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DYNAMIC STABILITY OF TIMOSHENKO BEAMS ON PASTERNAK VISCOELASTIC FOUNDATION

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Abstract. The dynamic stability problem of a Timoshenko beam supported by a generalized Pasternak-type viscoelastic foundation subjected to compressive axial loading, where rotary inertia is neglected, is investigated. Each axial force consists of a constant part and a time-dependent stochastic function. By using the direct Liapunov method, bounds of the almost sure asymptotic stability of a beam as a function of viscous damping coefficient, variance of the stochastic force, shear correction factor, parameters of Pasternak foundation, and intensity of the deterministic component of axial loading are obtained. With the aim of justifying the use of the direct Liapunov method analytical results are firstly compared with numerically obtained results using Monte Carlo simulation method. Numerical calculations are further performed for the Gaussian process with a zero mean as well as a harmonic process with random phase. The main purpose of the paper is to point at significance damping parameter of foundation on dynamic stability of the structure.

1. Introduction

The dynamic problem of the beam supported along their length by various types of foundations are very common in structural engineering. Various foundation models such as Winkler, Pasternak, Reissner, or Kerr are used to represent mechanical behavior of the soil. Winkler model in which the medium is taken as a system composed of infinitely close linear springs is the simplest one and is often adopted. In this model, the foundation applies only a reaction normal to the beam which is proportional to the beam deflection. However, such assumtation may sometimes cause the erroneous results, especially on the stresses inside the elastic continuum.

Different foundation models are developed as an effort to better capture the characteristics of an elastic continuum. One of them is a generalized Pasternak viscoelastic model. This foundation model can be successfully used to simulate a

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newly-developed track support. In this model one set of parallel springs and dashpots restricts any vertical displacement and another set prevents the system from rocking motion. Shear viscous layer between beam and the foundation provides the interaction between these viscoelastic elements.

The problem of dynamic response of Timoshenko beam with uniform cross-section supported by a generalized Pasternak-type viscoelastic foundation subjected to an arbitrary-distributed harmonic moving load is studied in [1]. The governing differential equations are derived using Hamilton principle and are translated in complex form by using Fourier transformation. The solution is directed to compute the deflection, bending moment and shear force along the beam length, and effect of velocity and frequency variation of the load on the beam response are investigated. Similarly, using the Hamilton principle in [2] the coupling nonlinear dynamic equations of a rotating, double-tapered, cantilever Timoshenko beam are derived. In the work of Jun et al. [3] the effects of axial force, shear deformation and rotary inertia are investigated according to stochastic response of an axially loaded composite Timoshenko beam with solid or thin-walled closed section exhibiting bending–torsion materially coupling under the stochastic excitations with stationary and ergodic properties.

The harmonic vibrations of a finite beam resting on a tensionless Pasternak foundation subjected to a concentrated harmonic load are investigated in [4]. The effects of the foundation stiffness parameters and the frequency parameter are illustrated by several numerical calculations. As an improved Winkler model, two-parameter Reissner foundation is used for studying contact lengths and deflection of a finite beam subjected to symmetric or asymmetric loading is studied in [5]. Zhang and Murphy [6] developed a general analytical model for the static response of a beam resting on a tensionless elastic foundation subjected to a lateral point load. The forced vibrations of an elastic beam resting on a non-linear tensionless Winkler foundation subjected to a concentrated dynamic load at its centre are considered in [7]. A Hamiltonian variational formulation to determine the energy minimizing boundary conditions of the tensionless contact problem for an Euler–Bernoulli beam resting on a Pasternak or a Reissner two-parameter foundation is given in [8]. The bending of a Timoshenko beam resting on a Kerr-type three-parameter elastic foundation is considered in [9]. The beam is elastically supported and presented solutions are given for one, two and three-parameters foundation models.

Free vibrations analysis of moderately thick rectangular plates resting on Winkler/Pasternak elastic foundation with various boundary conditions is discussed in [10]. The first-order shear deformation theory is employed to extract eigenvalue equation yielding the natural frequencies, and the effect of the foundation stiffness parameters is discussed. A refined higher order shear and normal deformation theory for exponential, power-low and sigmoid functionally graded material plates on elastic Pasternak foundation is presented in [11].

The purpose of the present paper is the investigation of the almost sure stability of a Timoshenko beam resting on generalized Pasternak viscoelastic foundation. Stability regions are numerically calculated as functions of viscous damping coefficient, variance of the stochastic force, shear correction factor, intensity of the deterministic component of axial loading and particularly of the parameters of Pasternak foundation. Very important indication in this paper is dominant role of foundation damping coefficient on dynamic stability of the beam.

The present paper is organized as follows. Partial differential equation of transverse motion of a beam based on Timoshenko beam theory is derived in section 2. For the governing differential equations of beam, the definition of almost-sure stability is given in the same section, and Liapunov functional is constructed in section 3 as a measure of distance between the solution and the trivial one. The conditions of almost-sure stability are obtained in sections 4. The numerical procedure of determining the boundaries of stability, as well as the analysis of obtained results, is given in section 5. Section 6 ends the paper with concluding remarks.

2. Problem formulation

The concept of the structures resting on elastic foundation is developed on the assumptation that the reaction forces of the foundation are proportional, at evry point, to the deflection of structure at that point. The vertical deformation characteristics of the foundation are defined by means continuous, closely spaced linear sprins with known constant of proportionality. This is Winkler's one parameter model which does not accurately represent the characteristics of many practical foundation. In order to eliminate the deficiency of this model, improved theories have been introduced on refinement of Winkler's model by various interconnections such as sher layers along the springs, [12–15]. The two-parameter Pasternak foundation assumes the existence of shear interactions between the spring elements. This may be accomplished by connecting the ends of the springs with a beam consisting of incompressible vertical elements which deform only by transverse shear. The stifnes of the springs and the shear rigidity of the beam are the two parameters of the foundation.

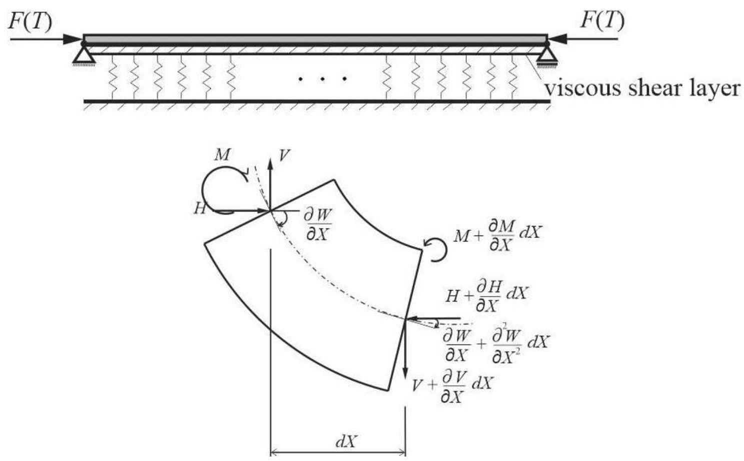

FIGURE 1. Timoshenko beam resting on Pasternak foundation and typical beam element

Figure 1 shows a uniform beam of length L resting on Pasternak foundation, subjected to transverse loading per unit length q, and the axial compressive load F. On beam axes is placed X-axis, and transversal loadings are parallel with Z-axis. A typical beam element is also shown in the figure (without the forces due to viscous damping).

The dynamic equilibrium of the element gives

$$\tag{2.1}\begin{aligned}\rho A\frac{\partial^2 W}{\partial T^2}+\beta_w\frac{\partial W}{\partial T}&=\frac{\partial V}{\partial X}+q,\\ \rho I\frac{\partial^2\Psi}{\partial T^2}+\beta_\Psi\frac{\partial\Psi}{\partial T}&=V-\frac{\partial M}{\partial X}+D\frac{\partial W}{\partial X}-c_\Psi\Psi,\end{aligned}$$

where ρ is mass density, A is cross-sectional area, $W=W(X,T)$ is transverse displacement, Ψ is the angle of rotation of cross-section due to bending, T is time, β_w and β_Ψ are viscous damping coefficients, c_Ψ is rocking stiffness, V is shear force due bending, I is the second moment of area of the cross-section for in-plane bending and M is bending moment.

For Pasternak viscoelastic foundation

$$q=-\bar{c}_w W+\bar{\mu}\frac{\partial^3 W}{\partial T\partial X^2},$$

where $\bar{c}_w$, $\bar{\mu}$ are foundation normal stiffness and damping coefficients.

According to the Timoshenko beam theory

$$\varepsilon_{XX}=-Z\frac{\partial\Psi}{\partial X},\quad \frac{\partial W}{\partial X}=\Psi+\varphi,$$

where ε_{XX} is normal strain, φ is the angle of rotation of cross-section due to shear. The shear force Q and bending moment M are defined as

$$\tag{2.2} Q=V+H\frac{\partial W}{\partial X}=\bar{k}GA\varphi=\bar{k}GA\Big(\frac{\partial W}{\partial X}-\Psi\Big),\quad M=-EI\frac{\partial\Psi}{\partial X},$$

where $\bar{k}$ is shear correction factor. Eliminating V and M from Eqs. (2.1) and (2.2) ones obtain two partial differential equations for transverse displacement W and angle Ψ

$$\tag{2.3}\begin{aligned}\rho A\frac{\partial^2 W}{\partial T^2}+\beta_w\frac{\partial W}{\partial T}-\bar{k}GA\Big(\frac{\partial^2 W}{\partial X^2}-\frac{\partial\Psi}{\partial X}\Big)+F(T)\frac{\partial^2 W}{\partial X^2}+\bar{c}_w W-\bar{\mu}\frac{\partial^3 W}{\partial T\partial X^2}&=0,\\ \rho I\frac{\partial^2\Psi}{\partial T^2}+\beta_\Psi\frac{\partial\Psi}{\partial T}-\bar{k}GA\Big(\frac{\partial W}{\partial X}-\Psi\Big)+c_\Psi\Psi-EI\frac{\partial^2\Psi}{\partial X^2}&=0.\end{aligned}$$

where $F(T)$ is time-dependent stationary stochastic process.

Due to the simply supported edges, boundary conditions have the form

$$\tag{2.4}\left.\begin{aligned}X&=0\\X&=L\end{aligned}\right\}\quad W=0,\quad \frac{\partial^2 W}{\partial X^2}=0.$$

The following parameters can be used to non-dimensionalize Eqs. (2.3)

$$T=k_t t,\qquad W_i=Lw_i,\qquad X=Lx,\qquad k_t=L^2\sqrt{\frac{\partial A}{EI}},$$

$$2\beta = \frac{\beta_w k_t}{\partial A}, \qquad f_o + f(t) = \frac{F(t)}{\rho A L^2}, \qquad c_w = \frac{\bar{c}_w L^4}{EI}, \qquad 2\mu = \frac{\bar{\mu}}{\sqrt{\rho A E I}},$$

where β is reduced damping coefficient, c_w and μ are reduced stiffness and shear viscosity coefficients of the elastic medium, f_o and $f(t)$ are reduced constant and stochastic component of axial force. Having in the mind that is influence of inertia of rotation cross-section is negligible comparing to transverse shear and omitting rotating damping coefficient and rocking stiffness ($c_\Psi = \beta_\Psi = 0$), Eqs. (2.3) have the form

$$\frac{\partial^2 w}{\partial t^2} + 2\beta \frac{\partial w}{\partial t} + c_w w - 2\mu \frac{\partial^3 w}{\partial t \partial x^2} + (f_o + f(t)) \frac{\partial^2 w}{\partial x^2} - k\Big(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \Psi}{\partial x}\Big) = 0 \tag{2.5}$$

$$k\Big(\frac{\partial w}{\partial x} - \Psi\Big) + \frac{\partial^2 \Psi}{\partial x^2} = 0, \tag{2.6}$$

By eliminating angle Ψ from relations (2.5) and (2.6) we obtain

$$\Big(1 - \frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)\Big[\frac{\partial^2 w}{\partial t^2} + 2\beta \frac{\partial w}{\partial t} + c_w w - 2\mu \frac{\partial^3 w}{\partial t \partial x^2} + (f_o + f(t)) \frac{\partial^2 w}{\partial x^2}\Big] + \frac{\partial^4 w}{\partial x^4} = 0. \tag{2.7}$$

The purpose of the present paper is the investigation of almost sure asymptotic stability of the double-nanobeam system subjected to stochastic time-dependent axial loads. To estimate perturbated solutions, it is necessary to introduce a measure of distance $\|.\|$ of solution of Eq. (2.7) with nontrivial initial conditions and the trivial one. Following Kozin [16], the equilibrium state of Eq. (2.7) is said to be almost surely stochastically stable, if

$$P\Big\{\lim_{t\to\infty} \|w(.,t)\| \to 0\Big\} = 1,$$

where P is probability.

3. Liapunov functional construction

The problem formulated in the previous chapter we will solve by using the Liapunov functional method. One of the first general methods of constructing Liapunov functional for deterministic systems was given by Parks–Pritchard [17]. Plaut and Infante [18] provided the construction of Liapunov functional for continuous systems subjected to random excitation, while Kozin [16] introduced the best functional. With the purpose of applying the Liapunov method, we can construct the functional by means of the Parks–Pritchard method [17]. Thus, let us write Eq. (2.7) in the formal form $\mathbf{L}\mathbf{w} = 0$ and introduce the linear operator $\mathbf{N}$ which is a formal derivative of the operator $\mathbf{L}$ with respect to $\partial/\partial t$

$$\mathbf{N}w = 2\Big(1 - \frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)\Big(\frac{\partial w}{\partial t} + \beta w - \mu \frac{\partial^2 w}{\partial x^2}\Big).$$

Integrating the scalar product $\mathbf{LwNw}$ on rectangular $C = [x : 0 \leqslant x \leqslant 1] \times [0, t]$ with respect to Eq. (2.7), it is clear

$$\int_0^1 \int_0^t \mathbf{LwNw}\, dt\, dx = 0. \tag{3.1}$$

After applying partial integration it obtains

$$2\int_0^1\!\!\int_0^t \Big[\Big(1-\frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)\Big(\frac{\partial^2 w}{\partial t^2}+2\beta\frac{\partial w}{\partial t}-2\mu\frac{\partial^3 w}{\partial t\partial x^2}\Big)\Big]$$
$$\Big[\Big(1-\frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)\Big(\frac{\partial w}{\partial t}+\beta w-\mu\frac{\partial^2 w}{\partial x^2}\Big)\Big]dt\,dx$$
$$=\int_0^1 \Big\{\Big[\Big(1-\frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)\Big(\frac{\partial w}{\partial t}+\beta w-\mu\frac{\partial^2 w}{\partial x^2}\Big)\Big]^2+\Big[\Big(1-\frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)\Big(\beta w-\mu\frac{\partial^2 w}{\partial x^2}\Big)\Big]^2\Big\}dx$$
$$+2\int_0^1\!\!\int_0^t \Big[\Big(1-\frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)\Big(\beta\frac{\partial w}{\partial t}-\mu\frac{\partial^3 w}{\partial t\partial x^2}\Big)\Big]\Big(\frac{\partial w}{\partial t}-\frac{1}{k}\frac{\partial^3 w}{\partial t\partial x^2}\Big)dt\,dx,$$

$$2c_w\int_0^1\!\!\int_0^t \Big[\Big(1-\frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)\Big(\frac{\partial w}{\partial t}+\beta w-\mu\frac{\partial^2 w}{\partial t\partial x^2}\Big)\Big]\Big(w-\frac{1}{k}\frac{\partial^2 w}{\partial x^2}\Big)dt\,dx$$
$$=c_w\int_0^1 \Big[w^2+\frac{2}{k}\Big(\frac{\partial w}{\partial x}\Big)^2+\frac{1}{k^2}\Big(\frac{\partial^2 w}{\partial x^2}\Big)^2\Big]dx$$
$$+2c_w\int_0^1\!\!\int_0^t \Big(w-\frac{1}{k}\frac{\partial^2 w}{\partial x^2}\Big)\Big[\Big(1-\frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)\Big(\beta w-\mu\frac{\partial^2 w}{\partial x^2}\Big)\Big]dt\,dx,$$

$$2f_o\int_0^1\!\!\int_0^t \Big[\Big(1-\frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)\Big(\frac{\partial w}{\partial t}+\beta w-\mu\frac{\partial^2 w}{\partial x^2}\Big)\Big]\Big(\frac{\partial^2 w}{\partial x^2}-\frac{1}{k}\frac{\partial^4 w}{\partial x^4}\Big)dt\,dx$$
$$=f_o\int_0^1 \Big(\frac{\partial^2 w}{\partial x^2}-\frac{1}{k}\frac{\partial^4 w}{\partial x^4}\Big)^2dx$$
$$+2f_o\int_0^1\!\!\int_0^t \Big(\frac{\partial^2 w}{\partial x^2}-\frac{1}{k}\frac{\partial^4 w}{\partial x^4}\Big)\Big[\Big(1-\frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)\Big(\beta w-\mu\frac{\partial^2 w}{\partial x^2}\Big)\Big]dt\,dx,$$

$$2\int_0^1\!\!\int_0^t \frac{\partial^4 w}{\partial x^4}\Big[\Big(1-\frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)\Big(\frac{\partial w}{\partial t}+\beta w-\mu\frac{\partial^2 w}{\partial x^2}\Big)\Big]dt\,dx$$
$$=\int_0^1\Big[\Big(\frac{\partial^2 w}{\partial x^2}\Big)^2+\frac{1}{k}\Big(\frac{\partial^3 w}{\partial x^3}\Big)^2\Big]dx+2\int_0^1\!\!\int_0^t \frac{\partial^4 w}{\partial x^4}\Big[\Big(1-\frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)\Big(\beta w-\mu\frac{\partial^2 w}{\partial x^2}\Big)\Big]dt\,dx.$$

Now, Eq. (3.1), can be written as

$$\mathbf{V}|_0^t-\int_0^t \frac{d\mathbf{V}}{dt}dt=0,$$

where

$$\mathbf{V}=\int_0^1 \Big\{\Big[\Big(1-\frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)\Big(\frac{\partial w}{\partial t}+\beta w-\mu\frac{\partial^2 w}{\partial x^2}\Big)\Big]^2$$
$$+\Big[\Big(1-\frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)\Big(\beta w-\mu\frac{\partial^2 w}{\partial x^2}\Big)\Big]^2$$
$$+c_w\Big[w^2+\frac{2}{k}\Big(\frac{\partial w}{\partial x}\Big)^2+\frac{1}{k^2}\Big(\frac{\partial^2 w}{\partial x^2}\Big)^2\Big] \tag{3.2}$$

$$- f_o\Big(\frac{\partial w}{\partial x} - \frac{1}{k}\frac{\partial^3 w}{\partial x^3}\Big)^2 + \Big(\frac{\partial^2 w}{\partial x^2}\Big)^2 + \frac{1}{k}\Big(\frac{\partial^3 w}{\partial x^3}\Big)^2\Big\}dx.$$

Time derivative of the functional (3.2) is

$$\begin{aligned}(3.3)\quad \frac{d\mathbf{V}}{dt} = 2\int_0^1 \Big\{&\Big(1 - \frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)\Big[\Big(\beta\frac{\partial w}{\partial t} - \mu\frac{\partial^3 w}{\partial t \partial x^2}\Big)\Big] \\ &+ c_w\Big(w - \frac{1}{k}\frac{\partial^2 w}{\partial x^2}\Big)\Big[\Big(1 - \frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)\Big(\beta w - \mu\frac{\partial^2 w}{\partial x^2}\Big)\Big] \\ &+ f_o\Big(\frac{\partial^2 w}{\partial x^2} - \frac{1}{k}\frac{\partial^4 w}{\partial x^4}\Big)\Big[\Big(1 - \frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)\Big(\beta w - \mu\frac{\partial^2 w}{\partial x^2}\Big)\Big] \\ &+ f(t)\Big(\frac{\partial^2 w}{\partial x^2} - \frac{1}{k}\frac{\partial^4 w}{\partial x^4}\Big)\Big[\Big(1 - \frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)\Big(\frac{\partial w}{\partial t} + \beta w - \mu\frac{\partial^2 w}{\partial x^2}\Big)\Big] \\ &+ \frac{\partial^4 w}{\partial x^4}\Big[\Big(1 - \frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)\Big(\beta w - \mu\frac{\partial^2 w}{\partial x^2}\Big)\Big]\Big\}dx.\end{aligned}$$

The measure of solution may be taken as $\|w\| = \sqrt{\mathbf{V}}$ and functional $\mathbf{V}$ must be positive definite. It will be fulfilled if

$$\begin{aligned}(3.4)\quad \mathbf{V} \geqslant \int_0^1 \Big\{&c_w\Big[w^2 + \frac{2}{k}\Big(\frac{\partial w}{\partial x}\Big)^2 + \frac{1}{k^2}\Big(\frac{\partial^2 w}{\partial x^2}\Big)^2\Big] \\ &- f_o\Big(\frac{\partial w}{\partial x} - \frac{1}{k}\frac{\partial^3 w}{\partial x^3}\Big)^2 + \Big(\frac{\partial^2 w}{\partial x^2}\Big)^2 + \frac{1}{k}\Big(\frac{\partial^3 w}{\partial x^3}\Big)^2\Big\}dx.\end{aligned}$$

According to the boundary conditions (2.4), we may write the solutions in the form

$$(3.5)\qquad w(x,t) = \sum_{m=1}^{\infty} T_m(t)\sin\alpha_m x,$$

where $\alpha_m = m\pi$, and relation (3.4) leads to

$$f_o < \frac{c_w}{\alpha_m^2} + \frac{\alpha_m^2}{1 + \frac{1}{k}\alpha_m^2},$$

which implies the condition that the beam is statically stable. When is neglected transverse shear, $k \to \infty$, we obtain static stability condition for Euler–Bernoulli beam resting on Pasternak foundation

$$f_o < \frac{c_w}{\alpha_m^2} + \alpha_m^2.$$

Liapunov functional is positive definite if deterministic component of axial loading is smaller than critical buckling load.

4. Almost-sure stability

Let a scalar function $\lambda(t)$ be defined as

$$(4.1)\qquad \frac{1}{\mathbf{V}}\frac{d\mathbf{V}}{dt} \leqslant \lambda(t).$$

As a minimum point is a particular case of the stationary point, we may write

$$\delta(\dot{\mathbf{V}} - \lambda \mathbf{V}) = 0. \tag{4.2}$$

The associated Euler's equations of the variational problem (4.2) are given by

$$\Big(1 - \frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)^{(2)} \Big[\Big(\lambda + 2\beta - 2\mu \frac{\partial^2}{\partial x^2}\frac{\partial w}{\partial t}\Big) + \Big(\lambda\beta - \mu\lambda\frac{\partial^2}{\partial x^2} + f(t)\frac{\partial^2}{\partial x^2}\Big)w\Big] = 0, \tag{4.3}$$

$$\begin{aligned} &\Big(1 - \frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)^{(2)}\Big(\lambda\beta - \mu\lambda\frac{\partial^2}{\partial x^2} + f(t)\frac{\partial^2}{\partial x^2}\Big)\frac{\partial w}{\partial t} \\ &\quad + \Big\{\lambda\Big[2\Big(1 - \frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)^{(2)}\Big(\beta - \mu\frac{\partial^2}{\partial x^2}\Big)^{(2)} + c_w\Big(1 - \frac{2}{k}\frac{\partial^2}{\partial x^2} + \frac{1}{k^2}\frac{\partial^4}{\partial x^4}\Big) \\ &\quad + f_o\Big(\frac{\partial^2}{\partial x^2} - \frac{2}{k}\frac{\partial^4}{\partial x^4} + \frac{1}{k^2}\frac{\partial^6}{\partial x^6}\Big) + \frac{\partial^4}{\partial x^4} - \frac{1}{k}\frac{\partial^6}{\partial x^6}\Big] + c_w\Big(1 - \frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)^{(2)}\Big(\beta - \mu\frac{\partial^2}{\partial x^2}\Big) \\ &\quad + 2f_o\frac{\partial^2}{\partial x^2}\Big(1 - \frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)^{(2)}\Big(\beta - \mu\frac{\partial^2}{\partial x^2}\Big) + 2f(t)\frac{\partial^2}{\partial x^2}\Big(1 - \frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)^{(2)}\Big(\beta - \mu\frac{\partial^2}{\partial x^2}\Big) \\ &\quad + 2\frac{\partial^4}{\partial x^4}\Big(1 - \frac{1}{k}\frac{\partial^2}{\partial x^2}\Big)\Big(\beta - \mu\frac{\partial^2}{\partial x^2}\Big)\Big\}w = 0, \end{aligned} \tag{4.4}$$

Introduction of solution (3.5) into Eqs. (4.3) and (4.4) yields

$$\begin{aligned} &\sum_{m=1}^{\infty}\big\{(\lambda + \ell_{1m})\dot{T}_m(t) + (\lambda\ell_{1m} - \alpha_m^2 f(t))T_m(t)\big\} = 0, \\ &\sum_{m=1}^{\infty}\Big\{(\lambda\ell_{1m} - \alpha_m^2 f(t))\dot{T}_m(t) \\ &\quad + \Big[(\lambda + 2\ell_{1m})\Big(c_w - f_o\alpha_m^2 + \frac{\alpha_m^4}{\ell_{2m}}\Big) + 2\ell_{1m}(\lambda\ell_{1m} - \alpha_m^2 f(t))\Big]T_m(t)\Big\} = 0, \end{aligned} \tag{4.5}$$

where

$$\ell_{1m} = \beta + \mu\alpha_m^2, \quad \ell_{2m} = 1 + \frac{1}{k}\alpha_m^2.$$

Non-trivial solutions for the functions $\dot{T}_m(t)$ and $T_m(t)$ exist if the determinant of system (4.5) vanishes, which yields a algebraic equation

$$\lambda_m^2 + 4\ell_{1m}\lambda_m + \frac{a_m}{b_m} = 0, \tag{4.6}$$

where

$$a_m = 4\ell_{1m}^2\ell_{2m}\Big(c_w - f_o\alpha_m^2 + \frac{\alpha_m^4}{\ell_{2m}} - \alpha_m^2 f(t)\Big) - \alpha_m^4\ell_{2m}f^2(t),$$

$$b_m = \ell_{2m}\Big(\ell_{1m}^2 + c_w - f_o\alpha_m^2 + \frac{\alpha_m^4}{\ell_{2m}}\Big).$$

Hence, from Eq. (4.6)

$$\lambda_m = \frac{\big|2(\beta + \mu\alpha_m^2)^2 + f(t)\alpha_m^2\big|}{\sqrt{(\beta + \mu\alpha_m^2)^2 + f_o\alpha_m^2 + \frac{\alpha_m^4}{1 + \kappa^{-1}\alpha_m^2}}} - 2(\beta + \mu\alpha_m^2).$$

By solving the differential inequality (4.1), we can estimate values of the functional $\mathbf{V}$

$$\frac{d\mathbf{V}}{dt} \leqslant \big(\max_m \lambda_m\big)\mathbf{V}, \tag{4.7}$$

and after integrating relation (4.7) with respect to time, we obtain

$$\mathbf{V} \leqslant \mathbf{V}_0 \exp\left[\frac{1}{t}\int_0^t \max_m \lambda_m(\tau)d\tau\right]t.$$

When the process $f(t)$ is ergodic and stationary, it can be concluded that the trivial solution of equation (2.7) is almost surely asymptotically stable if

$$E\big\{\max_m \lambda_m(t)\big\} < 0, \tag{4.8}$$

where $E\{\cdot\}$ denotes the operator of the mathematical expectation.

5. Numerical results and discussion

The relation (4.8) yields the possibility to obtain the almost sure stability domains of Timoshenko beams resting on Pasternak viscoelastic foundation. The stability domains of the beam are obtained for the Gaussian process and harmonic process with uniformly distributed phase in the range $[0, 2\pi]$. Assuming the harmonic process $f_h(t) = A_h \cos(\omega t + \theta)$, where the phase θ is random variable and the amplitude A_h is used to calculate the process variance $A_h^2/2$.

The almost sure asymptotic stability boundaries for Gaussian process and presented by full line, and for harmonic process by dashed line. Calculation was made by using Gauss–Christoffel quadratures; for the Gaussian process the parameters of Gauss–Hermite quadrature, and for harmonic process Gauss–Chebyshev quadrature are used.

In this section, analytical results are firstly compared with the numerical results obtained from the Monte Carlo simulation with the aim of justifying the use of the direct Liapunov method for this system. The Monte Carlo simulation method is widely used [19, 20] and it is very important in assessing the validity of the approximate analytical results.

Now, according to the numerical procedure presented in [21], relation (3.5) is substituted in Eqs. (2.7) and (3.3) which leads to the discretizated forms of these equations

$$\ddot{T}_m + 2(\beta + \mu\alpha_m^2)\dot{T}_m + \left[c_w + \frac{k\alpha_m^4}{k + \alpha_m^2} - (f_o + f(t))\alpha_m^2\right]T_m - 0, \tag{5.1}$$

$$\begin{aligned}\frac{d\mathbf{V}}{dt} = -2\Bigg[&\left(1 + \frac{\alpha_m^2}{k}\right)^2(\beta + \mu\alpha_m^2)\dot{T}_m + c_w\left(1 + \frac{\alpha_m^2}{k}\right)^2(\beta + \mu\alpha_m^2)T_m^2\\ &-f_o\alpha_m^2\left(1 + \frac{\alpha_m^2}{k}\right)^2(\beta + \mu\alpha_m^2)T_m^2 - f(t)\alpha_m^2\left(1 + \frac{\alpha_m^2}{k}\right)^2(\dot{T}_m + \beta T_m + \mu\alpha_m^2 T_m)T_m\\ &+\alpha_m^4\left(1 + \frac{\alpha_m^2}{k}\right)^2(\beta + \mu\alpha_m^2)T_m^2\Bigg],\end{aligned} \tag{5.2}$$

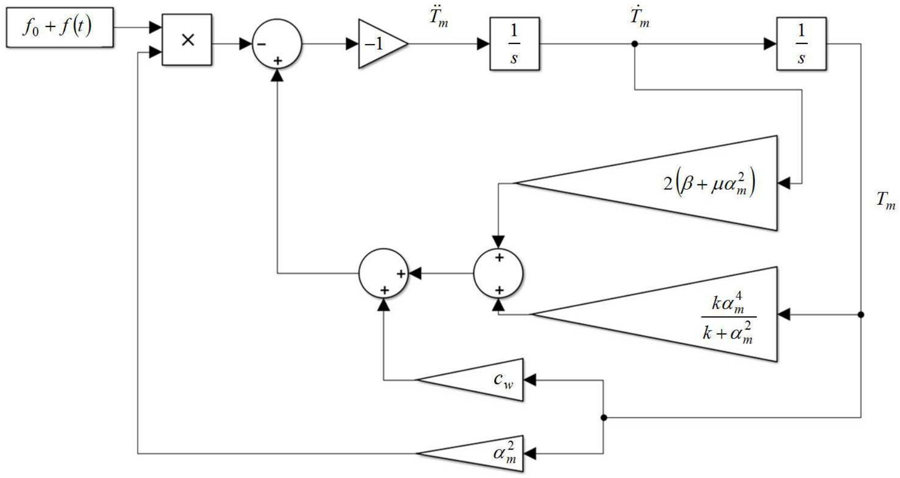

FIGURE 2. Block chart of the system presented by Eq. (5.1)

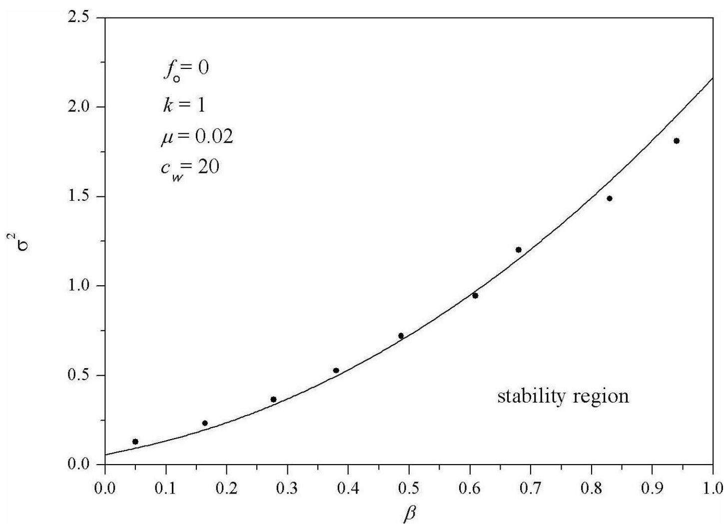

FIGURE 3. Numerical and analytical results comparison

where according to Eq. (5.1) the appropriate simulation scheme is given in Fig. 2.

By using the Monte Carlo simulation method, the system presented in Fig. 2 is simulated for different pairs of analytically obtained variances and damping coefficients where the estimated states from simulation are further replaced in Eq. (5.2) with the aim of calculating new pairs of variances and damping coefficients for the stability boundary. The comparison between analytical results (solid line) and the results obtained by the Monte Carlo simulation (dotted line) is presented in Fig. 3.

In Figs. 4, 5 and 6 stability regions for a beam are plotted in plane of variance σ^2 and viscous damping coefficient β as function of reduced parameters of the foundation, shear correction factor and deterministic load. It can be noted that stability regions are larger for Gaussian than for harmonic process.

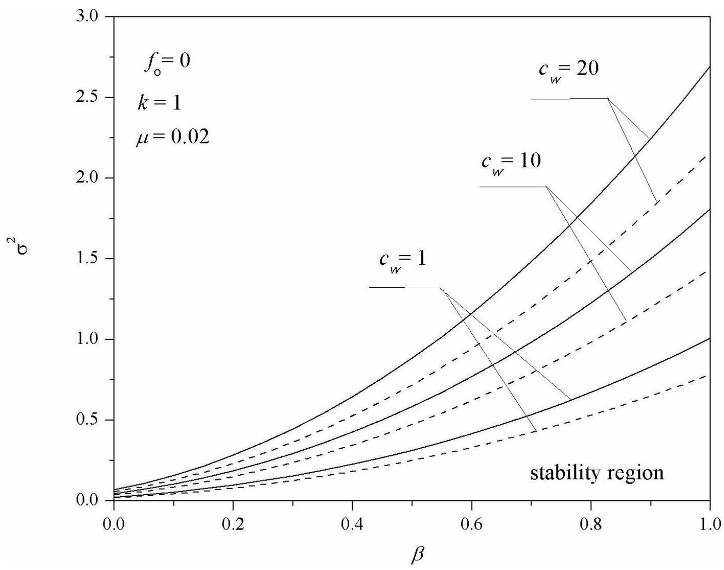

FIGURE 4. Stability regions of a Timoshenko beam as a function of elastic medium stiffness

Fig. 4 illustrate the effect of the elastic medium stiffness on stability regions for both processes, and it is evident the stability regions increase when stiffness c_w increases. For higher values of the stiffness, difference of largeness stability regions whose generate Gaussian process with respect to harmonical is larger.

Fig. 5 indicates that deterministic axial component acting on the beam, if it is compressive force, reduced stability regions, while tensile force enlarged stability regions. Lessening of the stability regions is more explicit when acts compressive force than enlargement stability regions when acts tensile force.

In Fig. 6 influence of transverse shear on dynamic stability of the beam is shown. It is evident that stability regions of short and thick beams where transverse shear can not be neglected, significally smaller with respect to the beam with high length to the smallest cross-section dimension ratio. Stability regions of Euler–Bernoulli beams, where influence of transverse shear can be neglected ($k \to \infty$), are remarkable larger than for Timoshenko beams.

It is evident influence of the foundation damping coefficient on enlarge regions of almost sure stability of the Timoshenko beam, Fig. 7. Numerical results shown that small increases of coefficient μ rapidly enlarged stability regions. Let we notice that this coefficient does not affect on static stability, but has particular important on dynamic stability.

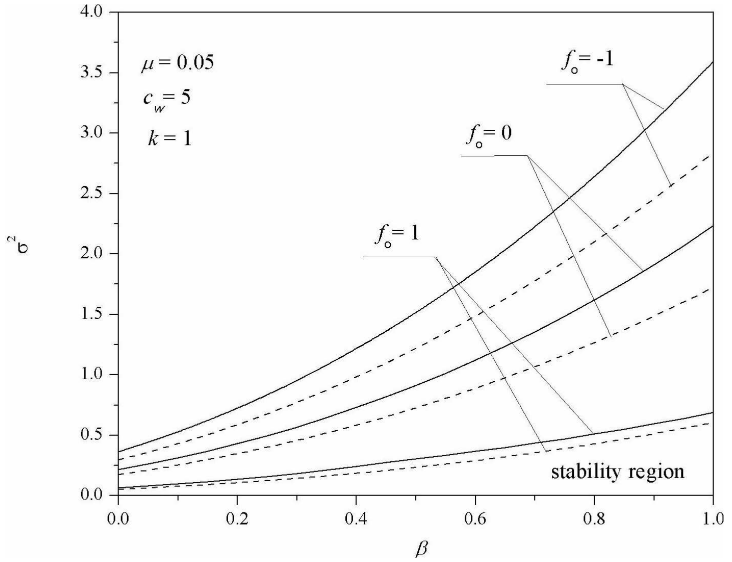

FIGURE 5. Stability regions of a Timoshenko beam as a function of deterministic axial load

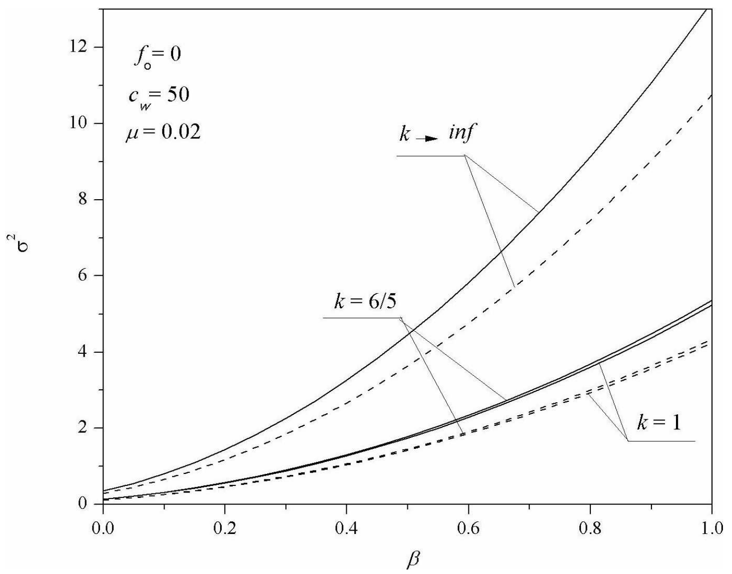

FIGURE 6. Stability regions of a Timoshenko beam as a function of the shear correction factor

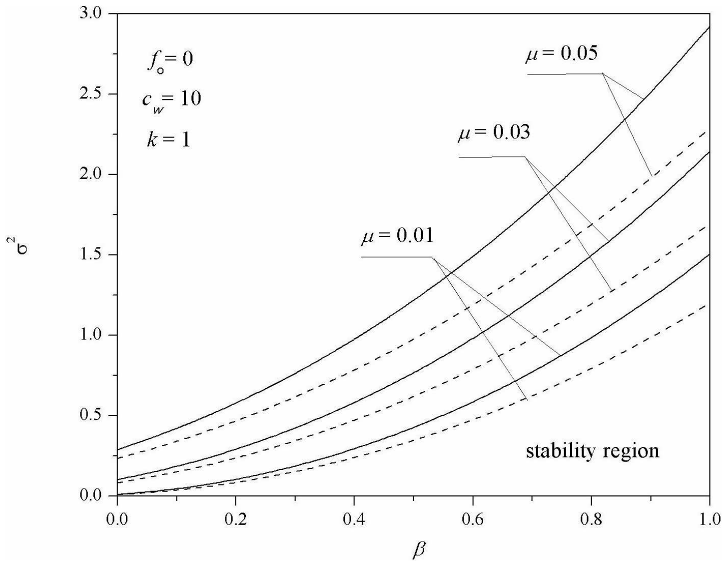

FIGURE 7. Stability regions of a Timoshenko beam as a function of the shear correction factor

6. Conclusions

According to the direct Liapunov method, on the basis Timoshenko beam theory, the stability of a beam resting on generalized Pasternak viscoelastic foundation and subjected to compressive axial loadings is studied. The axial forces acting on its ends consist of a constant part and a time-dependent stochastic function.

Regions of almost sure stability are obtained as a function of viscous damping coefficient, parameters of foundation, shear correction factor, variances of the stochastic forces, and intensity of the deterministic components of axial loading.

Transverse shear leads to decreasing of almost sure stability regions, for accustomed values of the shear correction factor ($k = 2/3$; $5/6$; 1) bounds of stability regions are very closed.

The increase of elastic medium stiffness leads to increasing the stability regions, while the growth of deterministic axial compressive force decreases stability regions. Generally, stability regions are larger for the Gaussian than the harmonic process.

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ДИНАМИЧКА СТАБИЛНОСТ ТИМОШЕНКОВИХ ГРЕДА НА ПАСТЕРНАКОВОЈ ВИСКОЕЛАСТИЧНОЈ ПОДЛОЗИ

Резиме. У раду је проучаван проблем динамичке стабилности Тимошенкове греде која лежи на генерализованој вискоеластичној подлози Пастернаковог типа, а подвргнута је дејству притисног аксијалног оптерећења. Имајући у виду ранија истраживања, занемарен је утицај инерције обртања попречног пресека носача. Свака аксијална сила се састоји од константне компоненте и временски променљиве стохастичке функције. Коришћењем директне методе Љапунова добијене су границе скоро сигурне асимптотске стабилности греде у функцији коефицијента вискозног пригушења, варијансе стохастичке силе, фактора корекције смицања, параметара Пастернакове подлоге и интензитета детерминистичке компоненте притисне силе. У циљу потврде коришћења директне методе Љапунова извршено је претходно тестирање аналитичких резултата нумеричким резултатима добијеним Монте Карло симулацијом. Нумеричка срачунавања су, надаље извршена за Гаусов процес са нултим математичким очекивањем као и за хармонијски процес са случајном фазом. Главни допринос рада се огледа у истицању значаја параметра пригушења подлоге на динамичку стабилност структуре.

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