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FRACTIONAL TELEGRAPHER'S EQUATION AS A CONSEQUENCE OF CATTANEO'S HEAT CONDUCTION LAW GENERALIZATION

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Abstract. Fractional telegrapher's equation is reinterpreted in the setting of heat conduction phenomena and reobtained by considering the energy balance equation and fractional Cattaneo heat conduction law, generalized by taking into account the history of temperature gradient as well. Using the Laplace transform method, fractional telegrapher's equation is solved on semi-bounded domain for the zero initial condition and solution is obtained as a convolution of forcing temperature on the boundary and impulse response. Some features of such obtained solution are examined.

1. Introduction and model formulation

Fractional telegrapher's equation, generalizing classical telegrapher's equation

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}u(x,t)+\frac{1}{D}\frac{\partial}{\partial t}u(x,t)=\frac{\partial^2}{\partial x^2}u(x,t), \tag{1.1}$$

where c is speed of wave propagation, D is diffusion coefficient and u is some physical quantity, x is the space coordinate and t is time, is applied in modelling various physical phenomena ranging from the anomalous diffusion and heat conduction to the wave propagation in different media: viscoelastic materials and guided media for electromagnetic signals, which has led to a broad range of generalizations of telegrapher's equation (1.1). Fractional telegrapher's equation, containing the fractional derivatives of orders $\alpha \in (0,2)$ and $\beta \in (0,1)$, instead of the second and first order partial derivatives in (1.1), is considered in [3,28]. Multi-term fractional telegrapher's equation, considered in [8], contain N terms of fractional derivatives of orders $n\alpha$, with $n \in \{1,\dots,N\}$ and $\alpha \in (0,1)$, while in [25] multi-term telegrapher's equation, with the highest order derivative being equal to one and fractional derivatives of order between zero and one, is used to describe the diffusion-type processes. As a special case of more general distributed-order diffusion-wave equation, fractional multi-term telegrapher's equation is analyzed in [4] in the setting

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of Volterra integral equations and solved in [5] using integral transform method. In [12], the second and the first order partial derivatives in (1.1) are respectively replaced by the fractional derivatives of orders $\alpha \in (1,2)$ and $\alpha - 1$, with the addition of term linear in u and forcing term. Problems on unbounded and bounded domains, modelled by fractional telegrapher's equation, are considered analytically and numerically in [7,9,20,22] without the presence of forcing term and in [15,16] with the presence of forcing term.

The aim is to reconsider fractional telegrapher's equation in the setting of heat conduction phenomena, as well as to obtain its solution on semi-infinite domain in the integral form for all values of the orders of fractional derivatives, using the approach presented in [11] and extending the results of [3], and to examine the features of such obtained solutions.

Fractional telegrapher's equation

$$\tau {}_0\mathrm{D}_t^{1+\alpha} \Delta T(x,t) + {}_0\mathrm{D}_t^{\beta} \Delta T(x,t) = \mathcal{D} \frac{\partial^2}{\partial x^2} \Delta T(x,t), \quad \alpha, \beta \in (0,1),\ \mathcal{D} = \frac{\lambda}{\rho c}, \tag{1.2}$$

where τ, ρ, c, and λ denote material constants: generalized relation time (measured in $\mathrm{s}^{1+\alpha-\beta}$), material density, specific heat capacity, and generalized thermal conductivity (measured in $\frac{\mathrm{J}}{\mathrm{K\,m\,s}^{\beta}}$); $\Delta T = T - T_{\mathrm{ref}}$ denotes temperature difference between the absolute temperature T and constant reference temperature T_{ref} as a function of spatial coordinate $x \in [0, \infty)$ and time $t > 0$; ${}_0\mathrm{D}_t^{n+\xi}$ is the operator of Riemann–Liouville fractional differentiation of order $n + \xi$, with $n \in \mathbb{N}_0$ and $\xi \in (0,1)$, defined by

$$ {}_0\mathrm{D}_t^{n+\xi} y(t) = \frac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}} \Big(\frac{t^{-\xi}}{\Gamma(1-\xi)} * y(t) \Big), \quad t > 0, $$

see [18], with $*$ denoting the convolution in time: $f(t) * g(t) = \int_0^t f(u) g(t-u) \mathrm{d}u$, as the model of heat conduction in one-dimensional semi-infinite rigid conductor will be obtained in Appendix A by considering: material heating, described by the energy balance equation

$$\rho c \frac{\partial}{\partial t} \Delta T(x,t) = -\frac{\partial}{\partial x} q(x,t), \tag{1.3}$$

where q denotes the heat flux, and heat conduction, described by the two-parameter generalization (parameters α, β) of fractional Cattaneo heat conduction law

$$\tau {}_0\mathrm{D}_t^{1-\beta} \big({}_0\mathrm{D}_t^{\alpha} q(x,t) \big) + q(x,t) = -\lambda {}_0\mathrm{D}_t^{1-\beta} \Big(\frac{\partial}{\partial x} \Delta T(x,t) \Big). \tag{1.4}$$

Note that system consisting of the energy balance equation (1.3) and constitutive equation (1.4) is equivalent to fractional telegrapher's equation (1.2) if the temperature gradient ($\frac{\partial}{\partial x} \Delta T$) and heat flux ($q$) are bounded at initial moment ($t = 0$), as well as if initial condition is zero ($\Delta T(x, 0) = 0$). The temperature difference with respect to the reference temperature is introduced in order to emphasize that equations (1.3) and (1.4), and therefore (1.2) as well, are valid only for temperatures much higher than the absolute zero, since equations (1.3) and (1.4) are invariant under the transformation $T = \Delta T + T_{\mathrm{ref}}$. In this setting, the zero initial condition means that the conductor has uniformly distributed reference temperature at initial moment, rather than zero absolute temperature.

The fractional Cattaneo constitutive equation

$$\tau_0 \mathrm{D}_t^{\alpha} q(x,t) + q(x,t) = -\lambda \frac{\partial}{\partial x} \Delta T(x,t), \tag{1.5}$$

is obtained from (1.4) when $\beta = 1$, since the fractional derivative of the zeroth order of a function is a function itself. Additionally, taking $\alpha = 1$ in (1.5), the classical Cattaneo law

$$\tau \frac{\partial}{\partial t} q(x,t) + q(x,t) = -\lambda \frac{\partial}{\partial x} \Delta T(x,t), \tag{1.6}$$

is obtained, since the fractional derivative of the first order is the classical first order derivative, while taking $\alpha = 0$, the Fourier law is recovered. On the other hand, assuming $\alpha = 0$ in (1.4), the one-parameter generalization (parameter β) of fractional Cattaneo law is obtained in the form

$$\tau_0 \mathrm{D}_t^{1-\beta} q(x,t) + q(x,t) = -\lambda_0 \mathrm{D}_t^{1-\beta} \big(\frac{\partial}{\partial x} \Delta T(x,t)\big), \tag{1.7}$$

since there is a fractional derivative acting not only on the heat flux, but also on the temperature gradient, having both derivatives of the same order. Additionally, taking $\beta = 0$ in (1.7), the generalization of classical Cattaneo law is obtained, since the left-hand-side of (1.7) is as in the classical Cattaneo law, while there is a time-derivative of temperature gradient on the right-hand-side.

The history of heat flux change is taken into account by the fractional Cattaneo law (1.5), since the fractional derivative of order $\alpha \in (0,1)$ is acting on the heat flux, while the one-parameter generalization of fractional Cattaneo law (1.7) takes into account the history of temperature gradient change as well, having the same orders $(1-\beta)$ of fractional derivatives acting on both heat flux and temperature gradient. In the two-parameter generalization of fractional Cattaneo heat conduction law (1.4), both type of history effects are taken simultaneously.

Considering one-and two-parameter generalizations of the fractional Cattaneo heat conduction law (1.7) and (1.4) in the hereditariness setting, i.e., considering the heat flux being dependent on the memory of temperature gradient in the form

$$q(x,t) = -\lambda K(t) * \frac{\partial}{\partial x} \Delta T(x,t) = -\lambda \int_0^t K(t-u) \frac{\partial}{\partial x} \Delta T(x,u) \mathrm{d}u,$$

where K is the memory kernel, one has (1.4) and (1.7) respectively rewritten in terms of the one-and two-parameter Mittag-Leffler functions

$$e_{\xi,\lambda}(t) = E_{\xi}(-\lambda t^{\xi}) = \mathcal{L}^{-1}\Big[\frac{s^{\xi-1}}{s^{\xi}+\lambda}\Big], \text{ with } E_{\xi}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\xi n + 1)}, \tag{1.8}$$

$$e_{\xi,\zeta,\lambda}(t) = t^{\zeta-1} E_{\xi,\zeta}(-\lambda t^{\xi}) = \mathcal{L}^{-1}\Big[\frac{s^{\xi-\zeta}}{s^{\xi}+\lambda}\Big], \text{ with } E_{\xi,\zeta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\xi n + \zeta)}, \tag{1.9}$$

see [14], as

$$q(x,t) = -\frac{\lambda}{\tau} e_{1+\alpha-\beta,\alpha,\frac{1}{\tau}}(t) * \frac{\partial}{\partial x}\Delta T(x,t) \quad \text{and} \tag{1.10}$$

$$q(x,t) = -\frac{\lambda}{\tau}\frac{\partial}{\partial x}\Delta T(x,t) + \frac{\lambda}{\tau}\Big(-\frac{\mathrm{d}}{\mathrm{d}t} e_{1-\beta,\frac{1}{\tau}}(t)\Big) * \frac{\partial}{\partial x}\Delta T(x,t). \tag{1.11}$$

Unlike the memory kernels

$$K_{cC}(t) = \frac{1}{\tau}\mathrm{e}^{-\frac{t}{\tau}} > 0 \quad \text{and} \quad K_{fC}(t) = -\tau\frac{\mathrm{d}}{\mathrm{d}t} e_{\alpha,\frac{1}{\tau}}(t) > 0,\ t > 0,$$

corresponding to the classical and fractional Cattaneo heat conduction laws respectively, that are completely monotonic functions (function f is completely monotonic if $(-1)^n f^{(n)}(t) \geqslant 0$, $n \in \mathbb{N}_0$), memory kernels

$$K_{opC}(t) = \frac{1}{\tau}\Big(\delta(t) + \frac{\mathrm{d}}{\mathrm{d}t} e_{1-\beta,\frac{1}{\tau}}(t)\Big) \quad \text{and} \quad K_{tpC}(t) = \frac{1}{\tau} e_{1+\alpha-\beta,\alpha,\frac{1}{\tau}}(t), \tag{1.12}$$

corresponding to one-and two-parameter generalizations of the fractional Cattaneo heat conduction law (1.11) and (1.10) respectively, are either sum of the term with Dirac δ-distribution (Fourier type kernel) and term containing negative memory kernel ($\frac{\mathrm{d}}{\mathrm{d}t} e_{1-\beta,\frac{1}{\tau}}(t) < 0$, $t > 0$), or not completely monotonic ($e_{\xi,\zeta,\lambda}$ is completely monotonic if $\xi \in (0,1]$ and $\xi \leqslant \zeta$, see [14]). Derivation of the memory kernels (1.12), along with the necessary assumptions, is given in Appendix B.

The review of heat conduction equations obtained via classical heat conduction laws, including Cattaneo (1.6), Jeffreys and many other, is given in [17]. Possibility of using the fractional Cattaneo law (1.5), along with its different variations, within the theory of anomalous transport processes is explored in [10]. By fractionalizing the classical Cattaneo constitutive model (1.6) in different manners and by combining such obtained heat conduction laws with the energy balance equation, classical telegrapher's equation (1.1) is generalized in [13, 23, 24, 27] and the corresponding problems on bounded, semi-bounded and unbounded domains are analyzed using analytical and numerical tools. Fractional Cattaneo law (1.5) is further generalized, either by considering its multi-term (or even distributed-order) version in [26], or by considering the spatial non-locality in [1, 6, 21, 29], or even by considering the non-locality in Cattaneo–Christov heat conduction law in [19]. In [2], several space- and time-fractional heat conduction problems are reviewed.

2. Analytical solution

Generalized fractional telegrapher's equation (1.2) will be solved using the Laplace transform method on the semi-bounded spatial domain $x \in [0,\infty)$ and for $t > 0$, assuming the boundary conditions

$$\Delta T(0,t) = T_0(t) \quad \text{and} \quad \lim_{x\to\infty} \Delta T(x,t) = 0, \quad t > 0, \tag{2.1}$$

as well as the zero initial condition

$$\Delta T(x,0) = 0, \quad \text{i.e.,} \quad T(x,0) = T_{\mathrm{ref}}, \quad x \in [0,\infty),$$

due to the requirement of equivalence between system of equations (1.3), (1.4), corresponding to heat conduction, and fractional telegrapher's equation (1.2), as described in the previous section.

Introducing the dimensionless quantities

$$\bar{x} = \frac{x}{x^*}, \quad \bar{t} = \frac{t}{t^*}, \quad t^* = \tau^{\frac{1}{1+\alpha-\beta}}, \quad x^* = \sqrt{\mathcal{D}(t^*)^\beta}, \quad \bar{T} = \frac{\Delta T}{T_{\mathrm{ref}}}, \quad \bar{T}_0 = \frac{T_0}{T_{\mathrm{ref}}}$$

into generalized fractional telegrapher's equation (1.2) and into boundary conditions (2.1), having the bars omitted, one obtains

$$ {}_0\mathrm{D}_t^{1+\alpha} T(x,t) + {}_0\mathrm{D}_t^{\beta} T(x,t) = \frac{\partial^2}{\partial x^2} T(x,t), \quad \text{subject to} \tag{2.2}$$

$$ T(0,t) = T_0(t) \quad \text{and} \quad \lim_{x\to\infty} T(x,t) = 0, \quad t > 0. \tag{2.3}$$

Applying the Laplace transform $\tilde{f}(s) = \mathcal{L}[f(t)](s) = \int_0^\infty f(t)\mathrm{e}^{-st}\mathrm{d}t$, $\operatorname{Re} s > 0$, to (2.2), for $x \in [0,\infty)$, $\operatorname{Re} s > 0$, one has

$$ s^{1+\alpha}\tilde{T}(x,s) + s^\beta \tilde{T}(x,s) = \frac{\partial^2}{\partial x^2}\tilde{T}(x,s), \quad \text{i.e.,}$$

$$ \frac{\partial^2}{\partial x^2}\tilde{T}(x,s) - (s^{1+\alpha} + s^\beta)\tilde{T}(x,s) = 0. \tag{2.4}$$

Solving (2.4), with the boundary conditions (2.3) taken into account, yields

$$ \tilde{T}(x,s) = \tilde{T}_0(s)\mathrm{e}^{-x\sqrt{s^{1+\alpha}+s^\beta}}, \quad x \in [0,\infty), \quad \operatorname{Re} s > 0, \tag{2.5}$$

since $\operatorname{Re}\sqrt{\psi(s)} > 0$, for $\operatorname{Re} s > 0$, with

$$ \psi(s) = s^{1+\alpha} + s^\beta, \quad s \in \mathbb{C}. \tag{2.6}$$

Namely, the imaginary part of ψ, after substitution $s = \rho\mathrm{e}^{\mathrm{i}\varphi}$, $\rho > 0$, $\varphi \in [0, \frac{\pi}{2})$ is made, is

$$ \operatorname{Im}\psi(\rho,\varphi) = \rho^{1+\alpha}\sin((1+\alpha)\varphi) + r^\beta \sin(\beta\varphi) \geqslant 0,$$

which implies $\arg\psi(s) \in (-\pi,\pi)$, i.e., $\operatorname{Re}\sqrt{\psi(s)} > 0$, for $\operatorname{Re} s > 0$, due to $\alpha, \beta \in (0,1)$ and $\operatorname{Im}\psi(s) = -\operatorname{Im}\psi(\bar{s})$, where bar denotes the complex conjugation.

Solution to the initial-boundary value problem corresponding to generalized fractional telegrapher's equation (2.2) is obtained by inverting the Laplace transform in (2.5) as the convolution of boundary condition T_0, given by $(2.3)_1$ and the impulse response T_δ (the response temperature to the prescribed Dirac delta temperature on the boundary) in the form

$$ T(x,t) = T_0(t) * T_\delta(x,t), \quad x \in [0,\infty), \quad \operatorname{Re} s > 0,$$

$$ \tilde{T}_\delta(x,s) = \mathrm{e}^{-x\sqrt{s^{1+\alpha}+s^\beta}}. \tag{2.7}$$

Definition of the inverse Laplace transform will be used to calculate the impulse response as

$$ T_\delta(x,t) = \mathcal{L}^{-1}[\tilde{T}_\delta(x,s)](x,t) = \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_0} \tilde{T}_\delta(x,s)\mathrm{e}^{st}\mathrm{d}s, \tag{2.8}$$

with Γ_0, parametrized by $s = p_0 + \mathrm{i}\rho$, $\rho \in (-\infty, \infty)$, being the Bromwich contour, by employing the Cauchy integral theorem $\oint_\Gamma f(z)\mathrm{d}z = 0$, where f is analytic function within the contour Γ, with the Bromwich contour Γ_0 as its part. Function ψ, given by (2.6), as it will be proved, may have zeros in the first Riemann sheet, implying that the Laplace transform of impulse response $\tilde{T}_\delta$ (2.7), due to the square root of ψ, may have branch points. This fact will affect the choice of the integration contour Γ. Note that $s = 0$ is also a branch point of $\tilde{T}_\delta$, due to the non-integer order power of s.

Substituting $s = \rho e^{\mathrm{i}\varphi}$, $\rho > 0$, $\varphi \in (-\pi, \pi]$, into function ψ (2.6), yields real and imaginary parts as follows

$$\begin{aligned}\operatorname{Re}\psi(\rho,\varphi) &= \rho^{1+\alpha}\cos((1+\alpha)\varphi) + \rho^\beta\cos(\beta\varphi),\\ \operatorname{Im}\psi(\rho,\varphi) &= \rho^{1+\alpha}\sin((1+\alpha)\varphi) + \rho^\beta\sin(\beta\varphi).\end{aligned}$$

The complex zeros of ψ, if they exist, are complex conjugated numbers having negative real part. The first property follows, since if $\varphi \to -\varphi$, then $\operatorname{Im}\psi(\rho,-\varphi) = -\operatorname{Im}\psi(\rho,\varphi)$. The second property follows, since if $\varphi \in (0, \frac{\pi}{2}]$, then $\operatorname{Im}\psi(\rho,\varphi) > 0$, due to $\alpha, \beta \in (0,1)$, while if $\varphi = 0$, then $\operatorname{Re}\psi(\rho, 0) \neq 0$ for $\rho > 0$.

Rewriting function ψ (2.6) as

$$\psi(s) = s^\beta(s^{1+\alpha-\beta} + 1), \quad s \in \mathbb{C}, \tag{2.9}$$

it becomes evident that:

- if $\alpha < \beta$, then function ψ, except for $s = 0$, does not have zeros in the complex plane;
- if $\alpha = \beta$, then function ψ, except for $s = 0$, has a negative real zero $s_0 = -1$;
- if $\alpha > \beta$, then function ψ, except for $s = 0$, has a pair of complex conjugated zeros having negative real part, given by $s_0 = e^{\mathrm{i}\frac{\pi}{1+\alpha-\beta}}$ and $\bar{s}_0 = e^{-\mathrm{i}\frac{\pi}{1+\alpha-\beta}}$.

This implies that $\sqrt{\psi(s)}$, appearing in $\tilde{T}_\delta$ (2.7), except for $s = 0$, in the first case does not have a branch point, while, in the second and third case, has negative real branch point and complex conjugated branch points with negative real part, respectively.

If $\alpha < \beta$, then in (2.9) one has the equation $s^\xi + 1 = 0$, $1 + \alpha - \beta = \xi \in (0,1)$, which, as it is well-known, has no zeros if $\arg s \in (-\pi, \pi]$, while if $\alpha = \beta$, then the solution $s_0 = -1$ is obvious. For $d > 0$ and $\zeta \in (1,2)$ the equation

$$s^\zeta + d = \rho^\zeta\cos(\zeta\varphi) + d + \mathrm{i}\rho^\zeta\sin(\zeta\varphi) = 0, \quad \text{implies} \quad s_0 = \frac{1}{\sqrt[\zeta]{d}} e^{\pm\mathrm{i}\frac{\pi}{\zeta}}.$$

If $\alpha > \beta$, then (2.9) is recovered for $d = 1$ and $\zeta = 1 + \alpha - \beta \in (1,2)$, so the solutions to (2.9) are $s_0 = e^{\pm\mathrm{i}\frac{\pi}{1+\alpha-\beta}}$. Note that $\varphi_0 = \frac{\pi}{\zeta} \in (\frac{\pi}{2}, \pi)$, i.e., the complex zero has negative real part.

In order to calculate the impulse response T_δ, using the inverse Laplace transform (2.8), the Cauchy integral theorem, stating that $\oint_\Gamma f(z)\mathrm{d}z = 0$ if f is analytic function in the region D having contour Γ as its border, will be used.

In the case when $\sqrt{\psi(s)}$ does not have branch point other than $s = 0$, i.e., when $\alpha < \beta$, the impulse response is already calculated in [3, Eqs. (3.8) and (3.9)] and it reads

$$T_\delta(x,t) = \frac{1}{2\pi \mathrm{i}} \int_0^\infty \left(\mathrm{e}^{-x\sqrt{\rho^{1+\alpha}\mathrm{e}^{-\mathrm{i}(1+\alpha)\pi} + \rho^\beta \mathrm{e}^{-\mathrm{i}\beta\pi}}} - \mathrm{e}^{-x\sqrt{\rho^{1+\alpha}\mathrm{e}^{\mathrm{i}(1+\alpha)\pi} + \rho^\beta \mathrm{e}^{\mathrm{i}\beta\pi}}} \right) \mathrm{e}^{-\rho t} \mathrm{d}\rho. \tag{2.10}$$

In the case when $\sqrt{\psi(s)}$, additionally to $s = 0$, has a pair of complex conjugated branch points $s_0 = \rho_0 \mathrm{e}^{\mathrm{i}\varphi_0}$ and $\bar{s}_0 = \rho_0 \mathrm{e}^{-\mathrm{i}\varphi_0}$, with $\rho_0 = 1$ and $\varphi_0 = \frac{\pi}{1+\alpha-\beta} \in \left(\frac{\pi}{2}, \pi\right)$, i.e., when $\alpha > \beta$, the Cauchy integral theorem $\oint_\Gamma \tilde{T}_\delta(x,s)\mathrm{e}^{st}\mathrm{d}s = 0$, with the contour $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_{3a} \cup \Gamma_{3b} \cup \Gamma_4 \cup \Gamma_{5a} \cup \Gamma_{5b} \cup \Gamma_6 \cup \Gamma_7 \cup \Gamma_8 \cup \Gamma_9$ chosen as in Figure 1, yields the impulse response

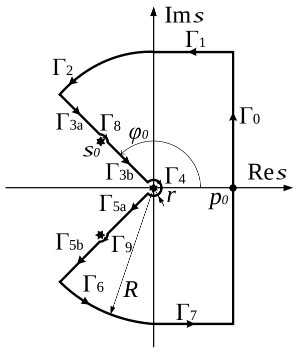

FIGURE 1. Contour Γ.

$$T_\delta(x,t) = \frac{1}{2\pi \mathrm{i}} \int_0^\infty \Big(\mathrm{e}^{-x\sqrt{\rho^{1+\alpha}\mathrm{e}^{\mathrm{i}(1+\alpha)\varphi_0} + \rho^\beta \mathrm{e}^{\mathrm{i}\beta\varphi_0}}} \mathrm{e}^{\mathrm{i}(\varphi_0 + \rho t \sin\varphi_0)} \\ - \mathrm{e}^{-x\sqrt{\rho^{1+\alpha}\mathrm{e}^{-\mathrm{i}(1+\alpha)\varphi_0} + \rho^\beta \mathrm{e}^{-\mathrm{i}\beta\varphi_0}}} \mathrm{e}^{-\mathrm{i}(\varphi_0 + \rho t \sin\varphi_0)} \Big) \mathrm{e}^{\rho t \cos\varphi_0} \mathrm{d}\rho, \tag{2.11}$$

since the integrals along contours $\Gamma_{3a} \cup \Gamma_{3b}$ and $\Gamma_{5a} \cup \Gamma_{5b}$ read

$$\lim_{\substack{R\to\infty,\\ r\to 0}} \int_{\Gamma_{3a}\cup\Gamma_{3b}} \tilde{T}_\delta(x,s)\mathrm{e}^{st}\mathrm{d} = \int_\infty^0 \mathrm{e}^{-x\sqrt{\rho^{1+\alpha}\mathrm{e}^{\mathrm{i}(1+\alpha)\varphi_0} + \rho^\beta \mathrm{e}^{\mathrm{i}\beta\varphi_0}}} \mathrm{e}^{\rho t \mathrm{e}^{\mathrm{i}\varphi_0}} \mathrm{e}^{\mathrm{i}\varphi_0} \mathrm{d}\rho$$
$$= -\int_0^\infty \mathrm{e}^{-x\sqrt{\rho^{1+\alpha}\mathrm{e}^{\mathrm{i}(1+\alpha)\varphi_0} + \rho^\beta \mathrm{e}^{\mathrm{i}\beta\varphi_0}}} \mathrm{e}^{\mathrm{i}(\varphi_0 + \rho t \sin\varphi_0)} \mathrm{e}^{\rho t \cos\varphi_0} \mathrm{d}\rho,$$
$$\lim_{\substack{R\to\infty,\\ r\to 0}} \int_{\Gamma_{5a}\cup\Gamma_{5b}} \tilde{T}_\delta(x,s)\mathrm{e}^{st}\mathrm{d}s = \int_0^\infty \mathrm{e}^{-x\sqrt{\rho^{1+\alpha}\mathrm{e}^{-\mathrm{i}(1+\alpha)\varphi_0} + \rho^\beta \mathrm{e}^{-\mathrm{i}\beta\varphi_0}}} \mathrm{e}^{\rho t \mathrm{e}^{-\mathrm{i}\varphi_0}} \mathrm{e}^{-\mathrm{i}\varphi_0} \mathrm{d}\rho$$
$$= \int_0^\infty \mathrm{e}^{-x\sqrt{\rho^{1+\alpha}\mathrm{e}^{-\mathrm{i}(1+\alpha)\varphi_0} + \rho^\beta \mathrm{e}^{-\mathrm{i}\beta\varphi_0}}} \mathrm{e}^{-\mathrm{i}(\varphi_0 + \rho t \sin\varphi_0)} \mathrm{e}^{\rho t \cos\varphi_0} \mathrm{d}\rho,$$

where contours $\Gamma_{3a} \cup \Gamma_{3b}$ and $\Gamma_{5a} \cup \Gamma_{5b}$ are parametrized by $s = \rho \mathrm{e}^{\mathrm{i}\varphi_0}$, $\rho \in (r, 1-r) \cup (1+r, R)$ and $s = \rho \mathrm{e}^{-\mathrm{i}\varphi_0}$, $\rho \in (r, 1-r) \cup (1+r, R)$, with $\varphi_0 = \frac{\pi}{1+\alpha-\beta}$, respectively, while the inverse Laplace transform of $\tilde{T}_\delta$ is given as in (2.8), and the integrals along contours Γ_1, Γ_2, Γ_4, Γ_6, Γ_7, Γ_8, and Γ_9 tend to zero as $R \to \infty$ and $r \to 0$.

In the case when $\sqrt{\psi(s)}$, apart form $s=0$, has a negative real branch point $s_0=-1$, i.e., when $\alpha=\beta$, the calculation of the inverse Laplace transform of $\tilde{T}_\delta$ will be the same as in the previous case with $\varphi_0=\frac{\pi}{1+\alpha-\beta}=\pi$, yielding the impulse response

$$T_\delta(x,t)=\frac{1}{2\pi\mathrm{i}}\int_0^\infty\left(\mathrm{e}^{-x\sqrt{\rho^\alpha\mathrm{e}^{-\mathrm{i}\alpha\pi}}\sqrt{1-\rho}}-\mathrm{e}^{-x\sqrt{\rho^\alpha\mathrm{e}^{\mathrm{i}\alpha\pi}}\sqrt{1-\rho}}\right)\mathrm{e}^{-\rho t}\mathrm{d}\rho. \tag{2.12}$$

It is left to prove that the above mentioned integrals tend to zero. The integral along contour Γ_1, parametrized by $s=p+\mathrm{i}R$, $p\in(0,p_0)$, with $R\to\infty$, is

$$I_{\Gamma_1}=\int_{\Gamma_1}\mathrm{e}^{-x\sqrt{s^{1+\alpha}+s^\beta}}\mathrm{e}^{st}\mathrm{d}s=\int_0^{p_0}\mathrm{e}^{-x\sqrt{(p+\mathrm{i}R)^{1+\alpha}+(p+\mathrm{i}R)^\beta}}\mathrm{e}^{(p+\mathrm{i}R)t}\mathrm{d}p.$$

Assuming $s=\rho\mathrm{e}^{\mathrm{i}\varphi}$, with $\rho=\sqrt{p^2+R^2}\sim R$ and $\varphi=\arctan\frac{R}{p}\sim\frac{\pi}{2}$ as $R\to\infty$, the previous integral is estimated by

$$\begin{aligned}\lim_{R\to\infty}\|I_{\Gamma_1}\|&\leqslant\lim_{R\to\infty}\int_0^{p_0}\left|\mathrm{e}^{-x\sqrt{R^{1+\alpha}\mathrm{e}^{\mathrm{i}\frac{(1+\alpha)\pi}{2}}+R^\beta\mathrm{e}^{\mathrm{i}\frac{\beta\pi}{2}}}}\right|\mathrm{e}^{pt}\mathrm{d}p\\&\leqslant\lim_{R\to\infty}\int_0^{p_0}\left|\mathrm{e}^{-xR^{\frac{1+\alpha}{2}}\mathrm{e}^{\mathrm{i}\frac{(1+\alpha)\pi}{4}}\sqrt{1+\frac{1}{R^{1+\alpha-\beta}}\mathrm{e}^{-\mathrm{i}\frac{(1+\alpha-\beta)\pi}{2}}}}\right|\mathrm{e}^{pt}\mathrm{d}p\\&\leqslant\lim_{R\to\infty}\int_0^{p_0}\mathrm{e}^{-xR^{\frac{1+\alpha}{2}}\cos\frac{(1+\alpha)\pi}{4}}\mathrm{e}^{pt}\mathrm{d}p=0,\end{aligned}$$

since $1+\alpha\in(1,2)$ implying $\cos\frac{(1+\alpha)\pi}{4}>0$. Similarly, $\lim_{R\to\infty}|I_{\Gamma_7}|=0$.

The contour Γ_2 is parametrized by $s=R\mathrm{e}^{\mathrm{i}\varphi}$, with $\varphi\in\left(\frac{\pi}{2},\varphi_0\right)$ and $R\to\infty$, so that the integral reads

$$I_{\Gamma_2}=\int_{\Gamma_2}\mathrm{e}^{-x\sqrt{s^{1+\alpha}+s^\beta}}\mathrm{e}^{st}\mathrm{d}s=\int_{\frac{\pi}{2}}^{\varphi_0}\mathrm{e}^{-x\sqrt{R^{1+\alpha}\mathrm{e}^{\mathrm{i}(1+\alpha)\varphi}+R^\beta\mathrm{e}^{\mathrm{i}\beta\varphi}}}\mathrm{e}^{Rt\mathrm{e}^{\mathrm{i}\varphi}}\mathrm{i}R\mathrm{e}^{\mathrm{i}\varphi}\mathrm{d}\varphi,$$

and therefore, in the limit when $R\to\infty$, its estimate is

$$\begin{aligned}\lim_{R\to\infty}|I_{\Gamma_2}|&\leqslant\lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\varphi_0}R\left|\mathrm{e}^{-xR^{\frac{1+\alpha}{2}}\mathrm{e}^{\mathrm{i}\frac{(1+\alpha)\varphi}{2}}\sqrt{1+\frac{1}{R^{1+\alpha-\beta}}\mathrm{e}^{-\mathrm{i}(1+\alpha-\beta)\varphi}}}\right|\mathrm{e}^{Rt\cos\varphi}\mathrm{d}\varphi\\&\leqslant\lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\varphi_0}R\mathrm{e}^{-xR^{\frac{1+\alpha}{2}}\cos\frac{(1+\alpha)\varphi}{2}}\mathrm{e}^{Rt\cos\varphi}\mathrm{d}\varphi=0,\end{aligned}$$

since, for $\varphi\in\left(\frac{\pi}{2},\varphi_0\right)$, $\cos\varphi<0$ and R tends to infinity faster than $R^{\frac{1+\alpha}{2}}$, regardless on the sign of $\cos\frac{(1+\alpha)\varphi}{2}$. Similar arguments imply $\lim_{R\to\infty}|I_{\Gamma_6}|=0$.

Parametrization of the contour Γ_4 is $s=r\mathrm{e}^{\mathrm{i}\varphi}$, with $\varphi\in(-\varphi_0,\varphi_0)$ and $r\to0$, yielding the integral

$$I_{\Gamma_4}=\int_{\Gamma_4}\mathrm{e}^{-x\sqrt{s^{1+\alpha}+s^\beta}}\mathrm{e}^{st}\mathrm{d}s=\int_{\varphi_0}^{-\varphi_0}\mathrm{e}^{-x\sqrt{r^{1+\alpha}\mathrm{e}^{\mathrm{i}(1+\alpha)\varphi}+r^\beta\mathrm{e}^{\mathrm{i}\beta\varphi}}}\mathrm{e}^{rt\mathrm{e}^{\mathrm{i}\varphi}}\mathrm{i}r\mathrm{e}^{\mathrm{i}\varphi}\mathrm{d}\varphi,$$

so that in the limit when $r\to0$ one has

$$\lim_{r\to0}|I_{\Gamma_4}|\leqslant\lim_{r\to0}\int_{-\varphi_0}^{\varphi_0}r\left|\mathrm{e}^{-x\sqrt{r^{1+\alpha}\mathrm{e}^{\mathrm{i}(1+\alpha)\varphi}+r^\beta\mathrm{e}^{\mathrm{i}\beta\varphi}}}\right|\mathrm{e}^{rt\cos\varphi}\mathrm{d}\varphi=0.$$

The integral along contour Γ_8 is

$$I_{\Gamma_8} = \int_{\Gamma_8} \mathrm{e}^{-x\sqrt{s^{1+\alpha}+s^{\beta}}} \mathrm{e}^{st} \mathrm{d}s = \int_{-\pi+\varphi_0}^{\varphi_0} \mathrm{e}^{-x\sqrt{(s_0+r\mathrm{e}^{\mathrm{i}\varphi})^{1+\alpha}+(s_0+r\mathrm{e}^{\mathrm{i}\varphi})^{\beta}}} \mathrm{e}^{(s_0+r\mathrm{e}^{\mathrm{i}\varphi})t} \mathrm{i}r\mathrm{e}^{\mathrm{i}\varphi} \mathrm{d}\varphi,$$

since contour Γ_8 is parametrized by $s - s_0 = r\mathrm{e}^{\mathrm{i}\varphi}$, with $\varphi \in (\varphi_0 - \pi, \varphi_0)$, with $r \to 0$, so that in the limit when $r \to 0$ it is estimated by

$$\lim_{r\to 0} |I_{\Gamma_8}| \leqslant \lim_{r\to 0} \int_{-\pi+\varphi_0}^{\varphi_0} r \left|\mathrm{e}^{-x\sqrt{s_0^{1+\alpha}+s_0^{\beta}}}\right| |\mathrm{e}^{s_0 t}| \mathrm{d}\varphi = \lim_{r\to 0} \int_{-\pi+\varphi_0}^{\varphi_0} r\,\mathrm{e}^{t\,\mathrm{Re}\,s_0} \mathrm{d}\varphi = 0,$$

having in mind $\psi(s_0) = s_0^{1+\alpha} + s_0^{\beta} = 0$. Similarly $\lim_{r\to 0} |I_{\Gamma_9}| = 0$.

3. Graphs of impulse response

The impulse response in transient regime, as a solution to fractional telegrapher's equation (2.2), is obtained analytically in integral form (2.10) if $\alpha < \beta$, (2.11) if $\alpha > \beta$, and (2.12) if $\alpha = \beta$. Evolution of the impulse response will be calculated and graphs of its time and space evolution will be presented, along with the discussion about their physical interpretation.

3.1. Time evolution of impulse response. Figures 2 and 3 present the evolution in time of impulse responses at discrete points of space, calculated by (2.10) and (2.12), displaying aperiodic character, if $\sqrt{\psi(s)}$, except for $s = 0$, does not have branch point, and critically aperiodic character, if $\sqrt{\psi(s)}$, except for $s = 0$, has a negative real branch point.

Graphs in Figure 4 display the damped periodic impulse response, obtained in the case when $\sqrt{\psi(s)}$, except for $s = 0$, has a pair of complex conjugated branch points, thus calculated by (2.11). Recall, the (dimensionless) quantity appearing

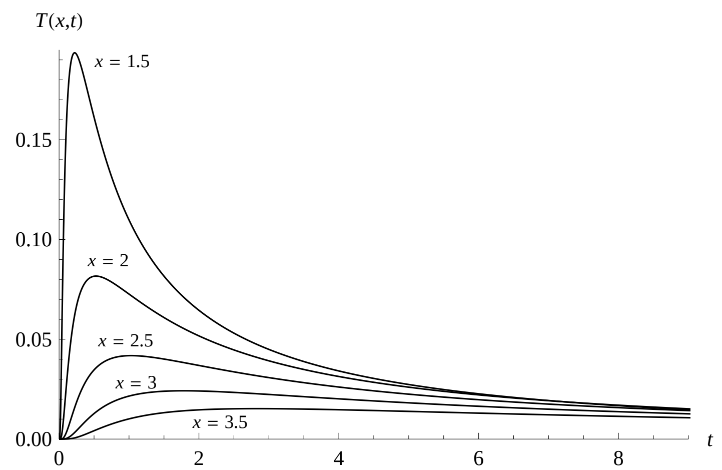

FIGURE 2. Impulse response as a function of time at discrete positions for $\alpha = 1/3$, $\beta = 2/3$.

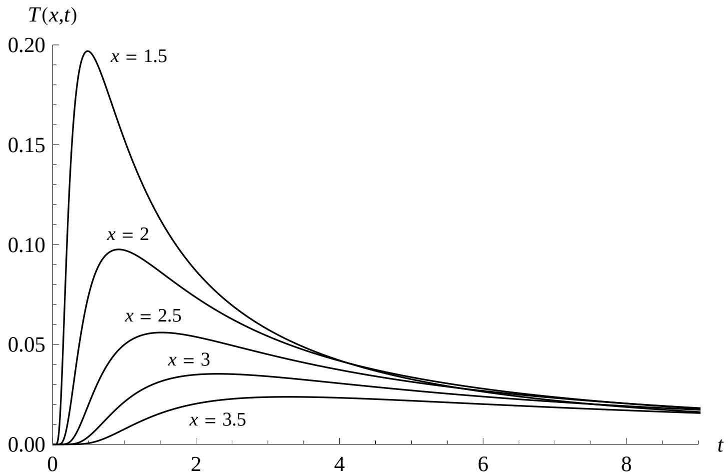

FIGURE 3. Impulse response as a function of time at discrete positions for $\alpha = 2/3$, $\beta = 2/3$.

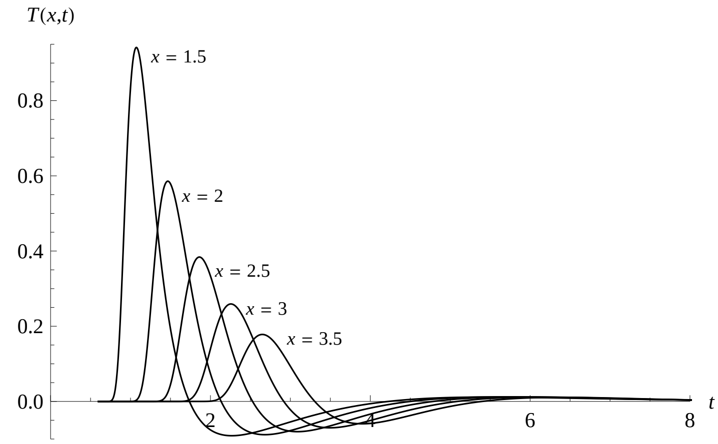

FIGURE 4. Impulse response as a function of time at discrete positions for $\alpha = 19/20$, $\beta = 1/5$.

in fractional telegrapher's equation (2.2) and its impulse response (2.11) as well, is the relative difference of the absolute and reference temperature ($\bar{T} = \frac{T}{T_{\mathrm{ref}}} - 1$) and its time evolution is depicted in Figure 4, so the temperature actually attains values above and below the reference temperature, while the relative temperature oscillates around zero.

3.2. Space profiles of impulse response. Figures 5–7 present graphs of impulse response's space profiles evolution in discrete time instances. Although the leading order of (fractional) differentiation in fractional telegrapher's equation

(2.2) is greater than one, space profiles shown in Figures 5 and 6, calculated by (2.10) and (2.12), correspond to the diffusion-type character (during time there is no spatial propagation of the Dirac delta disturbance applied at the boundary) of the impulse response, which is obtained as a consequence of the difference between orders of time differentiation $1+\alpha-\beta \in (0,1)$, i.e., when $\alpha \leqslant \beta$.

The wave-type character (there is spatial propagation of the Dirac delta disturbance during time) of the impulse response, calculated by (2.11), is evident form Figure 7 and it is a consequence of the difference between orders of time differentiation $1+\alpha-\beta \in (1,2)$, i.e., when $\alpha > \beta$. From Figure 7 one sees that, as time

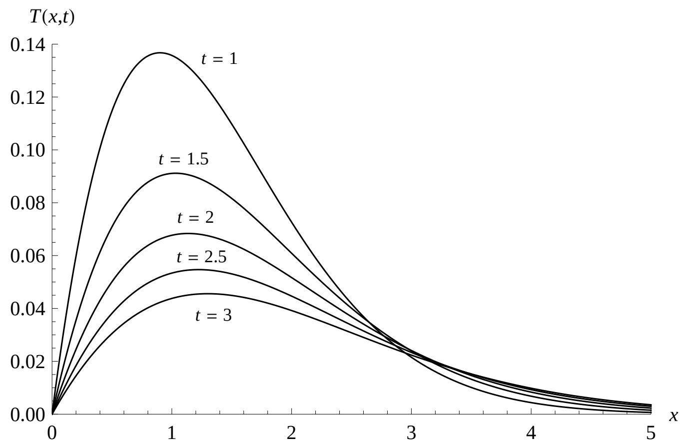

FIGURE 5. Impulse response as a function of position at discrete time instants for $\alpha = 1/3$, $\beta = 2/3$.

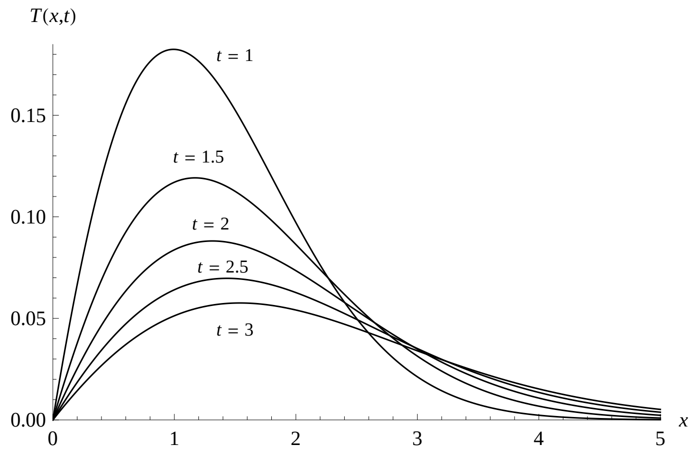

FIGURE 6. Impulse response as a function of position at discrete time instants for $\alpha = 2/3$, $\beta = 2/3$.

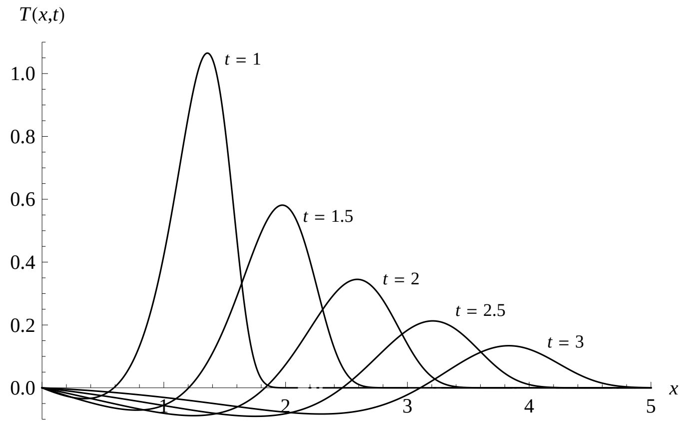

FIGURE 7. Impulse response as a function of position at discrete time instants for $\alpha = 19/20$, $\beta = 1/5$.

passes, the peaks of the impulse response propagate in space, while their height decreases and their width increases. There also exists a negative (more precisely the absolute temperature is less than the reference temperature) peak that travels during time, while its height increases up to a certain moment and then seemingly starts to decrease.

4. Conclusion

Fractional telegrapher's equation (1.2) is reinterpreted in terms of heat conduction theory and obtained by considering the energy balance equation (1.3) and the two-parameter generalization of fractional Cattaneo heat conduction law (1.4), taking into account not only the history of heat flux change, but also the history of temperature gradient. Rather than considering the absolute temperature, the difference with respect to the reference temperature is considered, implying that the equivalence between fractional telegrapher's equation (1.2) and system of equations (1.3) and (1.4) hold for a conductor having uniformly distributed reference temperature. It is also shown that if the heat flux is written in terms of temperature gradient history as (1.10), the memory kernel, corresponding to the two-parameter generalization of fractional Cattaneo heat conduction law (1.4), is non-completely monotonic two-parameter Mittag-Leffler function.

Dimensionless fractional telegrapher's equation (2.2), with the temperature difference relative to the reference temperature as the unknown function, is solved for the zero initial condition on semi-bounded domain and the solution is obtained as a convolution of the forcing (relative) temperature on the boundary and the impulse response. The solution is obtained using the Laplace transform method and the impulse response proved to have two different type of behaviour depending on the relation between the orders of fractional differentiation in (2.2). In the case when $\alpha \leqslant \beta$ in (2.2), the spatial profiles of impulse response show the diffusion-type character and the time evolution is aperiodic, while if $\alpha > \beta$ in (2.2), the spatial profiles of impulse response show the wave-type character and the time evolution is periodic.

Appendix A. Derivation of fractional telegrapher's equation (1.2)

Starting from the system consisting of the energy balance equation (1.3) and heat conduction law (1.4) and applying the fractional integral of order $1-\beta$ to the constitutive equation (1.4), one obtains

$$\tau\, {}_0\mathrm{D}_t^{\alpha} q(x,t) + {}_0\mathrm{I}_t^{1-\beta} q(x,t) = -\lambda \frac{\partial}{\partial x} \Delta T(x,t), \tag{A.1}$$

where the fractional integral of order $\xi > 0$ is defined by

$${}_0\mathrm{I}_t^{\xi} f(t) = \frac{t^{\xi-1}}{\Gamma(\xi)} * f(t), \quad t > 0.$$

The previous expression is obtained assuming that $\frac{\partial}{\partial x}\Delta T$ and q are bounded at initial moment, implying that ${}_0\mathrm{D}_t^{\alpha} q$ is bounded at $t = 0$ as well as that

$$\left[{}_0\mathrm{I}_t^{\beta}\left(\frac{\partial}{\partial x}\Delta T(x,t)\right)\right]_{t=0} = [{}_0\mathrm{I}_t^{\beta}({}_0\mathrm{D}_t^{\alpha} q(x,t))]_{t=0} = 0,$$

due to the composition property of fractional integral and Riemann–Liouville fractional derivative of the same order $\xi \in [0,1]$:

$${}_0\mathrm{I}_t^{\xi}({}_0\mathrm{D}_t^{\xi} f(t)) = f(t) - \frac{\mathrm{d}}{\mathrm{d}t}\, {}_0\mathrm{I}_t^{\xi}[{}_0\mathrm{I}_t^{1-\xi} f(t)]_{t=0}.$$

Taking the partial derivative of (A.1) with respect to coordinate x, and substituting $\frac{\partial}{\partial x} q$ from (1.3), yields

$$\tau\, {}_0\mathrm{D}_t^{\alpha}\left(\frac{\partial}{\partial t}\Delta T(x,t)\right) + {}_0\mathrm{I}_t^{1-\beta}\left(\frac{\partial}{\partial t}\Delta T(x,t)\right) = \mathcal{D}\frac{\partial^2}{\partial x^2}\Delta T(x,t), \quad \mathcal{D} = \frac{\lambda}{\rho c}.$$

Assuming the zero initial condition, i.e., $\Delta T(x,0) = 0$, one obtains fractional telegrapher's equation (1.2), due to the composition property of Riemann–Liouville fractional derivatives of orders $\xi, \zeta \in [0,1]$:

$$\begin{aligned}
{}_0\mathrm{D}_t^{\xi}\left({}_0\mathrm{D}_t^{\zeta} f(t)\right) &= {}_0\mathrm{D}_t^{\xi+\zeta} f(t) - \frac{\mathrm{d}^2}{\mathrm{d}t^2}\, {}_0\mathrm{I}_t^{1-\xi}[{}_0\mathrm{I}_t^{1-\zeta} f(t)]_{t=0}, \quad \text{i.e.,} \\
{}_0\mathrm{D}_t^{\xi}\left(\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right) &= {}_0\mathrm{D}_t^{\xi+1} f(t) - \frac{\mathrm{d}^2}{\mathrm{d}t^2}\, {}_0\mathrm{I}_t^{1-\xi}[f(t)]_{t=0},
\end{aligned}$$

since the zeroth order fractional integral of a function is a function itself (${}_0\mathrm{I}_t^{0} f(t) = f(t)$); due to the composition property of fractional integral and Riemann–Liouville fractional derivative of the orders $\xi, \zeta \in [0,1]$:

$$\begin{aligned}
{}_0\mathrm{I}_t^{\xi}({}_0\mathrm{D}_t^{\zeta} f(t)) &= {}_0\mathrm{D}_t^{\zeta}({}_0\mathrm{I}_t^{\xi} f(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\, {}_0\mathrm{I}_t^{\xi}[{}_0\mathrm{I}_t^{1-\zeta} f(t)]_{t=0}, \quad \text{i.e.,} \\
{}_0\mathrm{I}_t^{\xi}\left(\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right) &= \frac{\mathrm{d}}{\mathrm{d}t}({}_0\mathrm{I}_t^{\xi} f(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\, {}_0\mathrm{I}_t^{\xi}[f(t)]_{t=0},
\end{aligned}$$

and the fact that Riemann–Liouville fractional derivative of order $\xi \in [0,1]$ is defined by

$$ {}_0\mathrm{D}_t^\xi f(t) = \frac{\mathrm{d}}{\mathrm{d}t}({}_0\mathrm{I}_t^\xi f(t)). $$

For the properties of used compositions see [18].

Appendix B. Derivation of the memory kernels (1.12)

Applying the Laplace transform to the heat conduction law (1.4), one has

$$ \tau s^{1-\beta}\mathcal{L}[{}_0\mathrm{D}_t^\alpha q(x,t)](s) + \tilde{q}(x,s) = -\lambda s^{1-\beta}\frac{\partial}{\partial x}\Delta\tilde{T}(x,s), \quad \operatorname{Re} s > 0, \tag{B.1} $$

since the boundness of q at initial moment implies the boundness of ${}_0\mathrm{D}_t^\alpha q$, having $[{}_0\mathrm{I}_t^\beta({}_0\mathrm{D}_t^\alpha q(x,t))]_{t=0} = 0$ as a consequence and also the boundness of $\frac{\partial}{\partial x}\Delta T$ at $t=0$ implies $[{}_0\mathrm{I}_t^\beta(\frac{\partial}{\partial x}\Delta T(x,t))]_{t=0} = 0$, due to the Laplace transform of Riemann–Liouville fractional derivative of order $\xi \in [0,1]$:

$$ \mathcal{L}[{}_0\mathrm{D}_t^\xi f(t)](s) = s^\xi \tilde{f}(s) - [{}_0\mathrm{I}_t^{1-\xi} f(t)]_{t=0}. $$

By the same argumentation as above, (B.1) becomes

$$ \tau s^{1-\beta+\alpha}\tilde{q}(x,s) + \tilde{q}(x,s) = -\lambda s^{1-\beta}\frac{\partial}{\partial x}\Delta\tilde{T}(x,s), \quad \operatorname{Re} s > 0, $$

so that

$$ \tilde{q}(x,s) = -\frac{\lambda}{\tau}\frac{s^{1-\beta}}{s^{1-\beta+\alpha}+\frac{1}{\tau}}\frac{\partial}{\partial x}\Delta\tilde{T}(x,s), \quad \operatorname{Re} s > 0. \tag{B.2} $$

Using the Laplace transform of convolution and two-parameter Mittag-Leffler function (1.9), the previous expression (B.2) transforms into (1.10), with the memory kernel $(1.12)_2$.

For $\alpha = 0$, (B.2) takes the form

$$ \tilde{q}(x,s) = -\frac{\lambda}{\tau}\frac{s^{1-\beta}}{s^{1-\beta}+\frac{1}{\tau}}\frac{\partial}{\partial x}\Delta\tilde{T}(x,s), \quad \operatorname{Re} s > 0, $$

yielding (1.11), with the memory kernel $(1.12)_1$, since

$$ \frac{s^{1-\beta}}{s^{1-\beta}+\frac{1}{\tau}} = s\frac{s^{1-\beta-1}}{s^{1-\beta}+\frac{1}{\tau}} = s\mathcal{L}[e_{1-\beta,\lambda}(t)] = \mathcal{L}\Big[\frac{\mathrm{d}}{\mathrm{d}t}e_{1-\beta,\frac{1}{\tau}}(t) + \delta(t)\Big], $$

where the Laplace transform of derivative and one-parameter Mittag-Leffler function (1.8) are used.

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ФРАКЦИОНА ЈЕДНАЧИНА ТЕЛЕГРАФИЧАРА КАО ПОСЛЕДИЦА УОПШТЕЊА КАТАНЕОВОГ ЗАКОНА ПРОВОЂЕЊА ТОПЛОТЕ

Резиме. Фракциона једначина телеграфичара је реинтерпретирана у оквиру теорије провођења топлоте и добијена је коришћењем једначине енергетског биланса и фракционог Катанеовог закона провођења топлоте, који је уопштен узимањем у обзир и историје градијента температуре. Користећи се методом Лапласове трансформације, фракциона једначина телеграфичара је, уз нулте почетне услове, решена на полубесконачном просторном домену и решење је добијено као конволуција температуре на граници и импулсног одзива. Испитана су и нека својства добијеног решења.

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