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ON A PROPERTY OF DIFFERENTIAL EQUATIONS INTEGRABLE USING MEROMORPHIC DOUBLE-PERIODIC FUNCTIONS

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Given a general type of differential equations

$$F(y, y', y'', \dots, y^{(p)}) = 0 \tag{1}$$

of an arbitrary order, which does not contain x explicitly, one can propose to specify differential equations which belong to such type and that can be satisfied by meromorphic double-periodic functions. Here, I indicate a property of such equations which simplifies the given problem, without study in depth and which translates into a very simple and practical rule.

Suppose the equation is written as

$$\sum_{i=1}^{i=s} P_i y^{m_{0i}} y'^{m_{1i}} y''^{m_{2i}} \dots y^{(p)m_{pi}} = 0, \tag{2}$$

where the m's are nonnegative integers, the P_i's are some constants and such that one cannot have at the same time,

$$m_{oi} = m_{oj}, \quad m_{1i} = m_{1j}, \quad \dots, \quad m_{pi} = m_{pj}$$

for two different indexes i and j.
Let us consider the $2s$ nonnegative integers

$$\begin{aligned} M_i &= m_{0i} + m_{1i} + \dots + m_{pi}, \\ N_i &= m_{1i} + 2m_{2i} + \dots + pm_{pi}. \end{aligned} \tag{3}$$

We draw in a plane two axes, that of the M's and the N's, and indicate the s points (M_i, N_i) such that their indexes are carefully written next to each of them.

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If two or several of them coincide, then we will write next to such a point the indexes of all the points that coincide there.

We construct the polygonal line Π which is concave to OM, such that all the points (M_i, N_i) lie in its interior or on its boundary and is closed by lines, perpendicular to ON in the case that there are no vertices on that axis. Let

$$\begin{aligned} \gamma_{0i} &= m_{1i} + m_{2i} + \cdots + m_{pi}, \\ \gamma_{1i} &= m_{2i} + m_{3i} + \cdots + m_{pi}, \\ &\cdots\cdots\cdots\cdots \\ \gamma_{hi} &= m_{h+1,i} + \cdots + m_{pi}, \end{aligned} \tag{4}$$

and

$$A_i = \lambda^{\gamma_{0i}} (\lambda - 1)^{\gamma_{1i}} (\lambda - 2)^{\gamma_{2i}} \dots (\lambda - p + 1)^{\gamma_{p-1,i}}, \tag{5}$$

where λ is an arbitrary number.

It can happen that the polygonal line Π has one or several vertices in which two or several points (M_i, N_i) coincide. Such vertices are called *multiple vertices.*

Consider such a multiple vertex and let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the indexes of the terms of F which coincide there.

Let

$$A_{a_1} P_{a_1} + A_{a_2} P_{a_2} + \cdots + A_{a_n} P_{a_n} = 0; \tag{6}$$

which will be an equation in λ, of the form

$$\lambda^m + a_1 \lambda^{m-1} + a_2 \lambda^{m-2} + \cdots + a_{m-1} \lambda + a_m = 0,$$

and will be called *the equation in* λ *relative to the multiple vertex* $(\alpha_1, \alpha_2, \dots, \alpha_n)$. Each multiple vertex corresponds to an equation in λ, defined by (6).

It is possible that the differential equation (1) has meromorphic double-periodic integrals. *In such a case, it has the following properties*:

I. *The polygonal line* Π *has at least an edge with a negative integer angular coefficient or it has at least a multiple vertex such that its equation in* λ *has one or several negative integers root(s), that are between the values of the angular coefficients of the two edges that form the multiple vertex.*

Since a meromorphic double-periodic function cannot be holomorphic on the whole plane, in a neighborhood of a pole $x = a$, it can be written as

$$y = (x - a)^{\mu} f(x), \tag{7}$$

where μ is a negative integer and $f(x)$ is a holomorphic function in a neighborhood of $x = a$, which does not vanish at that point. On the other hand, I proved previously (*Ph.D thesis*) the following result: If μ has a determined value so that y, defined in (7), can satisfy the differential equation (1), then μ must be either one of the angular coefficients of the edges of the polygonal line Π, corresponding to the polygone F[1], or μ satisfies one of the equations in λ, relative to a multiple

[1]Translators' note: should be "the polynomial F".

vertex of Π and μ is between the angular coefficients of the two edges that form that multiple vertex.

The proposition I follows immediately.

II. *For all rational fractions $R(y)$ in y, the transform*

$$\Phi(z, z', z'', \ldots, z^{(p)}) = 0 \tag{8}$$

of $F = 0$ at $z = R(y)$ has the properties, given in the proposition I.

Since y is a meromorphic double-periodic function, so is z.

Notice at the same time that, if the conditions I are not satisfied by a transform (8) corresponding to a rational fraction $R(y)$, having more than two distinct poles, then *the equation $F = 0$ does not have any meromorphic integral.* Since the equation $\Phi = 0$ does not satisfy the conditions I, its integral z, which is also meromorphic if y is, will not become infinite for any value of x. Consequently, if a, b and c are three distinct poles of $R(y)$ in y, then three following conditions

$$y - a = 0, \quad y - b = 0, \quad y - c = 0$$

do not have any finite roots. It follows from the Theorem of M. Picard, that the integral y, (supposed to be meromorphic), is a constant.

III. *If we form a rational combination*

$$R(y, y', y'', \ldots, y^{(q)})$$

of y and its successive derivatives, such that the transform

$$\Psi(z, z', z'', \ldots, z^{(q)}) = 0 \tag{9}$$

of $F = 0$ to

$$z = R(y, y', y'', \ldots, y^{(q)}) \tag{10}$$

does not satisfy the conditions of Proposition I*, then the equation*

$$R(y, y', y'', \ldots, y^{(q)}) = \text{constant}, \tag{11}$$

plays the role of a first integral for the double-periodic meromorphic integrals of the equation $F = 0$ in the sense that any integral of such nature satisfies at the same time the equation $R =$ constant.

To prove the Proposition, it is sufficient to observe that y is meromorphic double periodic and z as well. Such a function does not have poles, thus reduces to a constant.

The consideration of the polygonal line Π and the corresponding equations in λ thus provides means of creating the first integrals on meromorphic double-periodic integrals to the given differential equation. Once these first integrals are known, the search for integrals in question is reduced to those common solutions to the two given differential equations, that we will do by differentiations and elimination of the successive derivatives of y. If $p > q$, we will differentiate the equation $R =$ constant with respect to x $(p - q)$ times. By eliminating $y^{(p)}$ from

$$F = 0, \tag{1}$$

$$\frac{d^{p-q}R}{dx^{p-q}} = 0, \tag{2}$$

we obtain the equation (3) of order less than p, having all common solutions of (1) and (2). By operating on (2) and (3) as on (1) and (2), we will replace one of the equations by another of a lower order and so on. We get a sequence

$$(1), (2), (3), \ldots, (m-2), (m-1), (m), \ldots \tag{Δ}$$

of differential equations. If the equation $F = 0$ actually admits meromorphic double periodic integrals not reducing to constants, we can always choose the constant in the first integral $R =$ constant so that the equations of the sequence (Δ) starting from a certain rank m reduce to identities. Any common integral to $F = 0$ and $R =$ constant is then an integral of the equation $(m-1)$. For $F = 0$ to have a double periodic meromorphic integral, it is necessary and sufficient that the equation $m-1$ admits double periodic meromorphic integrals and that among these integrals there are some that satisfy $F = 0$. The search for integrals of this kind is thus reduced to an equation of a lower order.

In particular, if the equation $m-1$ contains only y and y', this search is obtained easily by the Briot and Bouquet method.

These propositions make it possible in a large number of cases to simplify the search for conditions for a given type of differential equations to admit double periodic meromorphic integrals.

Notice that the polygonal line of the equation

$$P(y'') = Q(y),$$

where P and Q are polynomials of degree m and n, respectively, can contain only one edge with the negative angular coefficient and this coefficient is equal to

$$\frac{2m}{n-m}.$$

It is straightforward to see that the equation can only admit meromorphic double periodic integrals if n is of the form

$$n = m + \frac{2m}{k},$$

where k is a divisor of $2m$. For example, it is satisfied if $m = 1, n = 2$ or $m = 1, n = 3$, the coefficients of the polynomials P and Q are arbitrary; or if $m = 2, n = 4$ or $m = 2, n = 6$, and the coefficients of the polynomials P and Q being suitably selected, etc.

In general, in order for the equation

$$P(y^{(p)}) = Q(y),$$

to admit integrals in question, n must be of the form

$$n = m + \frac{pm}{k},$$

where k is a divisor of mp.

Notice also that the polygonal line of the equation

$$P(y^{(p)}) = Q(y)y'$$

has a single angular coefficient and it is equal to

$$\frac{mp-1}{n+1-m}.$$

We see that the existence of a meromorphic double periodic integral requires that

$$n = m - 1 + \frac{mp-1}{k},$$

where k is a divisor of $mp - 1$. It will actually be integrated in such functions for example if the coefficients of P and Q are arbitrary and $p = 3$, $m = 1$, $n = 1$ or $n = 2$, etc.

Considering equations of Briot and Bouquet

$$F(y, y') = 0,$$

for such an equation to be integrated through a meromorphic double periodic integral, its polygonal line must have at least an edge whose angular coefficient is a negative integer and at least one angular coefficient which is a positive integer and there are no edges whose angular coefficients are fractions. This proposition often greatly simplifies the question of specifying the equations, belonging to the general type of equations that can admit integrals of the kind considered. It results immediately on the one hand from the fact that a meromorphic double periodic function can not remain holomorphic in the whole plane and must vanish for an unlimited number of x values. On the other hand it is a consequence of the following proposition, that we demonstrated in an earlier work: let y be defined by

$$y = (x-a)^{\mu} f(x),$$

where f is a holomorphic function in a neighborhood $x = a$ that does not vanish for this value of x; for y to satisfy the equation

$$F(y, y') = 0,$$

it is necessary and sufficient that μ is equal to an angular coefficient of an edge of the polygonal line corresponding to the considered differential equation.

Notice also that, from the same proposition, it follows that for a Broit and Bouquet irreducible equation to be integrated by meromorphic functions in general, it is necessary that the polygonal line of the equation does not have any edge with fractional angular coefficient and it has at least one edge whose angular coefficient is different from zero. For if it were not so, the equation which cannot be integrated by other meromorphic functions except by rational or simply periodic functions and which can neither vanish nor be infinite for any value of x, would be reduced to a constant or one or several functions of the form:

$$He^{ax}.$$

In the latter case, the first member of the equation would be decomposable into factors of the form:

$$y + ay'$$

where a is a constant.

Moreover, if these necessary conditions for the existence of meromorphic integrals are fulfilled for the given equation, but that the polygonal line does not admit any edge with negative angular coefficient or any edge with positive angular coefficient, one easily ensures whether or not the equation admits meromorphic integrals. For in this case, the integral can only be rational or simply periodic, it can not, in the first case, have poles and in the second case zeros. Therefore, in the first case, it would be reduced to a polynomial in x or in e^{ax} and in the second case, it is the transform of the equation given in $\frac{1}{y}$ which must be reduced to such a polynomial and one easily completes the question by determining the degree and the coefficients of such a polynomial by the method of indeterminate coefficients.

О ЈЕДНОМ СВОЈСТВУ ДИФЕРЕНЦИЈАЛНИХ ЈЕДНАЧИНА КОЈЕ СЕ ИНТЕРГАЛЕ ПОМОЋУ МЕРОМОРФНИХ ДВОЈАКО-ПЕРИОДИЧНИХ ФУНКЦИЈА

Резиме. Разматрају се диференцијалне једначине произвољног реда општег облика које не зависе експлицитно од независне променљиве. Испитују се услови под којима такве једначине имају решење у облику двојако-периодичних мероморфних функција.

Такве неконстантне функције обавезно имају бар један пол, чији је ред неки природан број μ. Диференцијалним једначинама се додељује полигонална линија П, чија је својства изучавао аутор раније у својој докторској дисертацији. Темена полигона могу бити проста или вишеструка. Сваком вишеструком темену се додељује карактеристични полином. Ако диференцијална једначина има решење у облику двојако-периодичне мероморфне функције, онда њен полигон П има следеће својство: има ивицу чији је угаони коефицијент негативни цели број $-\mu$ или има вишеструко теме, тако да карактеристични полином тог темена има бар један целобројни корен $-\mu$ који се налази између угаоних коефицијената ивица које се сустичу у том темену.

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