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VARIATIONAL MODEL OF SCOLIOSIS

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Abstract. Scoliosis, being one of the most widespread spinal diseases among children, has been studied extensively throughout the history of medicine, yet there is no clear understanding of its initiating factors and the mechanogenesis of the monomorphic three-dimensional deformation due to its polyetiological nature. We present a novel mathematical model of the process of emergence of the three-dimensional deformation of the human spine based on variational principles. Typical scoliosis geometry is assumed to be described as minimal curves of a particular energy functional, which are shown to closely resemble actual scoliosis. We investigate the numerical properties of the first stage of scoliosis, which is shown to have the highest influence on the development of the disease.

1. Introduction

A successful treatment of any disease is always preceded by obtaining a clear understanding of the cause of the onset of that condition (pathogenesis), the features of its course and development, and what reaction the organism will demonstrate to arrest that process (sanogenesis) [13, 14]. There do exist a variety of methods of studying the human body that are used for this purpose. The study of idiopathic scoliosis is not an exception, but despite the whole array of results obtained, there is still no clear understanding of its initiating factors and the mechanogenesis of the monomorphic three-dimensional deformation due to its polyetiological nature. However, its direct connection with the process of growth is evident, as is the fact that it is always monomorphic (3D deformation) [1, 2, 4]. This treatment of the disease enables one to use mathematical methods in understanding its initiation and evolution, which is essential for the development of successful pathogenetic treatment strategies.

There do exist a number of mathematical models of biomechanical processes in the human spine, which can be classified into static and dynamic [5, 11, 15]. Static models are able to predict internal tension, deformations and other biomechanical properties of the spine under strain. In [3] a static model was used to reveal

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the stages of the evolution of a scoliosis-like deformation in a two-column model imitating a real spine. In this model, the spine is represented as two columns: the dorsal (back) column, consisting mainly of the spinal cord, and the ventral (front) column, that contains spine bones and muscles. The positions of these columns are shown in fig. 1 where one vertebra is schematically drawn. The spinal cord, forming the dorsal column, is in the centre. It is protected from mechanical damage by a vertebra which, in its turn, is a part of the second, ventral, column. The growth rates of these two columns are predetermined by different factors. Correspondingly, an imbalance in their lengths can appear. Taking into account that both columns are fixed jointly from the top and the bottom, one can understand that the columns should become deformed. More precisely, a 3D deformation should appear. The form of the spine in this initial stage of scoliosis is very important from the medical point of view, because it predetermines the following stages. The present paper is focused on the creation of a mathematical model for this stage of scoliosis.

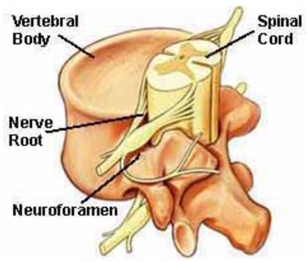

FIGURE 1. One segment of the human spine

Unfortunately, it turned out that the representation of the vertebral complex in the form of a set of two parts (columns) is not enough for a clearer and more detailed understanding of the development of the process of 3D deformation of the spinal column, since it does not reflect all the features of the structure and homeostasis of the vertebral complex. But it allows one to predict some dangers.

Models of another type, dynamic models, usually consider sequences of vertebrae, connected with spring bonds [10, 12]. These models, being fairly computationally simple [5], make it possible to investigate the dynamics of some parts of the spine. However, they are unable to display the dynamics of the spine as a whole [9]. To sum up, it can be concluded that currently all the known models cannot provide an adequate description of all the mechanical processes in the human spine.

In this work we present a new variational mathematical model of the three-dimensional deformation of the human spine.

2. The model

Having the whole physiological background in mind, we base the model on the following assumptions:

- The human body stays in a vertical state to ensure its normal functioning.

- Spine vertebrae have a limited relative movement ability, due to mechanical obstructions and muscles. This ability is different at different parts of the spine.
- Scoliosis occurs only after the spine becomes straight, and its first phase can be described as the ventral column turning around the dorsal column. The distance between the columns can be approximated by a linear or even a constant function.
- The main cause of scoliosis is a lack of balance in the growth of the columns in the two-column model.

Translating this into mathematical form, we obtain the following:

- Spine in the early stages of scoliosis is modelled as a curve on the surface of a cylinder, the cylinder's axis being the static dorsal column and the curve itself being the mobile ventral column. The cylinder's radius R represents the distance between the two columns. The curve is approximated as a function $\varphi(h)$, indicating the angle of the spine's deviation from a straight line. This function's argument h is the height of the corresponding point of the spine over its bottom, and belongs to the interval $[0, H]$, where H is the spine's height. $h = 0$ corresponds to the bottom of the spine, while $h = H$ is the top. The equation of the curve in cartesian coordinates is

$$\gamma(h) = \begin{pmatrix} R\cos\varphi(h) \\ R\sin\varphi(h) \\ h \end{pmatrix}$$

- The length of the curve (the ventral column) is bigger than the cylinder's height H (the dorsal column), thus forcing the curve to appear bent. The length is calculated as (the dot marks the derivative with respect to h):

$$L = \int_0^H \|\dot{\gamma}\| dh = \int_0^H \sqrt{R^2(\dot{\varphi}(h))^2 + 1}\, dh.$$

- The vertical appearance of the human body leads to the following boundary conditions for the curve:

$$\begin{aligned} &\varphi(0) = 0, \qquad &&\varphi(H) = 0, \\ &\dot{\varphi}(h), &&\dot{\varphi}(H) = 0. \end{aligned}$$

- Assuming that the spine tries to minimise the local curvature, the following functional is considered, inspired by works on solid mechanics [7]:

$$K = \int_0^H \alpha(h)(\kappa(h))^2 dh,$$

where $\alpha(h)$ is some empirical spine mobility coefficient and $\kappa(h)$ is the curvature of γ:

$$\kappa(h) = \frac{\|\dot{\gamma} \times \ddot{\gamma}\|}{\|\dot{\gamma}\|^3}.$$

After substitution for γ this becomes

$$K = \int_0^H \alpha(h) \frac{R^2(\dot\varphi(h))^4 + R^4(\dot\varphi(h))^6 + R^2(\ddot\varphi(h))^2}{(R^2(\dot\varphi(h))^2 + 1)^3}.$$

It is known [8] that the curvature of a thin beam is proportional to the beam's bending moment, and the total elastic energy is proportional to the integral of the squared curvature. Assuming that roughly the same mechanical processes happen in a bent human spine, the proposed functional K is conjectured to represent the *energy* of the system. This in turn suggests studying the model in the framework of classical mechanics using this energy functional.

3. Least-energy curves

Any classical mechanical system attempts to minimise its overall energy. This is impossible to achieve in pure closed systems due to the energy conservation law, which forces the system to oscillate near the local minimum or to follow some more complicated trajectories. However, real systems rarely appear closed, and energy loss due to interactions with the external environment typically forces them to move towards the local minimum. The human spine is definitely involved in mechanical interactions with almost any part of the body, prohibiting its treatment as a closed system. This suggests that the spine in scoliosis should stay at the local minimum of energy. Hence, curves of minimal energy are of special interest, as their comparison with real scoliosis data could provide evidence for the adequacy of the proposed model.

The problem now is: given H, R, L and α determine the curves φ that have the specified length, match the boundary conditions and locally minimise the functional K. A numeric approach was taken in order to compute these curves. The values of h were discretised, using finite differences to approximate the derivatives of φ, and finite sums to approximate the integrals. Then, gradient descent was used to find the constrained minima. Some of the found minima are presented in the figures below. The values used were $H = 30\,\text{cm}$ and $R = 4\,\text{cm}$, and α equal to inverse squared vertebrae maximal mobility angles, known from medical practice.

Figures 2 to 6 show some typically found minima for different values of L. The minima in figs. 2 to 4 and 6 correspond to C-shaped scoliosis, while the one in fig. 5 is S-shaped. C-shaped and S-shaped are the two most common forms of scoliosis, and any sufficiently adequate model should predict these forms. For $L = 30.003$ cm two curves (i.e. two local minima of the functional) of different forms are found (figs. 4 and 5), with the S-shaped one having about a two times smaller twisting angle. A bigger value of L can lead to a greater variety of different minimising curves (i.e., a larger number of local minima of the functional) with the same length due to having more possibilities of angular distribution.

Figures 7 and 8 show two minimising curves together with the X-ray images of real scoliosis for comparison. The minima show a tiny deviation in terms of angle (about $1^\circ - 4^\circ$), while the X-ray images show developed forms of scoliosis. Therefore, the X-ray images should be treated as the result of *the evolution* of the initial states represented by the found minima.

Summarising the above, it is clear that the found minima closely resemble a variety of forms of real scoliosis. Thus, one can hope that the model does indeed describe the system under study.

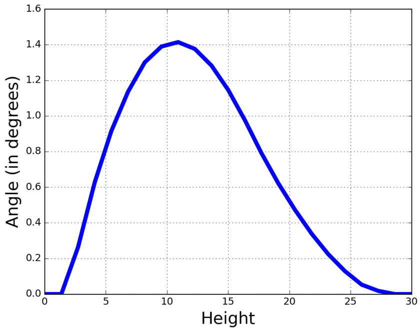

FIGURE 2. Minimising curve, $L = 30.001\,\mathrm{cm}$

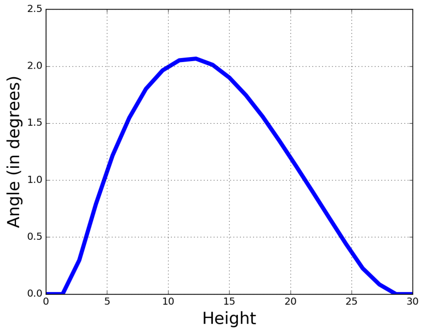

FIGURE 3. Minimising curve, $L = 30.002\,\mathrm{cm}$

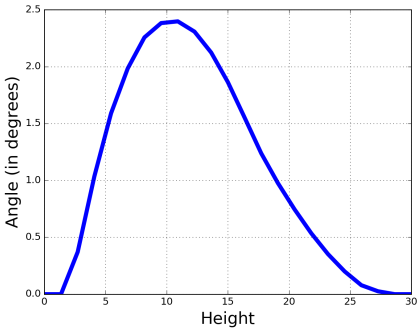

FIGURE 4. Minimising curve, $L = 30.003\,\mathrm{cm}$

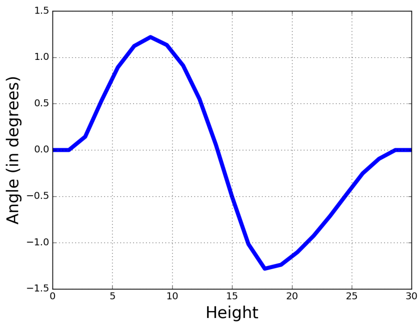

FIGURE 5. Minimising curve, $L = 30.003\,\mathrm{cm}$

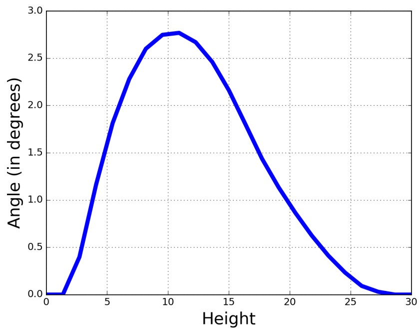

FIGURE 6. Minimising curve, $L = 30.004\,\mathrm{cm}$

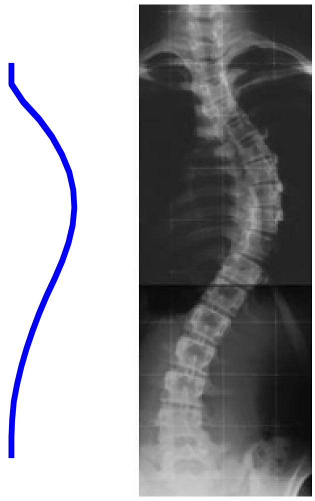

FIGURE 7. Comparison of a minimising curve with a real scoliosis X-ray, $L = 30.001\,\text{cm}$

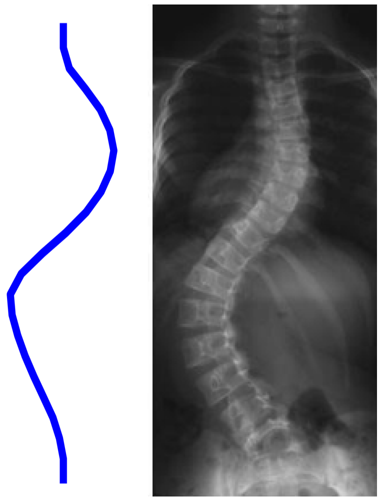

FIGURE 8. Comparison of a minimising curve with a real scoliosis X-ray, $L = 30.002\,\text{cm}$

4. Asymptotic behaviour in the first phases

It is evident from figs. 2 to 6 above that even a very small (about 0.01%) difference in column lengths can lead to significant spine twisting. Of course, such strong sensitivity is related to the ideal boundary condition in the model. In a real system the conditions are softer. But the property of the model reflects the high sensitivity of the corresponding real system. It shows the first stage of scoliosis, i.e., a small imbalance in the lengths. It is well-known that scoliosis is preceded by the state in which the spine is straight ($L = H$). It is the most unstable state. It determines the future evolution of the disease. Correspondingly, an asymptotic behaviour of the curve at $L \approx H$ is of special importance.

We remove the boundary conditions at $h = H$, since when the angle between the first and the last vertebra is less than 4° the spine doesn't attempt to turn back to compensate for the angle difference. Furthermore, in order to be able to study the system analytically, we approximate φ by a piecewise-linear function, which corresponds to a helix-shaped spine and is assumed to have a small curvature. We include a height range $[H_{\text{begin}}, H_{\text{end}}]$ where scoliosis takes place, so that at heights $[0, H_{\text{begin}}]$ the spine is vertically straight, then in the range $[H_{\text{begin}}, H_{\text{end}}]$ the spine assumes a helix-shaped form, and on the last interval $[H_{\text{end}}, H]$ the spine is again vertically straight. The formula for $\varphi(h)$ becomes

$$\varphi(h) = \begin{cases} 0, & h < H_{\text{begin}}, \\ \Phi \frac{h - H_{\text{begin}}}{H_{\text{end}} - H_{\text{begin}}}, & H_{\text{begin}} \leqslant h \leqslant H_{\text{end}}, \\ \Phi, & h > H_{\text{end}}, \end{cases}$$

where Φ is the maximum bending angle. Denote the size of the scoliosis height range as $D = H_{\text{end}} - H_{\text{begin}}$. To compute the length of this curve, break the domain of integration into three intervals and sum the results:

$$\begin{aligned}
L &= \int_0^{H_{\text{begin}}} dh + \int_{H_{\text{begin}}}^{H_{\text{end}}} \sqrt{\frac{R^2\Phi^2}{D^2} + 1}dh + \int_{H_{\text{end}}}^{H} dh \\
&= H_{\text{begin}} + (H_{\text{end}} - H_{\text{begin}})\sqrt{\frac{R^2\Phi^2}{D^2} + 1} + (H - H_{\text{end}}) \\
&= H - (H_{\text{end}} - H_{\text{begin}}) + (H_{\text{end}} - H_{\text{begin}})\sqrt{\frac{R^2\Phi^2}{D^2} + 1} \\
&= (H - D) + D\sqrt{\frac{R^2\Phi^2}{D^2} + 1} \\
&= (H - D) + \sqrt{R^2\Phi^2 + D^2}.
\end{aligned}$$

Denoting $\Delta L = L - H$, assuming that ΔL is small, and solving for Φ, we get

$$\begin{aligned}
&L = (H - D) + \sqrt{R^2\Phi^2 + D^2} \\
&L - H + D = \sqrt{R^2\Phi^2 + D^2} \\
&\Delta L + D = \sqrt{R^2\Phi^2 + D^2} \\
&(\Delta L)^2 + 2D\Delta L + D^2 = R^2\Phi^2 + D^2 \\
&(\Delta L)^2 + 2D\Delta L = R^2\Phi^2 \\
&\Phi = R^{-1}\sqrt{2D\Delta L + (\Delta L)^2} \approx R^{-1}\sqrt{2D\Delta L}.
\end{aligned}$$

The obtained formula and the square root behaviour near $\Delta L = 0$ (correspondingly, the singularity of the derivative) clearly show that, indeed, at the beginning of scoliosis even a small ΔL can cause significant twisting. This effect is more of a geometric nature, and is an expected feature of any model of this system. It should be further noted that ΔL is a quantity extremely difficult to measure, while Φ, D and R *can* be approximately measured for a concrete spine. Thus, the formula can be used to determine ΔL, opening new ways to predicting the evolution of the disease.

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References

1. K. Bagnall, *How can we achieve success in understanding the aetiology of AIS?*, Stud. Health Technol. Inform. **135** (2008), 61–74.
2. M. G. Dudin, M. V. Mikhailovsky, M. A. Sadovoy, D. Yu. Pinchuk, N. G. Fomichev, *Idiopathic scoliosis: who is to blame and what to do?*, Spine Surgery **2** (2014), 8–20.
3. M. G. Dudin, Yu. A. Baloshin, S. V. Bober, I. Yu. Pomortsev, *Mathematical modeling of the human spine*, Russian Journal of Biomechanics **20**(3) (2016), 272–282.
4. G. Duval-Beaupere, J. Dubousset, P. Queneau, A. Grossiord, *Pour unetheorie unique de l'evolution des scoliosis*, La Presse Médicale, **78**(25) (1970): 1141–1146.

5. T. Garcia, B. Ravani, *A biomechanical evaluation of whiplash using a multi-body dynamic model*, Journal of Biomechanical Engineering **125** (2003), 254–265.
6. C. Y. Greaves, M. S. Gadala, T. R. Oxland, *A three-dimensional finite element model of the cervical spine with spinal cord: an investigation of three injury mechanisms*, Annals of Biomedical Engineering **36** (2008), 396–405.
7. B. K. P. Horn, *The curve of least energy*, ACM Trans. Math. Softw. **9**(4) (1983), 441–460.
8. J. P. Hartog, *Strength of Materials*, Dover Publications, New York, 1949.
9. K. T. Huynh, I. Gibson, Zh. Gao, *Development of a Detailed Human Spine Model with Haptic Interface*, In: Haptics Rendering and Applications, A. El Saddik (Ed.), InTech, Shanghai, China, 2012.
10. N. Maurel, F. Lavaste, W. Skalli, *A three-dimensional parameterized finite element model of the lower cervical spine: study of the influence of the posterior articular facets*, Journal of Biomechanics **30** (1997), 921–931.
11. R. N. Natarajan, J. R. Williams, S. A. Lavender, et al., *Poro-elastic finite element model to predict the failure progression in a lumbar disc due to cyclic loading*, Comput. Struct. **85** (2007), 1142–1151.
12. Q. H. Zhang, H. W. Ng, E. C. Teo, *Development and validation of a C0-C7 FE complex for biomechanical study*, J. Biomech Eng. **127** (2005), 729–735.
13. S. M. Pavlenko, *The problem of sanogenesis*, Sovetskaia Meditsina **1** (1968), 3–7.
14. S. M. Pavlenko, *The problem of "disease-disease-recoverty" and sanogenesis*, Sovetskaia Meditsina **12** (1971), 3–8.
15. N. Yoganandan, S. Kumaresan, L. Voo, et al., *Finite element modeling of the C4-C6 cervical spine unit*, Medical Engineering Physics **18** (1996), 569–574.
16. M. G. Dudin, Yu. F. Sinitsky, *On the mechanism of torsion changes in scoliosis*, Journal Orthopaedics Traumatology and Prosthetics **2** (1984), 33–36.

ВАРИЈАЦИОНИ МЕТОДИ СКОЛИОЗЕ

Резиме. Сколиоза, као једна од најраспрострањенијих болести кичме код деце, интензивно је проучавана током читаве историје медицине. Ипак, још не постоји јасно разумевање њених узрочних фактора и механогенезе мономорфне тродимензионалне деформације због њене полиетиличке природе. У раду приказујемо нови математички модел процеса настанка тродимензионалне деформације човечије кичме заснован на варијационим принципима. Претпоставља се да је типична геометрија кичменог стуба описана као минимална крива одређена енергетским функционалом, за коју се показује да је блиска геометрији праве сколиозе. Испитана су нумеричка својства прве фазе сколиозе, за коју се претпоставља да има највећи утицај на развој болести.

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