundary (1.2) is integrable (respectively, non-integrable). Recall that constraint (1.2) is integrable if $f = g\dot{h}$, where $g\colon \Gamma \to \mathbb{R}$ is a positive smooth function, h is a smooth regular function defined on the configuration space M. By regularity we mean that $dh \neq 0$ at all points of M. In case of an integrable one-sided constraint, the motion of system $t \mapsto x(t)$ satisfies inequality $\dot{h} \geqslant 0$.

The dynamics of a system with a one-sided differential integrable constraint $\dot{h}(x) \geqslant 0$ should be distinguished from the dynamics of the corresponding holonomic system with a one-sided constraint $h(x) \geqslant c$, where c is a constant. In the first case, the system can move in any level set of the function h when $\dot{h} = 0$. In the second case, the system moves in the only possible level set. Therefore, in the second case, the impact on the boundary is possible. In the first case, the main phenomenon is the situation when the system leaves the constraint (or switches to a two-sided constraint).

In mechanics, we usually consider linear in velocities constraints

$$f = (a, \dot{x}),$$

where a is a non-zero covector field on the configuration space. However, in controlled systems, it is possible to impose constraints which are non-linear in velocities.

Below we consider the simplest case when only one constraint (1.3) is imposed on the system. In general, it is possible to consider more than one constraint. Yet, in this case, other constraints should be imposed as equalities. Otherwise, we face additional difficulties that stem from the analysis of motion in the 'corner points'.

When we consider the motion of a constrained system, it is important to keep in mind the following fact: the dynamics of such systems significantly depends on the realization of the constraints. This remark is especially relevant when we consider controlled systems (see [1]).

The main principles of dynamics of systems with one-sided holonomic (geometrical) constraints are well-known (a worth mentioning exposition of the history of this problem can be found in [2]). In our paper, we take the first steps in studying the dynamics of systems with one-sided non-integrable constraints.

It should be noted that the dynamics of systems with unilateral differential constraints has already been studied before. For instance, one can find references on the matter in the survey [4] by V. F. Zhuravlev. The name 'unilateral' has been proposed for constraint (1.2) if this constraint is active under the additional condition on the acceleration: $F(\ddot{x}, \dot{x}, x) \geqslant 0$. The constraint is removed when this condition does not hold. The reader can find some similarities of this definition with our approach which we present below. However, in the definition of a unilateral constraint, nothing was said on the motion of a system in a region of the phase space where inequality (1.3) holds. Therefore, the theory developed in our paper should not be confused with the theory of systems with unilateral constraints discussed in [4].

A ball rolling on a rough plane is the main example considered in [4]: the velocity of the contact point is zero provided $fN - |\tau| \geqslant 0$. Here N (respectively, τ) is the normal (respectively, tangential) component of the reaction at the point of contact, f is the dry friction coefficient. Another example considered in [4] is jumps of a ball rolling on a surface. In the first example, when the constraint is removed, the dimension of the phase space increases by two. In the second example it increases by four. As we mentioned above, these cases are not of interest here.

2. Non-holonomic model

In the region where $f > 0$, the mechanical system can be considered as free and the dynamics is described by the Lagrange equations

$$[T] = F, \tag{2.1}$$

where $[\,\cdot\,]$ is the so-called variational (or Lagrange) derivative:

$$[\Phi] = \frac{d}{dt}\frac{\partial \Phi}{\partial \dot{x}} - \frac{\partial \Phi}{\partial x}.$$

At first, we consider the simplest case when the system with constraint (1.2) is described by the classical non-holonomic model:

$$[T] = F + \lambda \frac{\partial f}{\partial \dot{x}}, \quad f = 0. \tag{2.2}$$

These equations for the non-linear constraints (1.2), apparently, were first proposed by N. G. Chetaev. Taking into account the constraint, λ can be presented as a function of state of the system, without solving the equations of motion.

It is needed to give a definition of the motion of a system with the one-sided constraint (1.3). Suppose that the system moves in accordance with the two-sided constraint (1.2) and then leaves this constraint; we should present conditions under which the system leaves (1.2) (the boundary of (1.3)). Then, after reversing the time $t \mapsto -t$, we obtain conditions under which the system switches to the constraint (1.2).

The determination of the moment when the system leaves two-sided constraint $f = 0$ is based on the following simple consideration: this constraint cannot prevent the system from switching to the free motion in region $f > 0$. To be more precise, let x_τ and $\dot{x}_\tau$ be the position and the velocity of the system at the moment $t = \tau$ and the following equality holds

$$f(\dot{x}_\tau, x_\tau) = 0.$$

Next, we consider the motion of the free system $t \mapsto x(t)$ such that $x(\tau) = x_\tau$, $\dot{x}(\tau) = \dot{x}_\tau$ for $t = \tau$. Function $x(\cdot)$ is a solution to the Lagrange equations (2.1). If

$$f(\dot{x}(t), x(t)) > 0$$

for small $t > \tau$, then we say that the mechanical system leaves the two-sided constraint (1.2) at the moment $t = \tau$. For $t > \tau$ (until the system is on the one-sided constraint again) the dynamics is described by the Lagrange equations (2.1).

How should we determine the moment when the system leaves the constraint? Let $t \mapsto z(t)$ be a solution to the non-holonomic equations with the same initial conditions for $t = \tau$: $x(\tau) = x_\tau$, $\dot{x}(\tau) = \dot{x}_\tau$. Functions $x(t)$ and $z(t)$ satisfy the following equations:

$$A\ddot{x} + B = 0, \quad A\ddot{z} + B = \lambda \frac{\partial f}{\partial \dot{z}}. \tag{2.3}$$

Here

$$A = \left\| \frac{\partial^2 T}{\partial \dot{x}^2} \right\| \tag{2.4}$$

is a positively defined operator and B is a function of x and $\dot{x}$.

If function $t \mapsto z(t)$ satisfies equation $f = 0$, then

$$\frac{\partial f}{\partial z}\dot{z} + \frac{\partial f}{\partial \dot{z}}\ddot{z} = 0. \tag{2.5}$$

Since the position and velocities of the free and constrained systems at the moment τ are the same, then from (2.3)–(2.5) we obtain

$$\frac{d}{dt} f(\dot{x}(t), x(t))\big|_{t=\tau} = -\lambda\Big(A^{-1}\frac{\partial f}{\partial \dot{x}}, \frac{\partial f}{\partial \dot{x}}\Big).$$

Therefore, if $\lambda(\tau) < 0$, then the system has already left the boundary of the one-sided constraint (1.3).

We come to the following rule: if for the moment $t = \tau$ the multiplier $\lambda(t)$ (from (2.2)) changes the sign from positive to negative, then $t = \tau$ is the moment when the system leaves the constraint. This rule has a clear mechanical interpretation. The force of reaction of the two-sided constraint $f = 0$ (the force that acts on the system from the constraint) can be presented as follows

$$R = \lambda \frac{\partial f}{\partial \dot{x}}.$$

If $\lambda < 0$, then the force R keeps the system on the constraint $f = 0$, which is impossible.

In the smooth (infinitely differentiable) case, zeros of the function λ can be non-isolated. This fact may cause additional technical difficulties for the determination of the moment when the system leaves the constraint. However, if all functions are analytic, there are no such difficulties.

3. Chaplygin sleigh

The Chaplygin sleigh is a well-studied non-holonomic system. Below we consider the simplest case in which the center of mass coincides with the contact point of the blade (Fig. 1, the star defines the orientation of the blade). Let u be the velocity of the contact point along the blade and v be the orthogonal projection of the velocity w.r.t. the blade. By ω we denote the angular velocity of the sleigh.

Let

$$v \geqslant 0$$

be the one-sided non-integrable constraint. The Chaplygin sleigh is a non-holonomic system with the two-sided non-integrable constraint $v = 0$.

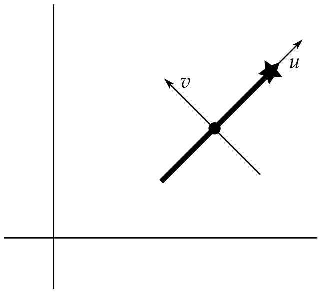

FIGURE 1.

Let us consider the simplest case when the sleigh is moving by inertia. It is a well-known fact that in this case the sleigh is moving in a circle and the radius of this circle depends on u and ω. If $\omega = 0$ then we have a degenerate circle—a straight line.

We omit detailed discussion concerning the mechanical realization of the one-sided constraint $v \geqslant 0$ for the Chaplygin sleigh. Note, however, that this type of constraint can be realized by means of blades having the form as shown in Fig. 2. Such a blade can freely slide to the left and cannot slide to the right. At the same time, the angular velocity of the sleigh with such a blade does not change with time.

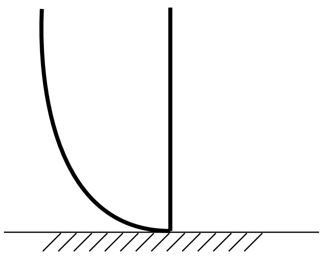

FIGURE 2.

Let us give the sleigh an initial motion along the blade and an initial clockwise rotation (as it is shown in Fig. 3). The one-sided constraint does not prevent the system from moving to the right from the blade oriented as shown in the picture. Initially we have $v = 0$. At this very moment the system leaves the two-sided constraint $v = 0$ (in accordance with the principle presented in Section 2). Then the contact point (which coincides with the center of mass of the sleigh) moves along a straight line with a constant velocity and the blade rotates uniformly. When the blade changes its orientation (makes a half turn), the motion along the straight line becomes impossible due to the one-sided constraint $v \geqslant 0$. After this, our non-holonomic sleigh continues its circular motion in accordance with the two-sided constraint $v = 0$, never leaving the circle (as it is shown in Fig. 3).

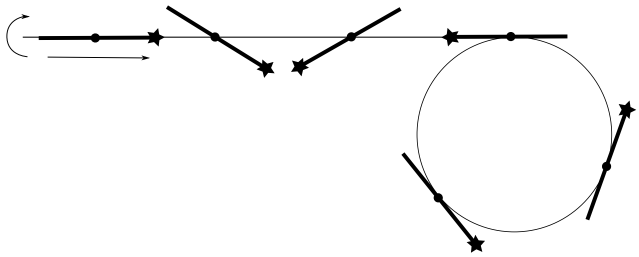

FIGURE 3.

From this elementary analysis, we obtain the following counter-intuitive conclusion: the system with a one-sided constraint is non-reversible. The dynamics of the sleigh is not invariant under the change of the direction of time $t \mapsto -t$. Indeed, if we reverse the motion of the sleigh, it will continue to move along the same circle but in the opposite direction and will never leave the circle (in accordance with the general principle from Section 2). It is worth mentioning that the equations of motion for systems with two-sided constraints and without constraints are reversible.

Moreover, the differential equations of motion of a mechanical system with a one-sided non-integrable constraint do not define a dynamical system. For instance, there is no uniqueness of the solution for such a system. Among other arguments, we mention that these problems are related to the fact that the dimension of the phase space of a system with a one-sided differential constraint is not constant.

4. Anisotropic friction and one-sided constraints

As it became known after a classical work by Caratheodory, the motion of a free system under the action of a viscous friction tends to the motion with a two-sided differential constraint as we infinitely increase the coefficient of the viscous friction [5]. A similar idea has been considered in the theory of holonomic systems with one-sided constraints: the 'forbidden' region of the configuration space is filled by the visco-elastic Kelvin-Foight continuum, then a limit passage, for which the coefficients of viscosity and elasticity are increased in a consistent manner, is applied. As a result, various models of impact and conditions under which the system leaves the constraints are obtained [6, 7].

We apply this idea in order to explain the paradox of non-reversibility of the Chaplygin sleigh with a one-sided constraint.

Let us consider the 'free' motion of the Chaplygin sleigh taking into account an additional viscous friction force F, which is directed orthogonally to the plane of the blade. In the moving coordinate system the equation of motion can be written as follows

$$\dot{u} = \omega v, \quad \dot{v} = -\omega u - \frac{F}{m}, \quad \dot{\omega} = 0. \tag{4.1}$$

Here m is the mass of the sleigh. The force of anisotropic friction is defined as follows:

$$F = mNv, \text{ if } v \leqslant 0; \quad F = 0, \text{ if } v > 0. \tag{4.2}$$

The positive coefficient of friction N will be increased to infinity. In other words, if the blade moves to the right (when $v < 0$), a large braking force appears.

Since $\omega = \text{const}$, the first two equations in (4.1) form an independent piecewise linear system of differential equations w.r.t. the variables u and v. The righthand sides of this system are continuous and satisfy the Lipschitz condition. Therefore, for this system, the theorem on the existence and uniqueness of a solution holds.

Let us now consider the solutions to this system with the initial data considered in Section 3. Let

$$u = u_0 > 0, \quad v = 0, \quad \omega = \omega_0 < 0$$

for $t = \pi/\omega_0 < 0$. Since system (4.1)–(4.2) is autonomous, the initial moment of time can be chosen arbitrarily. For $\pi/\omega_0 < t < 0$, the solution to (4.1) has the form

$$u = -u_0 \cos \omega_0 t, \quad v = u_0 \sin \omega_0 t.$$

In this time interval we have $v(t) > 0$; the point of contact moves along a straight line with a constant velocity.

Next it is needed to solve the following linear system

$$\dot{u} = \omega_0 v, \quad \dot{v} = -\omega_0 u - Nv$$

with the initial conditions $u = -u_0$, $v = 0$ for $t = 0$. The characteristic equation is as follows

$$\lambda^2 + \lambda N + \omega_0^2 = 0.$$

For large $N > 0$ this equation has the following real roots

$$\lambda_1 = -N + o(N), \quad \lambda_2 = -\frac{\omega_0^2}{N} + o\Big(\frac{1}{N}\Big). \tag{4.3}$$

Taking into account the initial conditions, we obtain

$$v(t) = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} \frac{u_0}{\omega_0} (e^{\lambda_1 t} - e^{\lambda_2 t}). \tag{4.4}$$

From (4.3), we have

$$\frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} = -\frac{\omega_0^2}{N} + o\Big(\frac{1}{N}\Big). \tag{4.5}$$

Then, for large N, we have

$$e^{\lambda_2 t} > e^{\lambda_1 t}$$

provided $t > 0$. Finally, from (4.4) and (4.5), for $N \gg 1$, we have for all $t > 0$

$$v(t) < 0 \text{ and } v(t) = O\Big(\frac{1}{N}\Big).$$

Here we take into account that ω_0 is negative. Therefore, for $t > 0$ and large N, the sleigh never leaves the region of the phase space where $v < 0$. If $N = \infty$, then the sleigh with anisotropic viscous friction (4.2) moves exactly along the trajectories considered in Section 3.

5. Variational principle

Let us now return to the case of motion of a mechanical system with a one-sided constraint

$$f(\dot{x}, x) \geqslant 0 \tag{5.1}$$

within the framework of the classical non-holonomic model. Let us have a continuously differentiable piecewise smooth path

$$x\colon [t_1, t_2] \to M, \tag{5.2}$$

which satisfies inequality (5.1). The property of piecewise smoothness means that there exists a finite number of points

$$\tau_1 = t_1 < \tau_2 < \cdots < \tau_k = t_2$$

such that at each interval (τ_p, τ_{p+1}) function $x(\cdot)$ is infinitely differentiable.

Let us formulate an integral variational principle that defines the motions of a mechanical system with a one-sided differential constraint in the form (5.1).

First we define the variation of the path

$$\delta x\colon [t_1, t_2] \to T_{x(t)}M.$$

As usual, we consider a family of piecewise smooth paths $x_\alpha(t)$, $-\varepsilon < \alpha < \varepsilon$, $t_1 \leqslant t \leqslant t_2$, where for $\alpha = 0$ we have the initial path (5.2). Moreover, we impose the following additional requirement

$$x_\alpha(\tau_p) = x(\tau_p) \tag{5.3}$$

for all $p = 1, \ldots, k$. By the variation of the path (5.2) we will understand

$$\delta x(t) = \frac{\partial x}{\partial \alpha}\Big|_{\alpha=0}.$$

It is clear that

$$(\delta x)^\cdot = \Big(\frac{\partial x}{\partial \alpha}\Big|_{\alpha=0}\Big)^\cdot = \frac{\partial \dot{x}}{\partial \alpha}\Big|_{\alpha=0} = \delta \dot{x}.$$

For our purposes, we can restrict ourselves to a special class of variations

$$x_\alpha(t) = x(t) + \alpha \delta x(t) + o(\alpha),$$

where functions $\delta x(\cdot)$ satisfy the following conditions:

(1) $\delta x(\tau_p) = 0$ (in accordance with (5.3)) and, in the intervals (τ_p, τ_{p+1}), functions $\delta x(\cdot)$ are infinitely differentiable,

(2) for $f(\dot{x}(t), x(t)) = 0$ and $t \neq \tau_p$, $\delta x(t)$ can be an arbitrary vector such that
$$\Big(\frac{\partial f}{\partial \dot{x}}\Big|_{x(t)}, \delta x(t)\Big) \geqslant 0,$$

(3) for $f(\dot{x}(t), x(t)) > 0$ and $t \neq \tau_p$, $\delta x(t)$ can be an arbitrary vector from $T_{x(t)}M$.

In particular, for the case 2), any vector from the hyperplane

$$\Big(\frac{\partial f}{\partial \dot{x}}\Big|_{x(t)}, \delta x(t)\Big) = 0 \tag{5.4}$$

can be the variation δx at the moment of time t.

There is the following extention of Hölder's variational principle.

Path (5.2) is the motion of a mechanical system with a one-sided constraint (5.1) iff

$$\delta \int_{t_1}^{t_2} T dt + \int_{t_1}^{t_2} (F, \delta x) dt \leqslant 0 \tag{5.5}$$

for all special variations δx defined above.

This proposition can be considered as a definition of motion for systems with one-sided differential constraints.

Let us give a short commentary/proof on/of this principle. First, note that the special variations can be chosen in such a way that $\delta x(t) = 0$ in all intervals of the subdivision of $[t_1, t_2]$ except one, for instance, (τ_p, τ_{p+1}). Then in (5.5) we can replace t_1 (t_2) by, respectively, τ_p (τ_{p+1}).

Next, inequality (5.5) can be written in the following form (taking into account the fixed endpoints of the variation):

$$\int_{\tau_p}^{\tau_{p+1}} (([T] - F), \delta x) dt \geqslant 0.$$

Since the variation can be localized (δx can be chosen to be zero outside an arbitrarily small neighbourhood of any moment of time) this integral inequality is equivalent to the inequality

$$(([T] - F), \delta x) \geqslant 0. \tag{5.6}$$

Let us rewrite this inequality in the form $(R, \delta x) \geqslant 0$, where R is the force of reaction.

This inequality holds if the system moves accordingly to the two-sided constraint $f = 0$. Moreover, this motion satisfies non-holonomic equation (2.2). Indeed, from the condition 2) for variations, inequality (5.6) holds for any vector $\delta x = \xi$ from the hyperplane (5.4). In particular, this inequality holds if we replace ξ by $-\xi$. Therefore, inequality (5.6) becomes an equality which defines the well-known D'Alambert-Lagrange principle for systems with two-sided constraints.

If inequality (5.6) does not hold (i.e., for some variation δx, satisfying condition 2), we have $(R, \delta x) < 0$), then it means that the force R 'keeps' the system on the one-sided constraint, which is impossible.

When the system is in the region $f > 0$, variations δx can be chosen to be arbitrary vectors. In this case, (5.6) becomes the Lagrange equation for a free system.

6. Béghin's servoconstraints

As it is known, constrained motions can be realized by means of a special control applied to the system. For instance, a system with the constraint $f(\dot{x}, x) = 0$ can be realized using control forces $\lambda\Phi$, where $\Phi(\dot{x}, x)$ is a given covector field and λ is a parameter.

$$[T] = F + \lambda\Phi, \quad f = 0.$$

As it was shown in [1], the controllability condition (determination of the multiplier λ) can be presented in the form

$$\Big(A^{-1}\Phi, \frac{\partial f}{\partial \dot{x}}\Big) \neq 0, \tag{6.1}$$

where the symmetrical operator A is defined by (2.4). This type of constraints was first introduced by Béghin (on this regard, see [8]). In [1], such constraints are called servoconstraints of type I.

Is it possible to realize motions with a one-sided constraint $f \geqslant 0$ in a similar way? The answer is proved to be positive. For this, it is needed to slightly modify the definition of variations (5.2) of a path satisfying condition (5.1). Special variations are defined as in Section 5, but property 2) should be replaced by the following

(2) on the boundary of the constraint, for $t \neq \tau_p$, we have

$$\Big(\Phi\big|_{x(t)}, \delta x(t)\Big) \geqslant 0 \quad (\leqslant 0),$$

when at this moment of time the following inequality holds

$$\Big(A^{-1}\Phi, \frac{\partial f}{\partial \dot{x}}\Big) > 0 \quad (< 0). \tag{6.2}$$

The following generalization of the variational principle from Section 5 holds.

A path (5.2) is a motion of a mechanical system with a servoconstraint of type I (5.1) iff

$$\delta \int_{t_1}^{t_2} T dt + \int_{t_1}^{t_2} (F, \delta x) dt \leqslant 0 \quad (\geqslant 0)$$

for all special variations δx.

Sign $\leqslant$ ($\geqslant$) is chosen in accordance with the sign in the inequality (6.2). When $\Phi = \partial f/\partial \dot{x}$, we have the variational principle from Section 5. The only difference from the principle in Section 5 is in the modification of the definition of variations (possible motions) of the system. The meaning of inequality (6.2) is the following: when this scalar product is positive, the covector Φ is directed toward the covector $\partial f/\partial \dot{x}$ (i.e., toward the 'allowed' region of the phase space).

The above variational principle can be explained in a similar way as we did it in Section 5. However, this principle can also be considered as the definition of motions of a system with a one-sided servoconstraint of I type.

The problem of realization of constraints by means of control forces can be considered from another, dual, point of view. We will explain this approach on the example of Béghin's servoconstraints. Suppose that there are control forces $\lambda\Phi$ acting on the system. Here, Φ is a given covector field, λ is a control parameter, which is supposed to be non-negative. We also suppose that the following inequality holds

$$\Big(A^{-1}\Phi, \frac{\partial f}{\partial \dot{x}}\Big) > 0.$$

Otherwise, we can change the sign of Φ. In particular, condition (6.2) is always satisfied. Our goal is to realize—by means of such a control (when $\lambda \geqslant 0$)—the dynamics of a mechanical system with a unilateral constraint $f(\dot{x}, x) \geqslant 0$. It is not hard to understand that the solutions to this problem with the additional condition $\lambda \geqslant 0$ are the paths defined by the above variational principle.

7. Vakonomic model and servoconstraints of type II

Besides the classical non-holonomic model, there is a whole family of models of systems with non-integrable two-sided constraints, which appear as a results of a forced anisotropy of the inertial properties of the systems. The simplest one (and, in some sense, the most natural) can be described by the Hamilton-Ostrogradsky variational principle

$$\delta \int_{t_1}^{t_2} T dt + \int_{t_1}^{t_2} (F, \delta x) dt = 0$$

for all variations with fixed endpoints satisfying the condition $f = 0$. These variations satisfy the following equation

$$\delta f = \frac{\partial f}{\partial \dot{x}} \delta \dot{x} + \frac{\partial f}{\partial x} \delta x = 0.$$

This is the so-called vakonomic model. For description, see [3].

There is another, and more general, model of motion for systems with servoconstraints of type II (in accordance with the terminology used in [1]). In this approach, the constraint is realized by means of a suitable control of the inertial properties of the system. Let us introduce and fix a function $\varphi\colon \Gamma \to \mathbb{R}$ and consider the system with the kinetic energy $T + \lambda\varphi$ (λ is a control parameter, an unknown function of time) under the action of the force F. Taking into account the two-sided constraint, the equations of motion have the form

$$[T] = F - \dot{\lambda} \frac{\partial \varphi}{\partial \dot{x}} - \lambda[\varphi], \quad f = 0.$$

Function $t \mapsto \lambda(t)$ is chosen to satisfy the constraint. The condition of realization of the two-sided constraint has the form

$$\Big(A^{-1} \frac{\partial \varphi}{\partial \dot{x}}, \frac{\partial f}{\partial \dot{x}}\Big) \neq 0.$$

In this case, λ satisfies a differential equation of first order with non-zero coefficients at $\dot{\lambda}$. In particular, for the determination of the motion it is required to give not only initial data x_0, $\dot{x}_0$ but also the initial value of λ. When $\varphi = f$ we have the vaconomic model.

As an example, let us consider a rigid body rotating about a fixed point in a three-dimensional Euclidean space. Let us fix in the moving space (associated to the moving body) two axes m and l. Suppose we are trying to realize the following non-integrable constraint: the projection of the angular velocity on m is zero. The body is controlled by a kinetic moment of a symmetric flywheel rotating around axis l. As it was shown in [1], if l and m are not orthogonal w.r.t. the metric defined by the kinetic energy, then the above two-sided constraint can be realized by a suitable rotation of the flywheel. When l and m coincide, we obtain vakonomic motions of the rigid body.

Similarly to Section 6, we can consider the problem of realization of a one-sided constraint by means of change of the inertial properties of the system (as mentioned above) assuming that $\lambda \geqslant 0$. For instance, in the example above, we may consider a restriction that the flywheel rotates only in one direction.

Suppose that

$$\Big(A^{-1}\frac{\partial\varphi}{\partial\dot x}, \frac{\partial f}{\partial\dot x}\Big) \neq 0.$$

Then, similarly to Section 5, consider a continuously differentiable piecewise smooth path (5.2) and its variations, which satisfy the following additional condition

$$\delta\varphi = \frac{\partial\varphi}{\partial\dot x}\delta\dot x + \frac{\partial\varphi}{\partial x}\delta x \geqslant 0. \tag{7.1}$$

If (5.2) is a motion of the system with a one-sided constraint, then

$$\delta\int_{t_1}^{t_2} T\,dt + \int_{t_1}^{t_2}(F,\delta x)dt \leqslant 0 \tag{7.2}$$

for all variations δx satisfying condition (7.1).

Indeed, in this case the reaction R equals

$$-\dot\lambda\frac{\partial\varphi}{\partial\dot x} - \lambda[\varphi].$$

Therefore, for variations with fixed endpoints satisfying (7.1), we have

$$\int_{\tau_p}^{\tau_{p+1}}(R,\delta x)dt = \int_{\tau_p}^{\tau_{p+1}}\Big(-\Big(\lambda\frac{\partial\varphi}{\partial\dot x}\delta x\Big)^{\cdot} + \lambda\frac{\partial\varphi}{\partial\dot x}\delta\dot x + \lambda\frac{\partial\varphi}{\partial x}\delta x\Big)dt = \int_{\tau_p}^{\tau_{p+1}}\lambda\delta\varphi\,dt \geqslant 0.$$

And, in this case, the inequality (7.2) holds.

Therefore, sufficient conditions that should be met along a motion of a mechanical system with one-sided differential constraints have the same form for different models. The only difference is in the definitions of the variations.

Acknowledgements

The research was funded by a grant from the Russian Science Foundation (Project No. 19-71-30012).

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О ДИНАМИЦИ СИСТЕМА СА ЈЕДНОСТРАНИМ НЕИНТЕГРАБИЛНИМ ВЕЗАМА

Резиме. У раду је направљен први корак у проучавању динамике система са једностраним диференцијалним везама које су дефинисане неједнакостима у фазном простору. Дата је општа дефиниија кретања система са таквим везама. У оквиру класичних нехоломних модела и система са сервовезама (дефинисаних Бенгином), дат је услов под којима систем напушта двострану диференцијалну везу. Посматра се пример Чаплигиновог сечива с једностраном везом која се реализује преко анизотропне силе вискозног трења. Представљени су и варијациони принципи за одређивање кретања система са једностраним диференцијалним везама.

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(Received 23.01.2019.)
(Revised 15.05.2019.)
(Available online 17.06.2019.)