THEORETICAL AND APPLIED MECHANICS
Volume 45 (2018) Issue 2, 177–188 DOI: https://doi.org/10.2298/TAM171017002B

AN EFFECT OF A PURELY DISSIPATIVE PROCESS OF MICROSTRESSES ON PLANE STRAIN GRADIENT PLASTICITY PROBLEMS

Adebowale Borokinni, Odunayo Fadodun, and Adegbola Akinola

Abstract. This article considers a plane strain gradient plasticity theory of the Gurtin–Anand model [M. Gurtin, L. Anand, *A theory of strain gradient plasticity for isotropic, plastically irrotational materials Part I: Small deformations*, J. Mech. Phys. Solids **53** (2005), 1624–1649] for an isotropic material undergoing small deformation in the absence of plastic spin. It is assumed that the system of microstresses is purely dissipative, so that the free energy reduces to a function of the elastic strain, while the microstresses are only related to the plastic strain rate and gradient of the plastic strain rate via the constitutive relations. The plane strain problem of the Gurtin–Anand model for a purely dissipative process gives rise to elastic incompressibility. A weak formulation of the flow rule is derived, making the plane strain problem suitable for finite element implementation.

1. Introduction

Strain gradient plasticity theories have emerged due to the inability of the classical theory to capture phenomena such as micro-identation, shear bands and size effects [1]. These effects are determined via the inclusion of the plastic strain gradient in the plastic flow rule. The inclusion of strain gradients in the flow rule would not have been needed if there had not been differences between the classical theory and various experimental results [2]. The inadequacy of the classical theory to model material behaviour at the micron scale necessitated the theory of strain gradient plasticity, which has been a field of study for more than three decades. In fact, it has always been a challenge to obtain good extensions of the classical theory to determine material behaviour at the infinitesimal length scales [3].

Among the most popular strain gradient theories is that of Gurtin and Anand [4], which presents the plastic flow rule as a nonlinear partial differential equation in the plastic strain, derived through the substitution of the constitutive relation

2010 *Mathematics Subject Classification:* 74C10.
Key words and phrases: plane strain gradient, microstresses, flow rule, weak formulation.

of microstresses into the microforce balance equation (obtained via the principle of virtual power).

The existence and uniqueness of the solution of the Gurtin–Anand model has been studied for hardening behaviour by Ebobisse et al. [5] and Reddy et al. [6], where an isotropic hardening function is included in the free energy function. However, in the treatment of the softening behaviour, Reddy et al. [6] included divergence of the plastic strain in the free energy function so as to arrive at a uniqueness result of the flow rule. Borokinni et al. [7] following the Gurtin and Anand model, studied the effects of internal microforces associated with the divergence of the plastic strain on plastic flow of materials, and showed that the energetic microforces are partial derivatives of the free-energy function with respect to the divergence of plastic strain [8]. Also, Lubarda [9] showed that for non-isothermal cases, an appropriate free-energy function can be chosen to determine the fraction of the rate of plastic work transformed into heat energy.

Plane strain problems are much simpler than three-dimensional problems, and provide rich information on the nature of solutions. We are motivated to consider the plane strain problem because there may be certain constraints that will further simplify the plane strain problems of viscoplasticity.

This work will consider the effect of the dissipative part of the microstresses in the plane strain gradient theory of plasticity. Section 2 will consider a simplified form of the Gurtin–Anand model in the absence of energetic polar microstress. In Section 3 we shall obtain the flow rule under plane strain conditions and determine the effect of the dissipative part of microstresses on the flow rule, while their energetic parts are absent for both fixed and varying values of the rate sensitivity parameter. Finally, Section 4 provides weak formulations of plane strain gradient problems for varying values of the rate sensitivity parameter.

2. Flow rule of the Gurtin–Anand Model of strain gradient plasticity

Let $\vec{u}(\vec{x},t)$ denote the displacement of an arbitrary point $\vec{x}$ in a region B. The classical theory of isotropic plastic solids undergoing small deformations is based on the decomposition

$$\nabla\vec{u} = \mathbf{H}^e + \mathbf{E}^p, \quad \mathrm{tr}(\mathbf{E}^p) = 0. \tag{2.1}$$

$\mathbf{H}^e$ represents rotation and stretching of the material structure, and $\mathbf{E}^p$ is the plastic distortion characterising the evolution of dislocations and other defects through the structure. As indicated by Gurtin and Anand [4], the plastic spin has been ignored. The elastic part of the displacement gradient can further be decomposed into elastic strain $\mathbf{E}^e$ and elastic rotation $\mathbf{W}^e$ in the form

$$\mathbf{H}^e = \mathbf{E}^e + \mathbf{W}^e$$

where

$$\mathbf{E}^e = \mathrm{sym}\,\mathbf{H}^e, \quad \mathbf{W}^e = \mathrm{skw}\,\mathbf{H}^e$$

are the symmetric and skew parts of $\mathbf{H}^e$ respectively.

2.1. Macroforce and Microforce balance. The basic rate-like variables in the Gurtin–Anand model are $\dot{\vec{u}}$, $\dot{\mathbf{H}}^e$ and $\dot{\mathbf{E}}^p$. These variables are not independent of each other by virtue of (2.1). The macroforce and microforce balances of the Gurtin–Anand model are obtained through the principle of virtual power following the assumptions that

- An elastic macrostress $\mathbf{T}$ is energy-conjugate to $\mathbf{H}^e$
- A plastic microstress $\mathbf{T}^p$ is energy-conjugate to $\mathbf{E}^p$
- A polar plastic microstress $\mathbb{K}^p$ is energy-conjugate to $\nabla\mathbf{E}^p$.

These assumptions allow the power expenditure due to internal agencies $W_{\text{int}}(P)$ over a sub-region P of B to be written in the form

$$W_{\text{int}}(P) = \int_P (\mathbf{T} : \dot{\mathbf{H}}^e + \mathbf{T}^p : \dot{\mathbf{E}}^p + \mathbb{K}^p \vdots \nabla\dot{\mathbf{E}}^p) dV.$$

Since $\mathbf{T}$ is symmetric, the internal power expenditure reduces to the form

$$W_{\text{int}}(P) = \int_P (\mathbf{T} : \dot{\mathbf{E}}^e + \mathbf{T}^p : \dot{\mathbf{E}}^p + \mathbb{K}^p \vdots \nabla\dot{\mathbf{E}}^p) dV,$$

where $\dot{\mathbf{E}}^p$ and $\mathbf{T}^p$ are symmetric deviatoric. Also, the polar microstress $\mathbb{K}^p$ and $\nabla\dot{\mathbf{E}}^p$ are symmetric deviatoric in the first two indices. The power expenditure $W_{\text{ext}}(P)$ due to body force $\vec{b}(\vec{x},t)$ in P, the macrotraction force $\vec{t}(\vec{n})$ and $\mathbf{K}(\vec{n})$ on ∂P is given by

$$W_{\text{ext}}(P) = \int_{\partial P} (\vec{t} \cdot \dot{\vec{u}} + \mathbf{K}(\vec{n}) : \dot{\mathbf{E}}^p) dA + \int_P \vec{b} \cdot \dot{\vec{u}}\, dV.$$

It is assumed that $\mathbf{K}(\vec{n})$ is symmetric deviatoric.

The principle of virtual power makes it possible to choose the kinematic-based virtual variables arbitrarily in such a way that these variables are consistent with (2.1). These kinematic virtual variables are denoted by $\delta\dot{\vec{u}}$, $\delta\dot{\mathbf{E}}^e$ and $\delta\dot{\mathbf{E}}^e$. The principle of virtual power implies that the internal and external virtual powers are the same. The consequences of this are the macroforce and microforce balances given respectively as

$$\operatorname{div}\mathbf{T} + \vec{b}(\vec{x},t) = 0 \tag{2.2}$$

and

$$\mathbf{T}_o = \mathbf{T}^p - \operatorname{div}\mathbb{K}^p. \tag{2.3}$$

$\mathbf{T}_o$ is the deviatoric part of the macrostress tensor $\mathbf{T}$. The macrotraction condition is given by

$$\vec{t}(\vec{n}) = \mathbf{T}\vec{n}$$

and the microtraction condition is given by

$$\mathbf{K}(\vec{n}) = \mathbb{K}^p\vec{n}.$$

2.2. Constitutive relations. The Helmholtz free energy ψ is assumed a function of the elastic strain $\mathbf{E}^e$ and the Burgers tensor $\nabla \times \mathbf{E}^p$. By the second law of thermodynamics, the free energy imbalance is written in the form

$$\dot{\psi} - \mathbf{T} : \dot{\mathbf{E}}^e - \mathbf{T}^p : \dot{\mathbf{E}}^p - \mathbb{K}^p \vdots \nabla \dot{\mathbf{E}}^p \leqslant 0.$$

The standard constitutive relation for the macrostress is given by

$$\mathbf{T} = \frac{\partial \psi}{\partial \mathbf{E}^e}. \tag{2.4}$$

The polar microstress can be decomposed into the energetic part $\mathbb{K}^p_{\text{en}}$ and the dissipative part $\mathbb{K}^p_{\text{dis}}$ written as

$$\mathbb{K}^p = \mathbb{K}^p_{\text{dis}} + \mathbb{K}^p_{\text{en}}.$$

At this point, it will be assumed that the energetic part of the polar microstress is absent, so that the microstresses are purely dissipative. Thus the local dissipative inequality reduces to

$$\mathbf{T}^p : \dot{\mathbf{E}}^p + \mathbb{K}^p_{\text{dis}} \vdots \nabla \dot{\mathbf{E}}^p \geqslant 0.$$

Following the works in [4] and [10], the constitution relations between the microstresses and their energy-conjugate are given by

$$\mathbf{T}^p = S\Big(\frac{d^p}{d_o}\Big)^m \frac{\dot{\mathbf{E}}^p}{d^p} \tag{2.5}$$

$$\mathbb{K}^p_{\text{dis}} = l^2 S_Y \Big(\frac{d^p}{d_o}\Big)^m \frac{\nabla \dot{\mathbf{E}}^p}{d^p} \tag{2.6}$$

$$\dot{S} = H(S) d^p, \quad S(\vec{x}, 0) = S_Y.$$

S is a stress-dimensioned internal variable quantity which defines the current resistance to plastic flow. S_Y is a constant denoting the initial yield strength, $H(S)$ is a hardening function, d^p is an effective flow rate defined by

$$d^p = \sqrt{|\dot{\mathbf{E}}^p|^2 + l^2 |\nabla \dot{\mathbf{E}}^p|^2},$$

where l is called the dissipative length scale. $d_o > 0$ is a constant called the reference flow rate and m is a constant called the rate-sensitivity parameter.

In this paper the free energy for an isotropic material following [4] and [11] is assumed to take the form

$$\psi(\mathbf{E}^e) = \mu |\mathbf{E}^e_o|^2 + \frac{1}{2}\kappa |\operatorname{tr} \mathbf{E}^e|^2, \tag{2.7}$$

where μ and κ are the elastic shear and the bulk modulus respectively. By using the definition of (2.4) and (2.7), the macrostress is given by

$$\mathbf{T} = 2\mu \mathbf{E}^e_o + \kappa \operatorname{tr}(\mathbf{E}^e)\mathbf{I}. \tag{2.8}$$

The microforce balance from (2.3) is given by

$$\mathbf{T}_o = \mathbf{T}^p - \operatorname{div} \mathbb{K}^p_{\text{dis}}. \tag{2.9}$$

Substituting (2.5) and (2.6) into (2.9), the flow rule for strain gradient plasticity in the absence of the energetic polar microstress becomes

$$\mathbf{T}_o = S\Big(\frac{d^p}{d_o}\Big)^m \frac{\dot{\mathbf{E}}^p}{d^p} - l^2 S_Y \,\mathrm{div}\,\Big(\Big(\frac{d^p}{d_o}\Big)^m \frac{\nabla\dot{\mathbf{E}}^p}{d^p}\Big). \tag{2.10}$$

3. Plane strain gradient formulation of a dissipative process

Suppose a body undergoes a plane strain so that the displacement vector $\vec{u}$ is written as

$$\vec{u} = u_1(x,y)\vec{e}_1 + u_2(x,y)\vec{e}_2,$$

where $\vec{e}_1$ and $\vec{e}_2$ are unit vectors parallel to the x and y axes respectively. Quantities are independent of the third coordinate z. The third component u_3 of the displacement vector is assumed to be zero. Therefore, in a plane strain, the components of the strain E_{13}, E_{23} and E_{33} are zeros where the components of the strain are defined by

$$E_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \qquad (i,j = 1,2,3).$$

Here it has been assumed that $x_1 = x$, $x_2 = y$ and $x_3 = z$ so that the non-zero components of the strain are defined by

$$E_{11} = \frac{\partial u_1}{\partial x}, \quad E_{22} = \frac{\partial u_2}{\partial y} \quad \text{and} \quad E_{12} = \frac{1}{2}\Big(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\Big).$$

Since the total strain are independent of the third coordinate then it can be assumed that the components of plastic strain are also independent of the third coordinate so that $E^p_{33} = 0$ and plastic incompressibility from (2.1) reduces to the form

$$E^p_{11} + E^p_{22} = 0.$$

Since the non-vanishing components of the plastic strain are E^p_{11}, E^p_{22} and E^p_{12}, the components of the plastic strain can be written as a column vector in the form

$$\mathbf{E}^p = \begin{pmatrix} E^p_{11} \\ E^p_{22} \\ E^p_{12} \end{pmatrix},$$

so that the laplacian of the plastic strain is defined as

$$\Delta\mathbf{E}^p = \begin{pmatrix} \frac{\partial^2 E^p_{11}}{\partial x^2} + \frac{\partial^2 E^p_{11}}{\partial y^2} \\ \frac{\partial^2 E^p_{22}}{\partial x^2} + \frac{\partial^2 E^p_{22}}{\partial y^2} \\ \frac{\partial^2 E^p_{12}}{\partial x^2} + \frac{\partial^2 E^p_{12}}{\partial y^2} \end{pmatrix}.$$

From (2.8) the components of the macrostress are defined by

$$T_{11} = \kappa\theta + 2\mu(E^e_{11} - \tfrac{1}{3}(E^e_{11} + E^e_{22}))$$

so that

$$T_{11} = \Big(\frac{3\kappa - 2\mu}{3}\Big)\theta + 2\mu E^e_{11},$$

where $\theta = E^e_{11} + E^e_{22} = E_{11} + E_{22}$.

Similarly,

$$T_{22} = \Big(\frac{3\kappa - 2\mu}{3}\Big)\theta + 2\mu E^e_{22},$$

and

$$T_{12} = 2\mu E^e_{12}.$$

Recall that under plane strain T_{33} is in general non-zero, where

$$T_{33} = \Big(\frac{3\kappa - 2\mu}{3}\Big)\theta. \tag{3.1}$$

The macroforce balance from (2.2) for the plane strain problem is defined by

$$\frac{\partial T_{11}}{\partial x} + \frac{\partial T_{12}}{\partial y} + b_1(x,y) = 0$$
$$\frac{\partial T_{12}}{\partial x} + \frac{\partial T_{22}}{\partial y} + b_2(x,y) = 0$$

with the traction conditions

$$T_{11}n_1 + T_{12}n_2 = t_1$$
$$T_{21}n_1 + T_{22}n_2 = t_2.$$

Let $M = \frac{3\kappa-2\mu}{3}$ and $N = \frac{3\kappa+4\mu}{3}$. The components of the macrostress in terms of M and N are defined by

$$T_{11} = N\frac{\partial u_1}{\partial x} + M\frac{\partial u_2}{\partial y} - 2\mu E^p_{11}. \tag{3.2}$$

$$T_{22} = N\frac{\partial u_2}{\partial y} + M\frac{\partial u_1}{\partial x} - 2\mu E^p_{22}. \tag{3.3}$$

$$T_{12} = 2\mu\Big(\frac{1}{2}\Big(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\Big) - E^p_{12}\Big). \tag{3.4}$$

and

$$T_{33} = M\Big(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\Big).$$

Following the definitions of the components of the macrostress (3.2)–(3.4), the macroforce balance in terms of M and N is written as follows

$$\frac{\partial}{\partial x}\Big(N\frac{\partial u_1}{\partial x} + M\frac{\partial u_2}{\partial y}\Big) + \mu\frac{\partial}{\partial y}\Big(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\Big) + b_1(x,y) = 2\mu\Big(\frac{\partial E^p_{11}}{\partial x} + \frac{\partial E^p_{12}}{\partial y}\Big).$$
$$\frac{\partial}{\partial y}\Big(N\frac{\partial u_2}{\partial y} + M\frac{\partial u_1}{\partial x}\Big) + \mu\frac{\partial}{\partial x}\Big(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\Big) + b_2(x,y) = 2\mu\Big(\frac{\partial E^p_{12}}{\partial x} + \frac{\partial E^p_{22}}{\partial y}\Big).$$

Recall that the deviatoric part of the macrostress $\mathbf{T}_o$ is defined by

$$\mathbf{T}_o = \mathbf{T} - \tfrac{1}{3}\,\mathrm{tr}(\mathbf{T})\mathbf{I},$$

where $\mathbf{I}$ is the rank-two unit tensor. The component of the deviatoric part of the macrostress can be obtained using (3.1) and (3.2)–(3.4). Clearly,

$$T_{o11} = T_{11} - \tfrac{1}{3}(T_{11} + T_{22} + T_{33})$$

$$T_{o11} = \frac{2}{3}\Big(N\frac{\partial u_1}{\partial x} + M\frac{\partial u_2}{\partial y} - 2\mu E^p_{11}\Big) - \frac{1}{3}\Big(N\frac{\partial u_2}{\partial y} + M\frac{\partial u_1}{\partial x} - 2\mu E^p_{22}\Big) - \frac{1}{3}M\Big(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\Big).$$

By simplifying it is obvious that

$$T_{o11} = 2\mu\Big(\frac{2}{3}\frac{\partial u_1}{\partial x} - \frac{1}{3}\frac{\partial u_2}{\partial y} - E^p_{11}\Big). \tag{3.5}$$

Similarly,

$$T_{o22} = 2\mu\Big(\frac{2}{3}\frac{\partial u_2}{\partial y} - \frac{1}{3}\frac{\partial u_1}{\partial x} - E^p_{22}\Big), \tag{3.6}$$

$$T_{o12} = 2\mu\Big(\frac{1}{2}\Big(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\Big) - E^p_{12}\Big). \tag{3.7}$$

and

$$T_{o33} = -\frac{2}{3}\mu\Big(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\Big).$$

From the flow rule (2.10), assume, for the sake of simplicity and without loss of generality, that $m = 1$ so that the flow rule reduces to the form

$$\mathbf{T}_o = \frac{S}{d_o}\dot{\mathbf{E}}^p - \frac{l^2}{d_o}S_Y\Delta\dot{\mathbf{E}}^p.$$

Thus the flow rule in component form using (3.5)–(3.7) is given by

$$0 = 2\mu E^p_{11} + \frac{S}{d_o}\dot{E}^p_{11} - 2\mu\Big(\frac{2}{3}\frac{\partial u_1}{\partial x} - \frac{1}{3}\frac{\partial u_2}{\partial y}\Big) - \frac{\partial}{\partial x}\Big(l^2\frac{S_Y}{d_o}\frac{\partial \dot{E}^p_{11}}{\partial x}\Big) - \frac{\partial}{\partial y}\Big(l^2\frac{S_Y}{d_o}\frac{\partial \dot{E}^p_{11}}{\partial y}\Big). \tag{3.8}$$

$$0 = 2\mu E^p_{22} + \frac{S}{d_o}\dot{E}^p_{22} - 2\mu\Big(\frac{2}{3}\frac{\partial u_2}{\partial y} - \frac{1}{3}\frac{\partial u_1}{\partial x}\Big) - \frac{\partial}{\partial y}\Big(l^2\frac{S_Y}{d_o}\frac{\partial \dot{E}^p_{22}}{\partial y}\Big) - \frac{\partial}{\partial x}\Big(l^2\frac{S_Y}{d_o}\frac{\partial \dot{E}^p_{22}}{\partial x}\Big). \tag{3.9}$$

and

$$0 = 2\mu E^p_{12} + \frac{S}{d_o}\dot{E}^p_{12} - \mu\Big(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\Big) - \frac{\partial}{\partial x}\Big(l^2\frac{S_Y}{d_o}\frac{\partial \dot{E}^p_{12}}{\partial x}\Big) - \frac{\partial}{\partial y}\Big(l^2\frac{S_Y}{d_o}\frac{\partial \dot{E}^p_{12}}{\partial y}\Big).$$

3.1. Effect of pure dissipation of the microstresses.

3.1.1. *Effect of pure dissipation for $m = 1$.* The addition of equations (3.8) and (3.9) (noting that $E^p_{11} + E^p_{22} = 0$) leads to

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0, \tag{3.10}$$

which corresponds to the fact that $T_{o33} = 0$. This implies that

$$E_{11} + E_{22} = \operatorname{tr}\mathbf{E} = \operatorname{tr}\mathbf{E}^e = E^e_{11} + E^e_{22} = 0. \tag{3.11}$$

Thus, pure dissipation of the polar microstress leads to a special case of elastic incompressibility given by (3.11).

Substituting (3.10) in (3.8) and (3.9) for each $k = 1,2$ will lead to (take $s_o = S_Y$)

$$0 = 2\mu E^p_{kk} + \frac{s_o}{d_o}\dot{E}^p_{kk} - 2\mu E^e_{kk} - \frac{\partial}{\partial x}\Big(l^2 \frac{s_o}{d_o}\frac{\partial \dot{E}^p_{kk}}{\partial x}\Big) - \frac{\partial}{\partial y}\Big(l^2 \frac{s_o}{d_o}\frac{\partial \dot{E}^p_{kk}}{\partial y}\Big).$$

Assume that $E^p_{11} = 0$, then a pure shear state is achieved as follows

$$E^p_{11} = 0 = E^p_{22}, \quad E^p_{12} \neq 0,$$

and

$$E^e_{11} = 0 = E^e_{22}, \quad E^e_{12} \neq 0.$$

Hence, if $E^p_{12} = 0$, then there is no plastic deformation experience by the body since the total strain becomes zero.

3.1.2. *Effect of pure dissipation for the general case of $m \geqslant 0$.* Now suppose the rate-sensitivity parameter $m \geqslant 0$ and letting $F(\dot{\mathbf{E}}^p, \nabla\dot{\mathbf{E}}^p) = \big(\frac{d^p}{d_o}\frac{1}{d^p}\big)$, then the flow rule is given by

$$\mathbf{T}_o = SF(\dot{\mathbf{E}}^p, \nabla\dot{\mathbf{E}}^p)\dot{\mathbf{E}}^p - l^2 S_Y \operatorname{div}(F(\dot{\mathbf{E}}^p, \nabla\dot{\mathbf{E}}^p)\nabla\dot{\mathbf{E}}^p). \tag{3.12}$$

Following the rules of differentiation, the flow rule will reduce to the form

$$\mathbf{T}_o = SF\dot{\mathbf{E}}^p - l^2 S_Y F\Delta\dot{\mathbf{E}}^p - l^2 S_Y \nabla F \cdot \nabla\dot{\mathbf{E}}^p.$$

In component form the flow rule becomes

$$T_{oij} = SF\dot{E}^p_{ij} - l^2 S_Y F\dot{E}^p_{ij,kk} - l^2 S_Y \dot{E}^p_{ij,k}F_{ij,k}.$$

In explicit form, the flow rule of the plane strain gradient plasticity becomes

$$T_{oij} = SF\dot{E}^p_{ij} - l^2 S_Y F\Big(\frac{\partial^2 \dot{E}^p_{ij}}{\partial x^2} + \frac{\partial^2 \dot{E}^p_{ij}}{\partial y^2}\Big) - l^2 S_Y\Big(\frac{\partial \dot{E}^p_{ij}}{\partial x}\frac{\partial F}{\partial x} + \frac{\partial \dot{E}^p_{ij}}{\partial y}\frac{\partial F}{\partial y}\Big)$$

Adding the components T_{o11} and T_{o22} together will result in the equation

$$T_{o11} + T_{o22} = 0. \tag{3.13}$$

Based on the definition of T_{o11} and T_{o22} from equations (3.5) and (3.6), then (3.13) leads to the equation

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0.$$

By the analysis provided in subsection 3.1.1 it is clear that for the general case of $m \geqslant 0$, the flow is a result of plastic shear strain provided $E^p_{11} = E^p_{22} = 0$. Thus the flow rule will reduce to the form

$$T_{o12} = SF\dot{E}^p_{12} - l^2 S_Y F\Big(\frac{\partial^2 \dot{E}^p_{12}}{\partial x^2} + \frac{\partial^2 \dot{E}^p_{12}}{\partial y^2}\Big) - l^2 S_Y\Big(\frac{\partial \dot{E}^p_{12}}{\partial x}\frac{\partial F}{\partial x} + \frac{\partial \dot{E}^p_{12}}{\partial y}\frac{\partial F}{\partial y}\Big)$$

4. Weak formulation for the flow rule of plane strain gradient plasticity

In this section the weak formulation of the macroforce and microforce balances will be developed. This will serve as a tool for formulating the finite element model for plane strain problems of plasticity. The weak formulation for $m = 1$ and the general case of $m \geqslant 0$ will be obtained. In order to solve linear problems, it can be assumed that $m = 1$ and that the hardening function $H(S) = 0$, so that $S = S_Y$. On the other hand, to solve general nonlinear problems based on the flow (2.10), we have placed no restriction on the dependent variables and rate-sensitivity parameter. We shall consider a state of pure shears with $E^p_{11} = E^p_{22} = 0$.

4.1. Weak formulation of the macroforce balance. Let $\omega_1(x, y)$ and $\omega_2(x, y)$ be weight functions satisfying the homogeneous boundary conditions of the flow rule. These weight functions have been chosen so that the force balances are satisfied for each approximate solution of the flow rule. The weight functions $\omega_1(x, y)$ and $\omega_2(x, y)$ are chosen such that

$$\begin{aligned}0 = &\int_\Omega \Big[\frac{\partial \omega_1}{\partial x}\Big(N\frac{\partial u_1}{\partial x} + M\frac{\partial u_2}{\partial y}\Big) + \mu\Big(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\Big)\frac{\partial \omega_1}{\partial y}\Big] dxdy \\ &+ \int_\Omega \Big[2\mu\frac{\partial E^p_{12}}{\partial y}\omega_1 - b_1\omega_1\Big] dxdy \\ &- \int_\Gamma \Big[\omega_1\Big(N\frac{\partial u_1}{\partial x} + M\frac{\partial u_2}{\partial y}\Big)\eta_x + \mu\omega_1\Big(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\Big)\eta_y\Big] dS\end{aligned}$$

and

$$\begin{aligned}0 = &\int_\Omega \Big[\frac{\partial \omega_2}{\partial y}\Big(N\frac{\partial u_2}{\partial y} + M\frac{\partial u_1}{\partial x}\Big) + \mu\Big(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\Big)\frac{\partial \omega_2}{\partial x}\Big] dxdy \\ &+ \int_\Omega \Big[2\mu\frac{\partial E^p_{12}}{\partial x}\omega_2 - b_2\omega_2\Big] dxdy \\ &- \int_\Gamma \Big[\omega_2\Big(N\frac{\partial u_2}{\partial y} + M\frac{\partial u_1}{\partial x}\Big)\eta_y + \mu\omega_2\Big(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\Big)\eta_x\Big] dS.\end{aligned}$$

4.2. Weak formulation of microforce balance.

4.2.1. *Weak formulation for a linear flow rule.* Let $m = 1$ and the hardening function $H(S) = 0$ so that $S = S_Y$. Assume that a symmetric second order tensor $\mathbf{W}$ is a test function having only three independent components and satisfies the homogeneous boundary conditions of the flow rule. From the flow rule (2.10), it is clear that an integral statement of the flow takes the form

$$0 = \int_\Omega \Big(\frac{S_Y}{d_o}\dot{\mathbf{E}}^p : \mathbf{W} - l^2\Delta\dot{\mathbf{E}}^p : \mathbf{W} - \mathbf{T}_o : \mathbf{W}\Big) dV,$$

so that by Gauss divergence theorems the weak formulation becomes

$$0 = \int_\Omega \Big(l^2 S_Y \nabla\dot{\mathbf{E}}^p \vdots \nabla\mathbf{W} + \frac{S_Y}{d_o}\dot{\mathbf{E}}^p : \mathbf{W} - \mathbf{T}_o : \mathbf{W}\Big) dV - \int_\Gamma l^2 S_Y \mathbf{W} : (\nabla\dot{\mathbf{E}}^p \cdot \vec{n}) dS,$$

where Ω is the region occupied by the body and Γ is the boundary of Ω. The weak formulation in terms of the component of the plastic strain would be given as

$$\begin{aligned} 0 = &\int_\Omega \Big(2\mu\omega_3 E^p_{12} + \omega_3 \frac{s_o}{d_o}\dot{E}^p_{12}\Big)dxdy - \int_\Omega 2\mu\omega_3\Big(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\Big)dxdy \\ &+ \int_\Omega \Big[l^2 \frac{S_Y}{d_o}\frac{\partial \omega_3}{\partial x}\frac{\partial \dot{E}^p_{12}}{\partial x} + l^2 \frac{S_Y}{d_o}\frac{\partial \omega_3}{\partial y}\frac{\partial \dot{E}^p_{12}}{\partial y}\Big]dxdy \\ &- \int_\Gamma \omega_3\Big[\Big(l^2\frac{S_Y}{d_o}\frac{\partial \dot{E}^p_{12}}{\partial x}\Big)\eta_x + \Big(l^2\frac{S_Y}{d_o}\frac{\partial \dot{E}^p_{12}}{\partial y}\Big)\eta_y\Big]dS. \end{aligned}$$

ω_3 is a component of $\mathbf{W}$ corresponding with the component E^p_{12} of the plastic strain tensor $\mathbf{E}^p$.

4.2.2. *Weak formulation for the non-linear flow rule.* Suppose $m \geqslant 0$ and $H(S)$ is a function of S, the flow takes the form of equation (3.12) given by

$$\mathbf{T}_o = SF\dot{\mathbf{E}}^p - l^2 S_Y \operatorname{div}(F\nabla\dot{\mathbf{E}}^p).$$

Given that $\mathbf{H}$ is a test function satisfying the homogeneous boundary condition of the flow rule, then the weak formulation of the flow rule is given by

$$0 = \int_\Omega (l^2 S_Y F\nabla\dot{\mathbf{E}}^p \vdots \nabla\mathbf{H} + SF\dot{\mathbf{E}}^p : \mathbf{H} - \mathbf{T}_o : \mathbf{H})dV - \int_\Gamma l^2 SF\mathbf{H} : (\nabla\dot{\mathbf{E}}^p \cdot \vec{n})dS.$$

In component form, the weak formulation of the flow rule is given by

$$\begin{aligned} 0 = &\int_\Omega \big(2\mu\omega_4 E^p_{12} + \omega_4 SF\dot{E}^p_{12}\big)dxdy - \int_\Omega 2\mu\omega_4\Big(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\Big)dxdy \\ &+ \int_\Omega \Big[l^2 \frac{S_Y}{d_o}\frac{\partial \omega_4}{\partial x}\frac{\partial \dot{E}^p_{12}}{\partial x} + l^2 \frac{S_Y}{d_o}\frac{\partial \omega_4}{\partial y}\frac{\partial \dot{E}^p_{12}}{\partial y}\Big]dxdy \\ &- \int_\Gamma \omega_4\Big[\Big(l^2 S_Y F\frac{\partial \dot{E}^p_{12}}{\partial x}\Big)\eta_x + \Big(l^2 S_Y F\frac{\partial \dot{E}^p_{12}}{\partial y}\Big)\eta_y\Big]dS. \end{aligned}$$

5. Conclusion

In a plane strain problem of the Gurtin–Anand model, it has been shown that during a purely dissipative process of the microstresses, the plastic flow is constrained by elastic incompressibility. In this work, the energetic part of the microstresses is absent, although in actual fact the work of [12] shows that each microstress has the energetic and dissipative parts. It would therefore be of interest to consider the effects of both parts of the microstresses on the flow for a plane strain problem.

References

1. C. Niordson, J. Hutchinson, *Basic strain gradient theories with application to constrained film deformation*, Journal of Mechanics of Materials and Structures **6** (2011), 395–416.
2. N. Fleck, G. M. Muller, M. F. Ashby, J. W. Hutchinson, *Strain gradient plasticity: theory and experiment*, Acta Metallurgica et Materialia **42** (1994), 475–487.
3. K. F. Nielsen, C. F. Niordson, *A numerical basis for strain-gradient plasticity theory: Rate-independent and rate-dependent formulation*, J. Mech. Phys. Solids **63** (2014), 113–127.

4. M. E. Gurtin, L. Anand, *A theory of strain-gradient plasticity for isotropic, plastically irrotational materials. Part I: Small deformations*, J. Mech. Phys. Solids **53** (2005), 1624–1649.
5. F. Ebobisse, A. T. McBride, B. D. Reddy, *On mathematical formulations of a model of gradient plasticity*, IUTAM-Symposium on theoretical, modelling and computational aspects of inelastic media (2008). Cape town: Springer-Verlag, 117–128.
6. B. Reddy, F. Ebobisse, A. McBride, *Well-posedness of a model of strain gradient plasticity for plastically irrotational materials*, Int. J. Plast. **24** (2008), 55–73.
7. A. S. Borokinni, A. P. Akinola, O. P. Akinola, O. O. Fadodun, *A new strain-gradient theory for an isotropic plastically deformed polycrystalline solid body*, Mathematics and Mechanics of Solid (2017), DOI: 10.1177/1081286517720842.
8. A. S. Borokinni, *Finite element procedure for one-dimensional viscoplastic problem of an isotropic slab*, Z. Angew. Math. Phys. **68**(2) (2017), 49, DOI: 10.1007/s00033-017-0793-9.
9. V. A. Lubarda, *On the recoverable and dissipative parts of higher order stresses in strain gradient plasticity*, Int. J. Plast. **78** (2016), 26–43.
10. G. Weber, L. Anand, *Finite deformation constitutive equations and time integration procedure for isotropic, hyperelastic-viscoplastic solids*, Comput. Methods Appl. Mech. Eng. **79** (1990), 172–202.
11. L. Anand, M. E. Gurtin, S. P. Lele, C. Gething, *A one dimensional theory of strain-gradient plasticity: formulation, analysis, numerical results*, J. Mech. Phys. Solids **53** (2005), 1789–1826.
12. M. E. Gurtin, *On framework for small-deformation viscoplasticity: free energy, microforce, strain gradients*, Int. J. Plast. **19** (2003), 47–90.

УТИЦАЈ ЧИСТО ДИСИПАТИВНОГ ПРОЦЕСА МИКРОНАПОНА НА РАВАНСКЕ ПРОБЛЕМЕ НАПРЕЗАЊА ГРАДИЈЕНТА ПЛАСТИЧНОСТИ

Резиме. У раду се разматра теорија раванског напрезања градијента пластичности Гуртин-Анандовог модела [M. Gurtin, L. Anand, , *A theory of strain gradient plasticity for isotropic, plastically irrotational materials Part I: Small deformations*, J. Mech. Phys. Solids **53** (2005), 1624–1649] за изотропни материјал који пролази кроз малу деформацију у одсуству пластичног спина. Претпоставља се да је систем микронапона чисто дисипативан, тако да се слободна енергија своди на функцију еластичне деформације, док су микронапони повезани само са променом пластичног напрезања и градијентом промене пластичног напрезања кроз конститутивне релације. Равански проблем напрезања модела Гуртин-Ананда за чисто дисипативан процес доводи до еластичне нестишљивости. Изводи се слаба формулација једначина тока, чинећи равански проблем напрезања погодним за примену коначних елемената.

(Received 17.10.2017.)
(Revised 19.1.2018.)
(Available online 02.03.2018.)

Distance Learning Institute
University of Lagos
Akoka
Nigeria
aborokini@unilag.edu.ng

Department of Mathematics
Obafemi Awolowo University
Ile-Ife
Nigeria
fadodun.odunayo@gmail.com

Department of Mathematics
Obafemi Awolowo University
Ile-Ife
Nigeria
aakinola@oauife.edu.ng