

UT VIS SIC TENSIO

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This paper is dedicated to the memory of Philippe Boulanger (1948–2017)

ABSTRACT. The mechanical properties of rubber-like materials have been offering an outstanding challenge to the solid mechanics community for a long time. The behaviour of such materials is quite difficult to predict because rubber self-organizes into mesoscopic physical structures that play a prominent role in determining their complex, history-dependent and strongly nonlinear response. In this framework one of the main problems is to find a functional form of the elastic strain-energy that best describes the experimental data in a mathematical feasible way. The aim of this paper is to give a survey of recent advances aimed at solving such a problem.

1. Introduction and Basic Equations

The theory of nonlinear elasticity plays a fundamental role in continuum mechanics: the statics of a huge class of materials (any *simple material* à la Noll) is given by the statics of a nonlinear elastic material [20]. Moreover, in recent times new important applications of such a theory have been found first of all in the biomechanics of soft tissues, see for example [7].

Real world rubber-like materials and soft tissues are more complex than idealised hyper-elastic materials. A complete model for such materials must contain more information than the one conveyed by isotropic elastic theory: anisotropy effects, Mullin's effect, viscoelastic phenomena, residual stresses, . . .

Here, I restrict my attention only to isotropic incompressible hyper-elastic materials. This class of materials is perfect and sufficient to communicate my ideas to a well educated but generic audience in mechanical sciences. The inclusion of any other effect of more complexity will only give more strength to my observations.

Let us consider a motion of a body $\mathcal{B}: \mathbf{X} \times [0, \infty] \rightarrow \mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ and let $\mathbf{F} = \text{Grad } \mathbf{x}$ be its gradient. Let $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ be the left Cauchy-Green deformation

2010 *Mathematics Subject Classification*: 74B20.

Key words and phrases: hyper-elasticity, simple extension, phenomenological reduction, non-Gaussian effects.

tensor because we consider only isochoric motions ($\det \mathbf{F} = 1$)

$$I_1 = \text{tr } \mathbf{B}, \quad I_2 = \text{tr } \mathbf{B}^{-1},$$

are its principal invariants. If a material is hyper-elastic, incompressible and isotropic we may introduce a strain-energy density function $W = W(I_1, I_2)$. The Cauchy stress tensor \mathbf{T} is given by the representation formula

$$(1.1) \quad \mathbf{T} = -p\mathbf{I} + 2W_1\mathbf{B} - 2W_2\mathbf{B}^{-1},$$

where p is a Lagrange multiplier associated with the constraint of incompressibility and $W_i = \partial W / \partial I_i$. The representation formula for the nominal stress is obtained from (1.1) by using the formula $\mathbf{P} = \mathbf{T}\mathbf{F}^{-T}$.

The theory of linear elasticity is obtained from (1.1) in two steps. We first consider $\mathbf{x} = \mathbf{X} + \mathbf{u}(\mathbf{X}, t)$, where the \mathbf{u} is displacement field, approximate the left Cauchy–Green deformation tensor as

$$\mathbf{B} \approx \frac{[\text{Grad } \mathbf{u} + (\text{Grad } \mathbf{u})^T]}{2}.$$

Then we linearise the stress-strain relationship (1.1). (In the linear case the $\mathbf{T} \approx \mathbf{P}$). In the incompressible case the linear stress-strain relationship contains only a constitutive parameter: the infinitesimal shear modulus μ .

Borrowing the term from Truesdell [19] the *Hauptproblem* for the theory of non-linear elasticity is to *find* a feasible explicit functional form for the strain-energy W able to *describe* experimental data. The real meaning of the verbs "to find" and "to describe" depends on the cultural background we bring with us in affording this problem. A material scientist is first of all interested in relating the constitutive parameters contained in the functional form of W with mesoscopic physical quantities. This kind of information is fundamental to manufacture rubber-like materials with desired mechanical properties. A mathematician is interested in the qualitative properties of the strain-energy function (convexity or poly-convexity, growth conditions, coercitivity, ...). This is because in the mind of many mathematicians a mechanical theory has to cope with Hadamard's notions of well-posedness. An engineer is interested in describing the experimental data and qualitative properties that ensure the possibility of perform reasonable numerical computations of problems of technical interest. Clearly a simulation in computational solid mechanics, irrespective of its degree of sophistication, is as good, *or as bad*, as the model it relies upon.

Despite of these differences in the goals of their investigation, all these scientists have to face a new complexity with respect what was going on in the linear theory: here the problem is not to work with some constitutive parameters (only μ in the incompressible and isotropic case, μ and λ in the compressible, or unconstrained, isotropic case) we are obliged to work with the full set of functional forms going from polynomials to special functions through exponentials, trigonometric, logarithmic, rational, ... and any possible kind of function.

For this reason in the literature a huge number of mathematical model for the strain-energy have been proposed. This is in contrast with our search of *universal* constitutive models i.e., a model able to describe the experimental data first of all

from a qualitative point of view and then from a quantitative point with acceptable relative errors of prediction with respect to the data.

To investigate the Hauptproblem following the recent paper by Destrade, Saccomandi and Sgura [3] (to which we refer for all the details) we consider two sets of uniaxial data in simple extension: one is the classical set by Treloar about natural rubber and one, denoted DC9, is a more recent one derived for a synthetic rubber-like material. In the various plot Treloar's data are red and DC9 data blue. We consider two set of data because Treloar's data are a classic, but sometimes people are worried about their precision. The DC9 set is obtained using modern testing machines and therefore in principle they must be more accurate.

The simple extension deformation is a homogeneous universal deformation

$$x = \lambda X, \quad y = \frac{1}{\sqrt{\lambda}} Y, \quad z = \frac{1}{\sqrt{\lambda}} Z,$$

where λ is the uniaxial stretch. Let us denote with $t = t(\lambda)$ the Cauchy stress in the tensile direction, with $\sigma = \sigma(\lambda)$ the nominal stress and with $g = g(z)$ where $z = 1/\lambda$ the Mooney force:

$$(1.2) \quad g(z) = W_1 + zW_2, \quad \text{where} \quad g(z) := \frac{\sigma(\lambda)}{2(\lambda - \lambda^{-2})}, \quad z := \lambda^{-1}.$$

The reason to introduce the Mooney plot is clear in the framework of the Mooney–Rivlin material, the rescaling gives linear relation between the stress and strain variables and fitting the data is easier. Moreover, we point out that this rescaling boosts the data for small stretches; this is a zone where relative errors can be quite important. The Mooney plot, i.e., just the rescaling of data as required by (1.2) allows to stress out three important *ranges* of deformation (we refer to any of of the figures containing data). First of all we point out the *finite but moderate* range of deformations for approximately $z \in]0.5, 1]$. Second, we have the *upturn* zone (approximately $z \in]0.2, 0.5]$) and then the last zone: the *limiting-chain* range where the data seems to blow up (approximately $z \in]0, 0.2[$). We shall discuss in the following pages the meaning of all these ranges of deformation.

There are many details that have to be fixed to have a rigorous discussion of this classical and simple example and once again we refer to Destrade, Saccomandi and Sgura [3]. The goal of my discussion is to give a general overview of all the underpinnings of this problem summarizing the work done by my co-authors and myself during a long period and contained mainly in the above mentioned paper and other two papers: the 2004 paper with Ogden and Sgura [12] and the recent review with Puglisi [16]. Moreover, in the present paper I will use a *historical* approach.

All my discussion is based on the deformation of simple extension and only in a couple of points I have to consider the universal homogeneous deformation of simple shear

$$x = X + KY, \quad y = Y, \quad z = Z,$$

where K is the amount of shear, and the non-universal inhomogeneous motion

$$(1.3) \quad x = X + u(Z, t), \quad y = Y + v(Z, t), \quad z = Z,$$

defining transverse waves. Here u and v are unknown function to be determined by the balance equations.

REMARK 1.1. The Hooke's law is *ut tensio, sic vis*. The reason because in the title you find the reverse will be clear at the end of next Section.

2. Focusing the Hauptproblem

Let us try to read the following sentence

Rdgnieg I blveiee cdnuolt aulactly taht was waht cloud uesdnatnrd.

This is not quite easy. It is possible to understand some of the words but it is hard to recover the meaning of this strange sentence. On the other hand if we read the following sentence build up with the same *words* everything seems to be directly understandable

I cdnuolt blveiee taht I cloud aulactly uesdnatnrd waht I was rdgnieg.

The only difference between the above sentences is the ordering. In the first sentence I have mixed the syntactic order, in the second the sentence, despite all the words are still misspelled, I have maintained the classical syntactic order.

The syntax of a language is exactly a non quantitative model: the words are the data and the syntactic rules are the relationships of the model. Maybe in the first sentence we are able to recover the meaning of the single words after some time, but the ordered sentence can be easily read, nearly as fast we read a correct sentence.

From this simple example we learn several important facts:

- experience by itself is not science;
- to every object there corresponds an ideally closed system of truths that are true of it and, on the other hand, an ideal system of possible cognitive processes by virtue of which the object and the truths about it would be given to any cognitive subject;
- thus, reality is not guaranteed for an isolated item, even when it seems to be giving us a reason to take it as the unified core attracting its manifold appearing to one hub of reference. The central location of the thing is dependent upon its real circumstances;
- the reality of "one" depends on "others"; i.e., on thing-connection.

These considerations are clearly not mine but are a summary of Edmund Husserl ideas [8]. My aim is to apply this point of view to the problem we are considering. To understand what we are doing it is not necessary to have a deep knowledge of Husserl's philosophy. It is sufficient to have in mind the simple example given by the above sentences.

First of all data by itself are not sufficient to concretise *knowledge*. Theory is based on experience but comes first with respect experiments. Experiments are dictated and shaped by the theory. For this reason a *good* model is able to survive to strong perturbation of the data and constitutive parameters. This because a good model links in a *universal* way the "one" with the "others". Too many models of rubber-like materials *seems* to be successful. The real status of the affair is that

they are mainly used in simulations that despite they are concretely descriptive, in the end, they apply only to some special phenomena. Here we are searching to point out more we are searching for a true *mathematical* model not a computational one.

3. A short guided history of the Hauptproblem

Let us sketch a sort of historical approach to the problem we are considering where I stress out some of the breakthroughs, i.e., important achievements, but also some of the criticalities.

3.1. Neo-Hookean material. The basic model of rubber-like mechanics is the neo-Hookean model for the strain-energy W

$$(3.1) \quad W = \frac{\mu}{2}(I_1 - 3).$$

Statistical mechanics is able to connect the only constitutive parameter μ (exactly the infinitesimal shear modulus) to the mesoscopic quantities of the polymeric network. The exact connection depends on the kind of average we consider to go from a single chain to the network. This is a purely entropic model, i.e., in the free energy we are discarding the internal energy, and it assumes that end-to-end distance of any chain composing the network is described by a Gaussian distribution function.

There is a certain *legend* which regards the neo-Hookean model as a good model for large but moderate stretches. This is clearly not true as is possible to

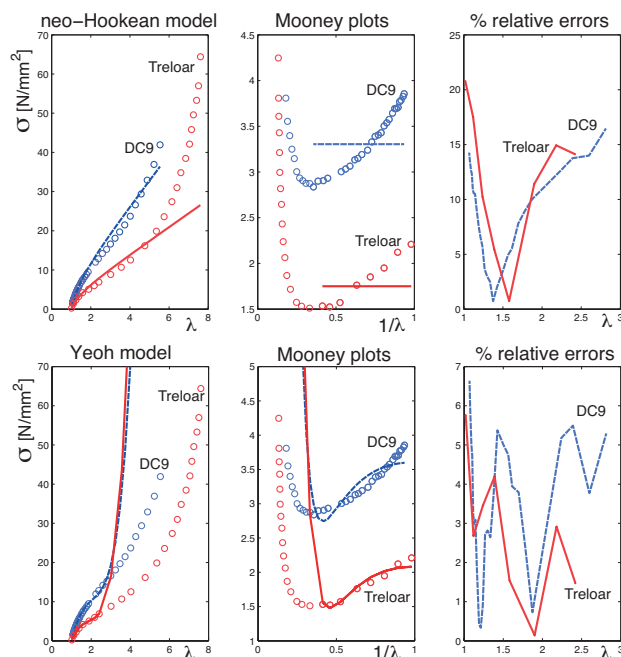


FIGURE 1. Neo-Hookean and generalized neo-Hookean materials (from [3])

test just looking at the corresponding Mooney plot. In general for all the class of generalized neo-Hookean materials, i.e., materials such that $W = W(I_1)$, the contrast with experimental data is very bad. Always in figure 1 it is possible to appreciate as bad is the Yeoh model is

$$W = c_1(I_1 - 3) + c_2(I_1 - 3)^2 + c_3(I_1 - 3)^3,$$

despite the three constitutive parameters c_1, c_2 and c_3 .

Generalized neo-Hookean materials are *interesting* for two reasons. First they are easily connected to statistical mechanics computations. The invariant I_1 is equal thrice the square of the stretch ratio of an infinitesimal line element averaged over all possible orientations. Therefore, it is natural to connect this invariant to the end-to-end distance of a chain. Second, the mathematics of generalized neo-Hookean materials is usually much simpler than the mathematics for the full model [6].

On the other hand, such models can be not suitable to describe experimental data. This fact can be quantitatively evaluated using a precious tools for experimentalist: the *universal relations* [13].

For isotropic materials universal relations are generated by the coaxiality among stress and strain $\mathbf{T}\mathbf{B} = \mathbf{B}\mathbf{T}$. This means that usually we have a maximum number of three universal relations, but for generalized neo-Hookean material a fourth universal relation is always possible. This universal relation is valid for any choice of the strain-energy $W = W(I_1)$ and represents a mean to justify quantitatively and in a direct experimental way the hypothesis $W_2 = 0$. For example, if we consider the simple shear deformation it is well known that from (1.1) we obtain

$$\begin{aligned} T_{11} &= -p + 2(1 + K^2)W_1 - 2W_2, & T_{22} &= -p + 2W_1 - 2(1 + K^2)W_2, \\ T_{33} &= -p + 2W_1 - 2W_2, & T_{12} &= 2K(W_1 + W_2), \quad T_{13} = T_{23} = 0. \end{aligned}$$

From these relations we obtain three universal relations: two are trivial $T_{13} = T_{23} = 0$ and one is the celebrated relation

$$T_{11} - T_{22} = KT_{12}.$$

Now let us pick p such that $T_{33} = 0$. This requirement is suggested by the experimental device used to test a material in simple shear. In so doing we fix $p = 2W_1 - 2W_2$ and obtain

$$\begin{aligned} T_{11} &= 2K^2W_1 - 2W_2, & T_{22} &= -2K^2W_2, \\ T_{12} &= 2K(W_1 + W_2), & T_{13} &= T_{23} = T_{33} = 0. \end{aligned}$$

Now, the new universal relation we obtain only for generalized neo-Hookean materials is given by $T_{22} = 0$. This is a simple requirement that we can test experimentally in a easy and direct way: to the best of my knowledge no real material satisfies in a exact way this relation.

3.2. Mooney-Rivlin material. The Mooney-Rivlin strain-energy density

$$(3.2) \quad W = \frac{1}{2}C_1(I_1 - 3) + \frac{1}{2}C_2(I_2 - 3),$$

plays a fundamental role in the history of polymer physics and the theory of non-linear elasticity for several reasons. First of all it is a purely phenomenological

theory stemming from the early tremendous effort devoted to rewrite the theory of Continuum Mechanics using the language of Tensor Algebra. For this reason this model led to the exploration of the non-linear theory of elasticity in deep and unexpected ways. Using (3.2) for the first time it is simple to write down the equation of nonlinear elasticity in their general format. Moreover, the balance equations corresponding to this model admit significative classes of non-homogeneous exact solutions and provided a new perspective to the interpretation of experimental data.

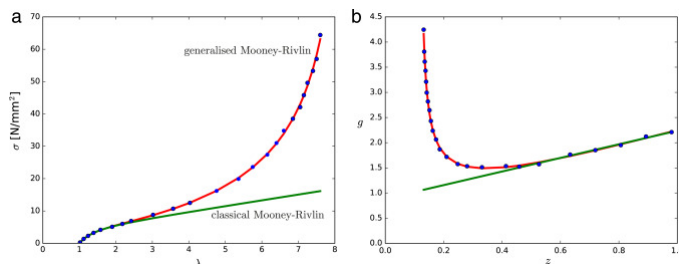


FIGURE 2. Mooney-Rivlin and generalised Mooney-Rivlin (from [10])

Mooney derived this model in searching for an exact linear relationship between the Cauchy shear stress component T_{12} and the amount of shear K in a simple shear experiment (or between the torque M and the twist ψ in a simple torsion experiment). Here we understand why the Mooney plot is so important for such a material, being

$$g(z) = C_1 + zC_2,$$

a straight line.

The Mooney-Rivlin material, introducing the C_2 term, improve the description of the experimental data: now it is true that for a moderate range of stretches we have a good agreement with the experimental data.

On the other hand, several drawbacks are connected with this model. First, of all it is not clear how to connect the various parameters with mesoscopic information, but the major problem is the following: Mooney has not realized that the model (3.2) is not the most general strain-energy density corresponding to a linear relationship $T_{12} = \mu K$. In a recent paper by Mangan, Destrade and Saccomandi [10] it has been shown that any generalized Mooney-Rivlin materials

$$(3.3) \quad W = \frac{1}{2}C_1(I_1 - 3) + \frac{1}{2}C_2(I_2 - 3) + H(I_1 - I_2),$$

where H is an arbitrary function of its argument, gets this linearity property. As it is possible to appreciate from the figure 2 we can use the function H to improve the fitting of the experimental data, but despite this fact all the generalized Mooney-Rivlin materials are in contrast with a simple and generic experimental observation derived from nonlinear acoustics [2].

In incompressible materials we can observe only the propagation of bulk transverse waves (1.3). The determining equations in a isotropic material for transverse waves are given by the system

$$(3.4) \quad \rho u_{tt} = [Q(u_z^2 + v_z^2)u_z]_z, \quad \rho v_{tt} = [Q(u_z^2 + v_z^2)v_z]_z,$$

where ρ is the constant density and Q is the *generalized shear modulus*.

For all the generalized Mooney-Rivlin (3.3) we have that $Q \equiv \mu$ the infinitesimal shear modulus [18]. Therefore for such a class of materials the equations (3.4) are uncoupled and linear: this fact seems to be not confirmed by experiments.

3.3. Ogden material. It is well known that the Ogden's strain-energy density

$$(3.5) \quad W = \sum_{i=1}^{\infty} \frac{\mu_i}{\alpha_i} (\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i}),$$

where $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches of the deformation and μ_i and α_i are the material constants, has been a major advancement in the possible solution to our problem.

If we consider a three or four term expansion of this formula then we have a very good agreement with experimental data and not only in simple extension. This is due of the particular form of the strain-energy (3.5) based on the Valanis-Landel hypothesis ($W = f(\lambda_1) + f(\lambda_2) + f(\lambda_3)$) which gives a special invariance in the stretch space: an invariance that seems to be fundamental in catching the experimental data in biaxial extension.

Despite this fact there is a major limitation associated with (3.5). This limitation is due to the fact that to fit the parameters α_i we need a nonlinear method and therefore there are several local extrema for the associated objective function and this gives the possibility to have non-uniqueness of the optimal parameter set. Moreover, these optimal sets of parameters which gives the same sets of relative errors on the other hand they give very different predictions. As it has been discussed into details in [12] this is a major problem, first of all when we use this model for simulation purposes.

3.4. Gent material. The limitations of the classical kinetic theory (i.e., the neo-Hookean model) were clear from the beginning and Treloar provides five possible reasons to understand the molecular significance of the deviation from statistical theory

- non-Gaussian effects both for the chains and the network;
- internal energy effects;
- chain entanglements;
- irreversible effects;
- non-random packing effects.

In the recent past the most investigated deviation that has been investigated is surely is that due to the non-Gaussian effects. In 1993 Arruda and Boyce proposed a multi-scale approach to take into account non-Gaussian effects [1]. Their methodological procedure is based on the use of the inverse Langevin function for the computation of the end-to-end distance for a single chain of the polymeric network, the eight chain network structure to perform the network average and the affine assumption imposing the coincidence of the network chain stretches and macroscopic stretches. The result is a generalized neo-Hookean model and thus a model that by sure is inapt to describe the data but despite this fact the model was very successful among the polymer mechanics community.

Alan Gent in 1996 published a short but very interesting note [4] showing that the idea of Arruda and Boyce can be obtained by considering a simple phenomenological modification of the neo-Hookean model. The Gent model is given by a two parameters strain-energy

$$(3.6) \quad W = -\frac{\mu}{2} J_m \ln \left(1 - \frac{I_1 - 3}{J_m} \right),$$

with one parameter being the infinitesimal shear modulus μ and the second parameter J_m is an average measure of the length of the polymeric chains composing the polymeric network. Indeed, in the limit $J_m \rightarrow \infty$ from (3.6) we recover the neo-Hookean material. Since the neo-Hookean model uses a Gaussian distribution for the end-to-end distance of the chains, and the support of the Gaussian distribution is not compact, the chains can be infinitely long (non-zero probability value for any end-to-end length).

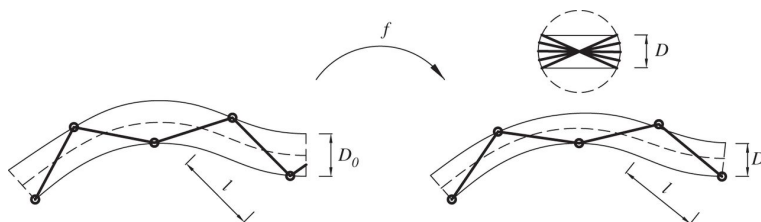


FIGURE 3. A cartoon of a single chain in a tube (from [16])

Following the cartoon in figure 3 it is simple to give to the Gent model a mesoscopic justification. Let us introduce a molecular chain composed by a large number N of rigid rods, each of the same length l , hinged together. Let us assume that this chain is confined to an ideal tube of diameter $D < l$. The projection of a single chain onto the axis of the tube is $\sqrt{l^2 - D^2}$ and because the total length is $L = Nl$ we find that in a simple one-dimensional setting thinking to a straight tube the end-to-end distance R of the chain is given as

$$R^2 = L^2 \left(1 - \frac{D^2}{l^2} \right)$$

On the other hand, it is well known that a rigid rod with a fixed point and free to have any orientation in the space can choose a number of configurations proportional to $4\pi l^2$, but here we are confined in a tube, then it is possible to choose only one set out of these orientations, proportional to D^2 .

Therefore, the entropy of a single rod can be computed (approximately) as $k_B \ln(D^2/l^2)$ and for the whole network using the most simple average and considering purely entropic contribution to the free energy we have

$$W \approx -k_B T \frac{L^2}{l^2} \ln \left(1 - \frac{R^2}{L^2} \right),$$

a model which is exactly a one-dimensional version of the Gent model.

Our derivation is more interesting than the Arruda-Boyce approach because it allows to introduce a term related to the second invariant I_2 term in a simple

way. To this end it is important to bear in mind that the mean square change in area is related to I_2 and if the cross-sectional area of the tube changes due to the macroscopic deformation, (for example due to the incompressibility conditions that reduce the *mobility* of the chains with the increase of the stretch), then it is possible to replicate our computations by introducing a multiplicative correction in the logarithmic term in the above expression. A similar idea has been introduced by Kroon [9], where the change of the microscopic tube dimensions has been shown to be proportional to the root mean square change of macroscopic area, i.e., to an averaged term $\sqrt{I_2}/2$. Pucci and Saccomandi [14] have indeed proposed the three-terms strain-energy

$$(3.7) \quad W = -C_1 J_m \ln \left(1 - \frac{I_1 - 3}{J_m} \right) + C_2 \ln \left(\frac{I_2}{3} \right).$$

A model that mix the Gent idea just illustrated and another term proposed by Gent and Thomas in the fifties to replace the C_2 term in the Mooney-Rivlin material. The model [14], up to now, gives the best fitting of data in uniaxial simple extension we know (see the next Section).

4. Toward a possible solution of the Hauptproblem

From the history of the problem under investigation we learn a lot of things. Using the Mooney plot it is clear that the Mooney-Rivlin material seems to be a good model in the first range of deformations. Then we have the *upturn* zone where the Mooney-Rivlin model fails: it seems that we need a *strain-hardening* term. In the last zone non-Gaussian effects are prominent: here it seems that we need to introduce a limiting chain parameter.

Summarizing our history, it is possible to hypothesize three fundamental steps in building a reliable model:

- dependence on I_2 ;
- introduction of at least three material parameters;
- the limiting chain effect.

Let us start with the assumption

$$W = \frac{1}{2}C_1(I_1 - 3) + \frac{1}{2}C_2f(I_2)$$

where several forms of $f(I_2)$ can be used:

$$f(I_2) = I_2 - 3, \quad f(I_2) = \sqrt{3}(\sqrt{I_2} - \sqrt{3}), \quad f(I_2) = 3 \ln(I_2/3),$$

these three terms are named respectively Mooney-Rivlin, Carroll and Gent-Thomas.

The first finding is that to model in a reliable way the finite but moderate range of deformation what we need is really a dependence on I_2 and no more than this. The figure 4 shows clearly that any functional form we choose for $f(I_2)$ gives similar results.

This is a very good news. In the spirit of Husserl phenomenological reduction it seems that the essence to catch the data in this range is the I_2 term we do not need to pay particular attention to the functional form. This is a great relief if we are worried about the robustness of the theory of nonlinear elasticity.

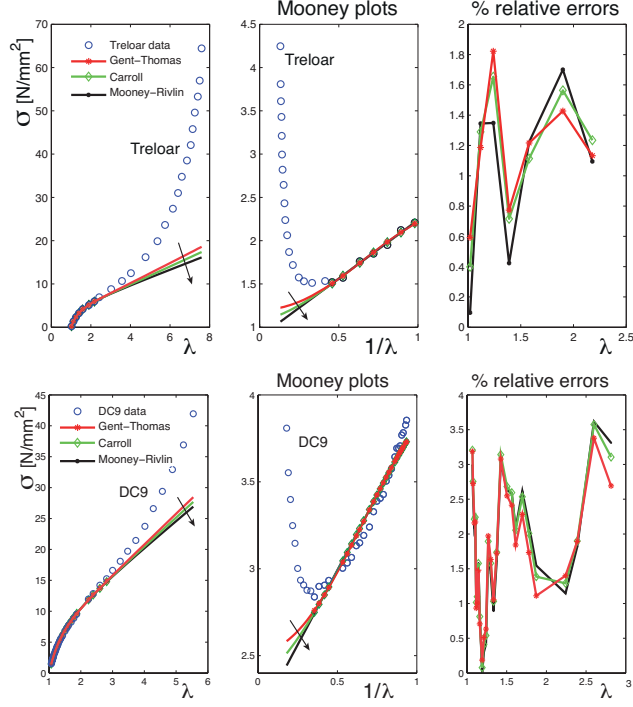


FIGURE 4. The moderate but finite range (from [3])

If we move to the possibility to model the upturn in the Mooney plot our first idea is to read the upturn as a clear manifestation of the nonlinearity. Then, our goal is to *reinforce* the nonlinearity and this can be done by introducing a third parameter

$$W = \frac{1}{2}C_1(I_1 - 3) + \frac{1}{2}C_2f(I_2) + C_3(I_1^n - 3^n),$$

where n is an exponent that must be determined a priori. This requirement is necessary to avoid the possibility of a nonlinear fitting procedure. In statistical physics there is a result due to Pincus for a macromolecular chain in a good solvent that fixes the values of n at around 2.5. Here we are not in a good solvent and we discover that it is sufficient to improve the neo-Hookean just a little bit to have again the possibility to model the upturn with any proposed functional form of $f(I_2)$. In the figure 5 we can appreciate this situation with $n \approx 1.6$ for the Treloar's data and $n \approx 1.1$ for the DC9 data.

When we arrive to the full range now it is clear why the Pucci and Saccomandi model [14] gives a wonderful performance. The model contains I_2 , we have three parameters, it has a strain-hardening effect (hidden in the \ln term with I_1) and we have the limiting chain effect. Clearly this is not the only possibility. Other good models can be built up with the same philosophy and this has been done (see [3]). Moreover, there is the possibility of other approaches as the implicit theory of elasticity [11] and mesoscopic modelling [17].

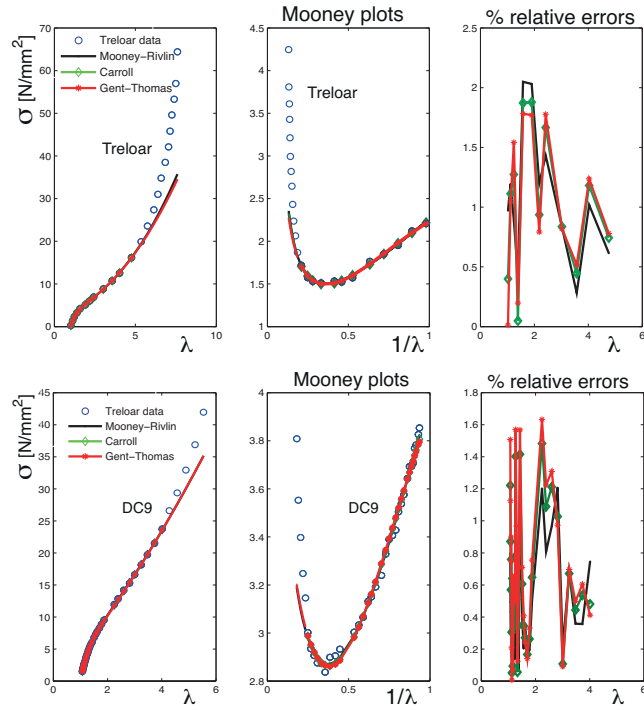


FIGURE 5. The upturn (from [3])

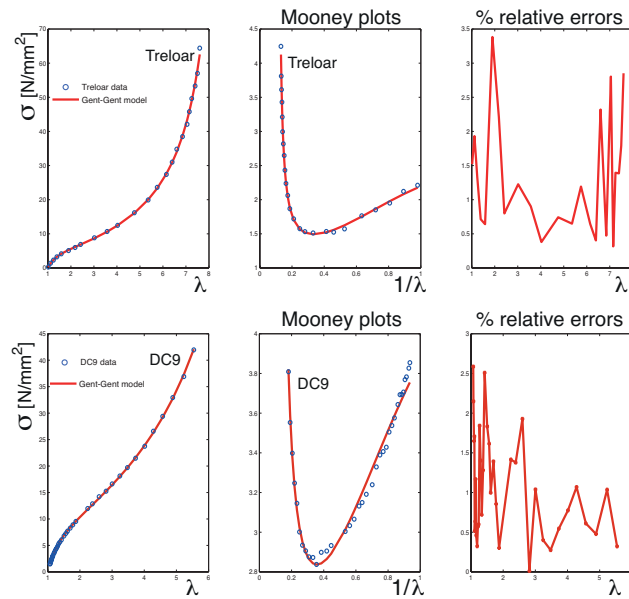


FIGURE 6. Full range (from [3])

5. Concluding remarks

When we deal with a nonlinear theory we face a *mare magnum*: any functional form can be a priori a constitutive equation. Experimental data are not sufficient to restrict in a definitive way the possible functional forms. Only the power of hypothetical deductive method of mathematical modelling can truly help.

A possibility to overcome this problem is to do as it is usual in nonlinear acoustics [5]: to identify the nonlinear theory with the *fourth-order theory* of elasticity. If one considers linear elasticity as the Taylor expansion to second order in the strain of the elastic potential, the fourth-order elasticity is obtained pushing to fourth order such an expansion. In so doing, nonlinearity is just some constants: the Landau constants of third and fourth order. This kind of models maybe sufficient to explain some experimental facts arising in nonlinear acoustics, but they are quite poor to describe, for example, the simple extension data. There are situations where a rough but honest and feasible model and can be sufficient to recover and consider the information you need. On the other hand, there are many situations where we need much more.

The strategy we propose can be a reasonable way to advance in the problem we have stressed out. Our strategy is based not only in reading the data but in trying to build a framework where the data must be read. If you read only the data you are just trying to understand the words that are misspelled. Our strategy try to correct the words using a syntax.

Clearly, a deeper analysis is required to cope with more complex set of data (for example biaxial data). This is our future plan. At the moment we have solved a little piece of the whole problem and with Albert Camus we can say

Et c'est bien là le génie: l'intelligence qui connaît ses frontières.

Acknowledgments. This paper is based on a invited lecture I have delivered during the 6th International Congress of Serbian Society of Mechanics (Mountain Tara, Serbia, June 19–21, 2017). I thanks the organisers for their kind invitation. Both the lecture I have delivered and therefore also this paper are based mainly on the findings generated by my longstanding collaboration with Michel Destrade, Ray Ogden, Giuseppe Puglisi and Ivonne Sgura. Without their help and their collaboration nothing would have been possible. My knowledge in general continuum mechanics is due to Kumbakonam Rajagopal and Maurizio Vianello. My basic knowledge in finite elasticity is due to Mike Hayes and Niall Horgan. My actual knowledge in nonlinear elasticity is due, not only to the above mentioned scientists, but also to Millard Beatty, Philippe Boulanger, Luis Dorfmann, Yibin Fu, Alain Goriely, Jerry Murphy, Edvige Pucci, Domenico de Tommasi, Luigi Vergori, Alan Wineman and Giuseppe Zurlo. I was a lucky man to met all these wonderful scientists and friends.

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UT VIS SIC TENSIO

РЕЗИМЕ. Механичке особине материјала типа гуме пружају изузетан дугогодишњи изазов истраживачима у области механике чврстог тела. Понашање таквих материјала је прилично тешко предвидети јер се гуме самоорганизују у мезоскопске физичке структуре које играју истакнуту улогу у одређивању њиховог сложеног и јаког нелинеарног одговора, који зависи од историје деформације. У датом оквиру, један од главних проблема је проналажење функционалног облика еластичне напонске енергије која најбоље описује експерименталне податке на математички изводљив начин. Циљ рада је да се да преглед напретка у решавању датог проблема.

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(Received 03.07.2017.)
(Revised 28.10.2017.)
(Available online 03.11.2017.)