THEORETICAL AND APPLIED MECHANICS
Volume 44 (2017) Issue 2, 255–270 DOI: https://doi.org/10.2298/TAM171107017S

STABILITY OF LEVITRON^{TM} REVISITED

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Dedicated to Professor Aleksandar Bakša, teacher and friend

Abstract. This note discusses some issues related to stability of stationary motion of hovering magnetized top in a homogeneous magnetic field. Stability of synchronous motion is analyzed using the simplified model in which the hovering motion of the center of mass is ignored. Stability boundaries are derived using Lyapunov direct method. In particular, it is shown that, for a given angle Δ between magnetic moment dipole and principal axis of the top, there is an interval of stationary values of nutation angle θ_0 for which the stationary synchronous motion is stable.

1. Introduction

Levitron^{TM} is a commercial product consisted of the permanent magnetic base and a top which is also a magnetic dipole. It is possible to produce stable levitating motion of the top above the magnetic base, akin to stable precession of an ordinary gyroscope. Although its inventor, Ray Harrigan, was discouraged by the academic community, he persisted in the efforts to construct a hovering top. This remarkable toy attracted the attention of physicists because Earnshaw's theorem rules out stable magnetic levitation of static magnetic dipoles [1]. Harrigan's discovery called for reasonable explanation of the phenomenon and stability analysis of stationary motion.

The papers of Simon et al. [2] and Berry [3] provided persuasive explanation of the phenomenon of hovering motion of magnetic top. Among different interesting problems, the stability of stationary motion attracted much attention. Not pretending to give an exhaustive, and still growing list of references, the ones that inspired the present work are mentioned in the sequel. Flanders et al. [4] analyzed the stability of synchronous motion assuming that the centre of mass is fixed in laboratory frame, i.e., motion in the space $TSO(3)$. Gov et al. [5] analyzed the stability of stationary rotation about vertical axis, which is also the symmetry axis of the magnetic field, taking into account all six degrees of freedom, i.e.,

2010 *Mathematics Subject Classification:* 70E50, 70K20.
Key words and phrases: stability, Lyapunov method, Levitron^{TM}.

in $T(\mathbb{R}^3 \times SO(3))$. Dullin and Easton [6] studied stability in Hamiltonian framework, i.e., in $T^*(\mathbb{R}^3 \times SO(3))$, proved linear stability of relative equilibrium and improved Berry's results [3]. Genta et al. [7] discussed lower and upper stability limits of the spin velocity. Krechetnikov and Marsden [8] analyzed the influence of dissipative and positional forces on the stability boundary; it turned out that both forces appear in the case of Levitron$^{\text{TM}}$ due to eddy currents. Recently, Bonsioli and Delprete [9] analyzed stability taking into account the aerodynamic drag torque. Common feature of all the studies mentioned above is that their results are based upon linear stability analysis. They are reliable when determining the boundaries of instability regions in parameter space. However, their applicability may be limited—they do not provide a decisive answer about stability when there are eigenvalues with zero real part, since the influence on non-linear terms could become important.

This study presents the results of stability analysis of synchronous stationary motion based upon simplified model, exploited also in [4], in which the hovering motion of the mass center is ignored, i.e., the center of mass is treated as a fixed point. The stability is analyzed using Lyapunov direct method, since in the considered model one eigenvalue of the linearized system is zero. Thus, in this case the linear stability does not imply stability of the stationary motion (relative equilibrium). It is shown that appropriate Lyapunov function cannot be constructed using energy-like first integral solely, but rather as a linear combination of two independent first integrals–generalized energy integral and angular momentum integral. A new result, obtained using this approach, shows that, for a given angle Δ between magnetic moment dipole and the principal axis of the top, there is an interval of stationary values of nutation angle θ_0 for which stationary synchronous motion is stable.

The paper is organized as follows. In Section 2 the mathematical model is derived using Lagrangian equations, along with corresponding first integrals (conservation laws); stationary motion of the top is analyzed and the first result about the existence of stationary solution is obtained. Section 3 is technical and contains variational equations of the problem, both exact (non-linear) and linear. In Section 4 the results of linear stability analysis, obtained previously by Flanders et al. [4], are recovered. Section 5 exposes the main results of this study based upon non-linear stability analysis using Lyapunov's direct method. These results are compared with the results of linear stability analysis in Section 6. The paper is closed by some concluding remarks.

2. Stationary motion of the magnetic top

Equations of motion of the magnetic top, with moment $\mathbf{m}$ of magnetic dipole, in a homogeneous magnetic field $\mathbf{H}$, will be given in the form of Lagrangian equations. It will be assumed that the moment of magnetic dipole is constant, fixed in the body and forms constant angle Δ with the symmetry axis of the top. Without loss of generality, it will be assumed $\Delta \geqslant 0$. It will also be assumed that the center of mass of the top is fixed, i.e., its hovering motion will be ignored. This assumption is physically meaningful when the gravitational force is equilibrated by the constant magnetic field. It could serve as a reasonable approximation of more realistic situations, as well. This model of hovering magnetic top was studied in [4], where Lagrangian equations of motion could be found. In the following text the governing equations and stationary solution will be given for the completeness of the exposition and to fix the notation.

Introducing standard Euler's angles of precession, nutation and rotation (ψ, θ and φ), moment of magnetic dipole $\mathbf{m}$ could be expressed as

$$\mathbf{m} = m\cos\Delta\mathbf{e}_\zeta - m\sin\Delta\mathbf{e}_\eta; \quad m = \text{const.}, \tag{2.1}$$

and magnetic field is assumed to be constant, having vertical direction $\mathbf{e}_z$

$$\mathbf{H} = -H\mathbf{e}_z = -H(\sin\theta\sin\varphi\mathbf{e}_\xi + \sin\theta\cos\varphi\mathbf{e}_\eta + \cos\theta\mathbf{e}_\zeta); \quad H = \text{const.}, \tag{2.2}$$

where $\{\mathbf{e}_\xi, \mathbf{e}_\eta, \mathbf{e}_\zeta\}$ is the standard basis of the moving frame $O\xi\eta\zeta$ (see Fig. 1).

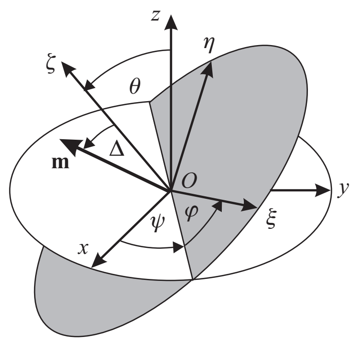

FIGURE 1. Coordinate frames of the hovering top problem.

Corresponding potential energy of the magnetic field reads

$$\Pi = -\mathbf{m}\cdot\mathbf{H} = mH(\cos\Delta\cos\theta - \sin\Delta\sin\theta\cos\varphi).$$

It is assumed that the body is axially symmetric, with axial moments of inertia $J_\xi = J_\eta \neq J_\zeta$, so that Lagrangian function of the magnetic top in homogeneous magnetic field can be written in the form

$$\begin{aligned} L = E_k - \Pi = \tfrac{1}{2}J_\xi(\dot\theta^2 + \dot\psi^2\sin^2\theta) + \tfrac{1}{2}J_\zeta(\dot\varphi + \dot\psi\cos\theta)^2 \\ - mH(\cos\Delta\cos\theta - \sin\Delta\sin\theta\cos\varphi), \end{aligned} \tag{2.3}$$

where an overdot denotes the time derivative. The Lagrangian equations $\left(\frac{d}{dt}\right)\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\partial L}{\partial q} = 0$ for coordinates $q = \psi, \theta, \varphi$ read

$$\ddot\psi(J_\xi\sin^2\theta + J_\zeta\cos^2\theta) + J_\zeta\ddot\varphi\cos\theta + (J_\xi - J_\zeta)\dot\psi\dot\theta\sin 2\theta - J_\zeta\dot\varphi\dot\theta\sin\theta = 0; \tag{2.4}$$

$$\begin{aligned} J_\xi\ddot\theta - \dot\psi^2(J_\xi - J_\zeta)\sin\theta\cos\theta + J_\zeta\dot\varphi\dot\psi\sin\theta \\ - mH(\cos\Delta\sin\theta + \sin\Delta\cos\theta\cos\varphi) = 0; \end{aligned} \tag{2.5}$$

$$J_\zeta(\ddot\varphi + \ddot\psi\cos\theta - \dot\psi\dot\theta\sin\theta) + mH\sin\Delta\sin\theta\sin\varphi = 0. \tag{2.6}$$

Note that governing equations have two first integrals, Jacobi's (generalized energy) and cyclic integral for coordinate ψ since $\partial L/\partial t = 0$ and $\partial L/\partial \psi = 0$. They have the following form

$$E = \tfrac{1}{2}J_\xi(\dot\theta^2 + \dot\psi^2\sin^2\theta) + \tfrac{1}{2}J_\zeta(\dot\varphi + \dot\psi\cos\theta)^2 + mH(\cos\Delta\cos\theta - \sin\Delta\sin\theta\cos\varphi) = \text{const.} \tag{2.7}$$

$$C = (J_\xi\sin^2\theta + J_\zeta\cos^2\theta)\dot\psi + J_\zeta\dot\varphi\cos\theta = \text{const.} \tag{2.8}$$

The results of this study will be presented in dimensionless form. To that end the following dimensionless quantities will be introduced

$$J = J_\xi/J_\zeta; \qquad \tau = t\hat\omega; \qquad \hat\omega = \sqrt{mH/J_\zeta},$$

where $\hat\omega$ is the reference quantity which has unit s^{-1}. Using these quantities, Lagrangian equations (2.4)–(2.6) can be given in dimensionless form

$$\psi''(J\sin^2\theta + \cos^2\theta) + \varphi''\cos\theta + (J-1)\psi'\theta'\sin 2\theta - \varphi'\theta'\sin\theta = 0; \tag{2.9}$$

$$(J-1)\sin\theta\cos\theta + \varphi'\psi'\sin\theta - (\cos\Delta\sin\theta + \sin\Delta\cos\theta\cos\varphi) = 0; \tag{2.10}$$

$$\varphi'' + \psi''\cos\theta - \psi'\theta'\sin\theta + \sin\Delta\sin\theta\sin\varphi = 0, \tag{2.11}$$

where prime denotes the derivative with respect to dimensionless time variable τ. At the same time, dimensionless form of the first integrals (2.7)–(2.8) reads

$$\hat E = \tfrac{1}{2}J(\theta'^2 + \psi'^2\sin^2\theta) + \tfrac{1}{2}(\varphi' + \psi'\cos\theta)^2 + (\cos\Delta\cos\theta - \sin\Delta\sin\theta\cos\varphi) = \text{const.} \tag{2.12}$$

$$\hat C = (J\sin^2\theta + \cos^2\theta)\psi' + \varphi'\cos\theta = \text{const.} \tag{2.13}$$

Since Lagrangian (2.3) of the system has only one cyclic coordinate, this model of magnetic top has a peculiar stationary solution, different from the one of usual top, which describes so-called synchronous motion [4].

LEMMA 2.1. *Governing equations* (2.9)–(2.11) *admit stationary solution*

$$\psi'(t) = \omega_0; \quad \theta(t) = \theta_0; \quad \varphi(t) = \varphi_0 (= 0), \tag{2.14}$$

where ω_0, θ_0 and φ_0 are real constants, provided they satisfy the constraint

$$\omega_0^2(J-1)\sin\theta_0\cos\theta_0 + \sin(\Delta + \theta_0) = 0. \tag{2.15}$$

PROOF. Stationary solution (2.14) satisfies Eq. (2.9) identically, while Eq. (2.11) yields $\sin\Delta\sin\theta_0\sin\varphi_0 = 0$. We shall adopt the solution $\varphi_0 = 0$, since it will be shown in the sequel that other possibilities ($\Delta = 0$ or $\theta_0 = 0$) are ruled out by stability conditions. Finally, the non-trivial constraint (2.15) to the stationary solution is a consequence Eq. (2.10). □

Given Δ and θ_0, the stationary value of precessional angular velocity ω_0 can be determined from equation (2.15)

$$\omega_0 = \sqrt{\frac{\sin(\Delta + \theta_0)}{(1-J)\sin\theta_0\cos\theta_0}}. \tag{2.16}$$

In real situations angles Δ and θ_0 are small and (2.16) can be approximated as

$$\omega_0 \approx \omega_{\min}\sqrt{1+\frac{\Delta}{\theta_0}}; \quad \omega_{\min} = \sqrt{\frac{1}{1-J}}. \tag{2.17}$$

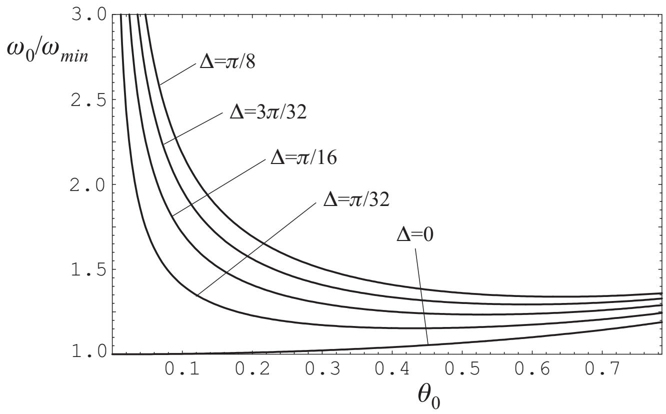

FIGURE 2. Stationary angular velocity $\omega_0/\omega_{\min}$ vs. θ_0 for different Δ.

The last result calls for a brief explanation. The value $\omega_{\min}$ was given by Flanders et al. [4, Eq. (8)] as a minimal value of precessional angular velocity needed for the existence of synchronous stationary motion. This statement can be supported by the following arguments. For small values of angles Δ and θ_0 equation (2.15) can be solved for θ_0

$$\theta_0 \approx \frac{\Delta}{\omega_0^2(1-J)-1}. \tag{2.18}$$

Since Δ and θ_0 are assumed positive, the following inequality has to be satisfied

$$\omega_0 > \sqrt{\frac{1}{1-J}} = \omega_{\min}, \tag{2.19}$$

thus confirming the observation of [4] with constraint $J < 1$ (i.e., $J_\xi < J_\zeta$). Actually, ω_0 determined by (2.16) satisfies this inequality (see Fig. 2), except for $\Delta = \theta_0 = 0$, and (2.19) can be regarded as a necessary condition for the existence of stationary solution.

3. Variational equations

Stability analysis relies on the variational equations—governing equations for perturbations. They will be given in the form of the system of first-order of ordinary differential equations (ODE's).

Before embarking on derivation of variational equations, let us consider some global aspects of the problem, i.e., the reduction of the system. Since precession angle ψ is cyclic (ignorable) coordinate, and implies existence of cyclic integral (2.13) (or (2.8)), the order of the system can be reduced by one. In a more formal way, the system is defined on $TSO(3)$ (or $T^*SO(3)$ in Hamiltonian formulation), and it is invariant with respect to S^1-action (rotation about vertical axis). This invariance permits reduction to $(TSO(3))/S^1$ (which is diffeomorphic to $\mathbb{R}^3 \times S^2$ in Hamiltonian formulation). The reduction in Hamiltonian formulation is straightforward and stationary motion (2.14) can be regarded as relative equilibrium of the reduced system in $\mathbb{R}^3 \times S^2$. As a consequence, the stationary motion is a particular solution of the governing equations (2.9)–(2.11) of the form

$$\begin{aligned} &\hat{\psi}'(t) = \omega_0; \\ &\hat{\theta}(t) = \theta_0; \qquad \hat{\theta}'(t) = 0; \\ &\hat{\varphi}(t) = 0; \qquad \hat{\varphi}'(t) = 0. \end{aligned} \tag{3.1}$$

To derive the variational equations, the perturbations of the stationary solution (3.1) have to be introduced

$$\begin{aligned} &\psi'(t) = \omega_0 + x_1; \\ &\theta(t) = \theta_0 + x_2; \qquad \theta'(t) = x_4 = x_2'; \\ &\varphi(t) = x_3; \qquad \varphi'(t) = x_5 = x_3'. \end{aligned} \tag{3.2}$$

By inserting (3.2) into (2.9)–(2.11), the following set of non-linear variational equations is obtained

$$\begin{aligned} x_1' &= \frac{1}{J}\{\sin\Delta \cot(\theta_0 + x_2)\sin x_3 \\ &\qquad + (1-2J)(\omega_0 + x_1)\cot(\theta_0 + x_2)x_4 + \csc(\theta_0 + x_2)x_4 x_5\}; \\ x_2' &= x_4; \\ x_3' &= x_5; \\ x_4' &= \frac{1}{J}\{\cos(\theta_0 + x_2)\cos x_3 \sin\Delta + \sin(\theta_0 + x_2)\cos\Delta \\ &\qquad - (1-J)\cos(\theta_0 + x_2)\sin(\theta_0 + x_2)(\omega_0 + x_1)^2 \\ &\qquad\qquad - \sin(\theta_0 + x_2)(\omega_0 + x_1)x_5\}; \\ x_5' &= \frac{\csc^2(\theta_0 + x_2)}{J}\{(\cos^2(\theta_0 + x_2) + J\sin^2(\theta_0 + x_2)) \\ &\qquad \times ((\omega_0 + x_1)x_4 - \sin\Delta \sin x_3)\sin(\theta_0 + x_2) \\ &\qquad - ((1-J)\sin(2(\theta_0 + x_2))(\omega_0 + x_1) \\ &\qquad\qquad + \sin(\theta_0 + x_2)x_5)\cos(\theta_0 + x_2)x_4\}. \end{aligned} \tag{3.3}$$

System (3.3) can be written in a compact form $\mathbf{x}' = \mathbf{F}(\mathbf{x})$, where $\mathbf{x} = (x_1, \dots, x_5)^T$ is the vector of perturbations. Stationary solution (3.1) corresponds to a trivial solution (stationary point) $\mathbf{x} = \mathbf{0}$ of (3.3).

Linearized variational equations are obtained by expanding r.h.s. of Eq. (3.3) in the neighborhood of the unperturbed state $x_i = 0$, $i = 1, \dots, 5$

$$\begin{aligned} x_1' &= \frac{\cot\theta_0}{J}(x_3 \sin\Delta + (1-2J)\omega_0 x_4); \\ x_2' &= x_4; \end{aligned}$$

$$
\begin{aligned}
x_3' &= x_5; \\
x_4' &= \frac{1}{J}\{-(1-J)\omega_0 x_1 \sin 2\theta_0 \\
&\qquad + (\cos(\Delta+\theta_0) - (1-J)\omega_0^2 \cos 2\theta_0)x_2 - \omega_0 x_5 \sin\theta_0\}; \qquad (3.4) \\
x_5' &= \frac{1}{J}\Big\{ -\frac{\sin\Delta}{\sin\theta_0}(\cos^2\theta_0 + J\sin^2\theta_0)x_3 \\
&\qquad + \frac{\omega_0}{\sin\theta_0}(-(1-2J)\cos^2\theta_0 + J\sin^2\theta_0)x_4\Big\}.
\end{aligned}
$$

The system (3.4) can be written in a compact form $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = D\mathbf{F}(\mathbf{0})$, i.e., the matrix of coefficients of the r.h.s. of variational equations (3.4) which will be omitted for the sake of brevity.

By introducing variational equations, stability analysis of stationary motion (3.1) becomes the stability analysis of trivial solution $\mathbf{x} = \mathbf{0}$ of variational equations (3.3), or (3.4). In the light of introductory remarks about reduction, it may be noticed that variational equations form a reduced system with respect to governing equations (2.9)–(2.11). To that end, the stability of stationary motion (3.1) can be interpreted as the stability of relative equilibrium in the reduced space $\mathbb{R}^3 \times S^2$.

4. Linear stability analysis

Linear stability analysis is based upon analysis of the eigenvalues of coefficient matrix $\mathbf{A}$, see Bakša and Vesković [10], Merkin [11] or Khalil [12]. If all the eigenvalues have negative real parts, unperturbed solution is asymptotically stable. If there is at least one eigenvalue with positive real part, unperturbed solution is unstable. Finally, if there are some eigenvalues with zero real part, while real parts of the other ones are negative, unperturbed solution is said to be marginally (neutrally) stable. Conclusion about stability in the first two cases does not depend on higher order terms in variational equations. However, in the case of marginal stability, linear stability analysis could not provide a conclusive answer: higher order terms could make the unperturbed solution to retain (neutral) stability, to become asymptotically stable or even to be unstable.

These limitations of linear stability analysis come on their own in the study of the synchronous motion of hovering magnetic top. Namely, characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ is of the fifth degree, but can be reduced to the following special form

$$\lambda(a\lambda^4 + b\lambda^3 + c\lambda^2 + d\lambda + e) = 0, \qquad (4.1)$$

where

$$
\begin{aligned}
a &= 2J^3; \\
c &= J\{J(-2\cos\Delta\cos\theta_0) + (2 + J - J\cos 2\theta_0)\csc\theta_0\sin\Delta \\
&\qquad + 2(J^2 + (1 - 3J + 2J^2)\cos^2\theta_0)\omega_0^2\}; \qquad (4.2) \\
e &= -\tfrac{1}{2}J\csc\theta_0\sin\Delta\{2(1 + J + (1-J)\cos 2\theta_0)\cos(\Delta+\theta_0) \\
&\qquad - (1-J)(2(1+J)\cos 2\theta_0 - (1-J)(-3 + \cos 4\theta_0))\omega_0^2\}, \qquad (4.3)
\end{aligned}
$$

while $b = d = 0$. Obviously, one eigenvalue is $\lambda = 0$ and marginal stability is the best one can expect from linear stability analysis.

In order to recover the results of Flanders et al. [4], it will be assumed that Δ and θ_0 are small. Consequently, due to (2.18) they are not independent, and the following relation holds

$$\frac{\Delta}{\theta_0} \approx \omega_0^2(1-J) - 1. \tag{4.4}$$

Introducing this relation into (4.2), approximate values of the coefficients (independent of Δ and θ_0) are obtained

$$\begin{aligned} a &= 2J^3; \\ c &= 2J(-2J + (1-2J+2J^2)\omega_0^2); \\ e &= 2J(1-(1-J)\omega_0^2)^2, \end{aligned} \tag{4.5}$$

which simplify the linear stability bounds.

THEOREM 4.1. *Under the smallness assumption* (4.4), *and* $0 < J < 1$, *synchronous motion* (2.14) *is linearly marginally stable if stationary precessional angular velocity* ω_0 *satisfy the necessary condition for existence* (2.19)

$$\omega_0 > \sqrt{\frac{1}{1-J}} = \omega_{\min}. \tag{4.6}$$

PROOF. Since one eigenvalue is $\lambda = 0$, we have to determine conditions for non-positivity of the real parts of the remaining ones. Non-zero eigenvalues are obtained as solutions of bi-quadratic equation $a\lambda^4 + c\lambda^2 + e = 0$. Solutions λ^2 ought to be real and negative in order to satisfy conditions of marginal stability. Since

$$\lambda^2 = \frac{1}{2a}\big(-c \pm \sqrt{c^2 - 4ae}\big),$$

and $a > 0$, coefficients of the characteristic equation have to satisfy the following conditions

(a) $c > 0$;
(b) $c^2 - 4ae > 0$;
(c) $-c \pm \sqrt{c^2 - 4ae} < 0 \Rightarrow 4ae > 0$.

Inequalities (a) and (b) impose the following lower bounds for stationary angular velocity

$$\omega_0 > \omega_1 = \left(\frac{2J}{1-2J+2J^2}\right)^{1/2}; \quad \omega_0 > \omega_2 = 2\sqrt{J}, \tag{4.7}$$

while the inequality (c) is satisfied for any ω_0. Taking into account the value of $\omega_{\min}$ given by (4.6), it is a matter of simple analysis to prove the following inequalities

$$\left(\frac{\omega_1}{\omega_{\min}}\right)^2 = \frac{1}{1-2J+2J^2} - 1 \leqslant 1; \quad \left(\frac{\omega_2}{\omega_{\min}}\right)^2 = 4J(1-J) \leqslant 1;$$
$$\left(\frac{\omega_1}{\omega_2}\right)^2 = \frac{1}{2(1-2J+2J^2)} \leqslant 1,$$

where equalities hold for $J = 1/2$. In conjunction with (4.6) they imply

$$\omega_0 > \omega_{\min} \geqslant \omega_2 \geqslant \omega_1, \tag{4.8}$$

which proves the Theorem.
Inequalities (4.8) are graphically illustrated in Fig. 3. □

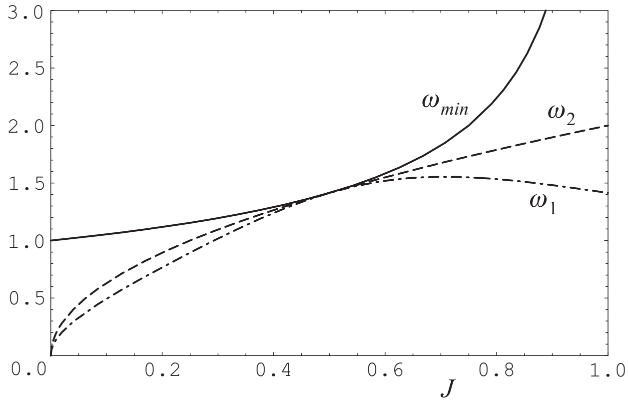

FIGURE 3. Stability bounds for angular velocity by linear theory.

Theorem 4.1 states that stationary solution ω_0, determined by (2.16), is marginally stable in the sense of linear stability analysis if it exists, i.e., if it satisfies the necessary condition (4.6). Unfortunately, this conclusion cannot be regarded as a complete proof of stability of synchronous stationary motion for the reasons stated above. However, for the purpose of comparison, inequality $(4.7)_2$ could be written in dimensional form

$$\frac{mH}{\omega_0^2 J_\xi} < \frac{1}{4}\Big(\frac{J_\zeta}{J_\xi}\Big)^2,$$

which was given in [4] as condition for existence of real roots. On the other hand, inequality $(4.7)_1$ have the following dimensional form

$$\frac{mH}{\omega_0^2 J_\xi} < \frac{1}{2}\Big(\frac{J_\zeta}{J_\xi}\Big)^2 - \frac{J_\zeta}{J_\xi} + 1,$$

which is a condition for existence of negative roots in [4].

5. Non-linear stability analysis by Lyapunov method

The problem encountered in this study is typical for systems in which some kind of energy conservation exists. As already mentioned, this model of hovering magnetic top has quadratic first integral (2.12), which is actually the (generalized) energy integral. It usually suffices for the construction of Lyapunov function and application of the direct method. Nevertheless, in some situations second order expansion of the energy integral, in the neighborhood of unperturbed solution, also contains linear terms, and thus does not fulfill the condition of definiteness. This is just the case in the present problem.

To resolve the problems of this kind Chetayev proposed the procedure for the construction of Lyapunov function using a combination of first integrals [10, 11], if there exist more then one. The main problem consists in finding suitable combination of first integrals which is definite in the neighborhood of unperturbed solution. As a consequence, non-linear stability can be proved since the derivative of Lyapunov function, constructed in this way, is identically zero. This procedure is especially promising when the system possesses linear first integral, apart from quadratic one. In that sense, Chetayev's procedure is quite similar to the energy-momentum method (see [13, Chapter 5]).

In our problem, cyclic integral (2.13) can be adjoined to Jacobi one (2.12) to construct the Lyapunov function in the following way

$$V = \hat{E} - \hat{E}_0 + \kappa(\hat{C} - \hat{C}_0), \tag{5.1}$$

where $\kappa \in \mathbb{R}$ is constant to be determined, and $\hat{E}_0$ and $\hat{C}_0$ denote first integrals evaluated on unperturbed stationary solution (2.14). Introducing perturbations (3.2) into Lyapunov function (5.1), one obtains

$$\begin{aligned} V(\mathbf{x}) = {} & \tfrac{1}{2}J(x_4^2 + (\omega_0 + x_1)^2 \sin^2(\theta_0 + x_2)) + \tfrac{1}{2}(x_5 + (\omega_0 + x_1)\cos(\theta_0 + x_2))^2 \\ & + (\cos\Delta\cos(\theta_0 + x_2) - \sin\Delta\sin(\theta_0 + x_2)\cos x_3) \\ & - \tfrac{1}{2}J\omega_0^2\sin^2\theta_0 - \tfrac{1}{2}\omega_0^2\cos^2\theta_0 - (\cos\Delta\cos\theta_0 - \sin\Delta\sin\theta_0) \\ & + \kappa\{(J\sin^2(\theta_0 + x_2) + \cos^2(\theta_0 + x_2))(\omega_0 + x_1) \\ & \quad + x_5\cos(\theta_0 + x_2) - (J\sin^2\theta_0 + \cos^2\theta_0)\omega_0\}. \end{aligned} \tag{5.2}$$

By expanding (5.2) in Taylor series up to second order in the neighborhood of $x_i = 0$, $i = 1, \dots, 5$, it is found out that linear terms disappear, either identically, or by means of relation (2.15), when $\kappa = -\omega_0$. Thus, the following quadratic form is obtained

$$\begin{aligned} V(\mathbf{x}) = {} & \tfrac{1}{2}\{(J\sin^2\theta_0 + \cos^2\theta_0)x_1^2 + 2\cos\theta_0 x_1 x_5 + x_5^2\} \\ & + \tfrac{1}{2}((1 - J)\omega_0^2\cos 2\theta_0 - \cos(\Delta + \theta_0))x_2^2 \\ & + \tfrac{1}{2}\sin\Delta\sin\theta_0 x_3^2 + \tfrac{1}{2}Jx_4^2 + O(\|\mathbf{x}\|^3). \end{aligned} \tag{5.3}$$

where $\|\mathbf{x}\|$ is the Euclidean norm of perturbation vector. If this quadratic form is positive definite, there will exist a neighborhood of unperturbed solution $x_i = 0$ in which Lyapunov function (5.2) is positive definite. Consequently, according to Lyapunov theorem, unperturbed solution will be stable with respect to small perturbations of initial conditions, since the derivative of (5.2) with respect to variational equations (3.3) is zero.

Theorem 5.1. *Stationary synchronous motion* (2.14) *of the magnetic top is non-linearly stable with respect to small perturbations of initial consditions, if the following conditions hold*

$$\Delta > 0; \quad \theta_0 > 0; \tag{5.4}$$

$$\Delta > -\theta_0 + \arctan\left(\tfrac{1}{2}\tan 2\theta_0\right) = \Delta_L. \tag{5.5}$$

Proof. To prove the non-linear stability, we have to determine the conditions of positive-definiteness of the quadratic form (5.3). This will be checked in several steps. First, note that (5.3) is not a complete quadratic form—only the perturbations x_1 and x_5 are coupled, whereas the definiteness of the remaining combination of pure squares depend on the sign of their coefficients only. Hence, it is obvious that coefficient of x_4^2 is positive since $0 < J < 1$. Since $\Delta \geqslant 0$ by assumption, positivity of the coefficient of x_3^2 leads to $\Delta > 0$ and $\theta_0 > 0$, thus proving (5.4).

Definiteness of quadratic form of x_1 and x_5 can be analyzed using Sylvester criterion, which says that matrix (quadratic form) is positive definite if and only if all the determinants associated with upper-left submatrices are positive. In our case the matrix reads

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{15} \\ c_{51} & c_{55} \end{pmatrix} = \begin{pmatrix} J\sin^2\theta_0 + \cos^2\theta_0 & \cos\theta_0 \\ \cos\theta_0 & 1 \end{pmatrix}$$

and Sylvester's criterion is reduced to the following inequalities

$$J\sin^2\theta_0 + \cos^2\theta_0 > 0; \quad J\sin^2\theta_0 > 0, \tag{5.6}$$

that are satisfied due to $(5.4)_2$.

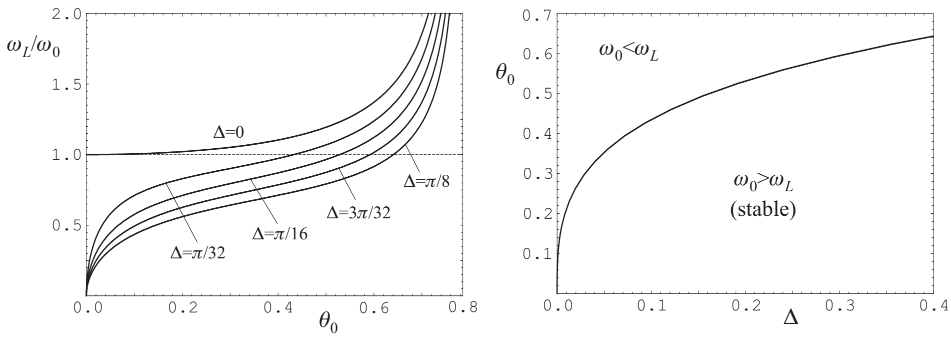

Figure 4. Stability bounds for angular velocity by Lyapunov method and critical curve in (Δ, θ_0) plane.

Finally, for the definiteness of (5.3), the coefficient of x_2^2 also has to be positive. This leads to the following constraint for angular velocity

$$\omega_0 > \left(\frac{\cos(\Delta+\theta_0)}{(1-J)\cos 2\theta_0}\right)^{1/2} = \omega_L. \tag{5.7}$$

The value of ω_L has to be compared with ω_0 determined by Eq. (2.16). This provides a restriction on the values of Δ and θ_0 for which the stationary solution (2.14) is stable

$$\frac{\sin(\Delta+\theta_0)}{\sin\theta_0\cos\theta_0} > \frac{\cos(\Delta+\theta_0)}{\cos 2\theta_0}, \tag{5.8}$$

which leads to stability condition (5.5) and concludes the proof. □

Remark 5.1. The inequality $\theta_0 > 0$, that is the part of stability conditions, means that the axis of the top has to be tilted with respect magnetic field **H**. It is in accordance with the existence of stationary angular velocity from Eq. (2.16).

Remark 5.2. It is clear that for $\Delta = 0$ the conclusion about stability cannot be drawn, since the definiteness is lost (see the coefficient of x_3^2). This justifies an assumption that the moment **m** of magnetic dipole deviates from the symmetry axis $O\zeta$ of the top.

The conclusions of Theorem 5.1 are illustrated in Fig. 4. Curves ω_L/ω_0 are drawn for different values of Δ. It can be observed that, given Δ, inequality (5.7), i.e., $\omega_L/\omega_0 < 1$, is satisfied only for certain range of values of θ_0. Consequently, inequality (5.5) determines the critical curve $\Delta_L(\theta_0)$ in (Δ, θ_0) plane, which bounds the stability region, at least for small values of Δ and θ_0.

The results of the non-linear stability analysis, by means of Lyapunov direct method, are new to the best author's knowledge. So far, the linear stability analysis was the main tool for determination of stability bounds. The results drawn from non-linear stability analysis shed new light on the problem and should be compared with the known results of linear stability.

6. Linear versus non-linear stability

In the final part of the study, new stability results obtained by Lyapunov method will be compared with the results of linear stability analysis, given in Section 4. At first sight, it seems that they are incomparable. Stability bound (5.5), derived from Eq. (5.7), does not depend on J—the ratio of moments of inertia. On the other hand, inequalities (4.7) give rise to a conclusion (4.8) which is independent of nutation angle θ_0 and deviation angle Δ. The origin of this contradiction lies in the fact that linear stability results are approximate—they were derived with assumption that θ_0 and Δ are small and (2.18) holds. This led to a characteristic equation with coefficients (4.5) independent of θ_0 and Δ.

Refined picture can be obtained if the approximation (2.18) is dropped. Then, complete expressions for the coefficients (4.2) have to be used, but they have to obey the same inequalities (a), (b), and (c), derived in Section 4. They will be carefully examined in the sequel and compared with main non-linear stability result, Eq. (5.5).

Consider the characteristic equation (4.1), whose non-zero solutions satisfy

$$a\lambda^4 + c\lambda^2 + e = 0, \tag{6.1}$$

with coefficients determined by Eq. (4.2). They have to satisfy stability conditions, i.e., the inequalities used in the proof of Theorem 4.1. Inequality (a), $c > 0$, leads to the following stability condition

$$\omega_0^2 > \frac{2J\cos\Delta\cos\theta_0 - (2 + J - J\cos 2\theta_0)\csc\theta_0\sin\Delta}{2(J^2 + (1 - 3J + 2J^2)\cos^2\theta_0)}.$$

By inserting (2.16) into last inequality, after some straightforward calculation one obtains the following stability bound for deviation angle Δ

$$\Delta > -\arctan\Big(\frac{2(J^2+(1-4J+3J^2)\cos^2\theta_0)\tan\theta_0}{3-4J+3J^2+(1-4J+3J^2)\cos 2\theta_0}\Big). \tag{6.2}$$

Although this bound contains inertia factor J, apart from nutation angle θ_0, it can be shown that it is negative for its entire range of values $(0 < J < 1)$ and for small values of θ_0. Therefore, this inequality practically does not impose any stability bound whatsoever.

Inequality (b), $c^2 - 4ae > 0$, leads to the following stability condition

$$k_0\omega_0^4 + k_2\omega_0^2 + k_4 > 0. \tag{6.3}$$

It can be shown that discriminant $D = k_2^2 - 4k_0k_4 < 0$, with $k_0 > 0$, for the whole range $0 < J < 1$ and small values of angles θ_0 and Δ. Consequently, stability condition (6.3) is unconditionally satisfied.

Finally, inequality (c), $4ae > 0$, which is reduced to $e > 0$ since $a = 2J^3 > 0$, gives the following stability condition

$$\omega_0^2 > \frac{2(1+J+(1-J)\cos 2\theta_0)\cos(\Delta+\theta_0)}{(1-J)(2(1+J)\cos 2\theta_0 - (1-J)(-3+\cos 4\theta_0))}.$$

By inserting (2.16) into last inequality, another stability bound for deviation angle Δ is obtained

$$\Delta > -\theta_0 + \arctan\Big(\frac{2(1+J+(1-J)\cos 2\theta_0)\cos\theta_0\sin\theta_0}{2(1+J)\cos 2\theta_0 - (1-J)(-3+\cos 4\theta_0)}\Big) = \Delta_{\text{lin}}. \tag{6.4}$$

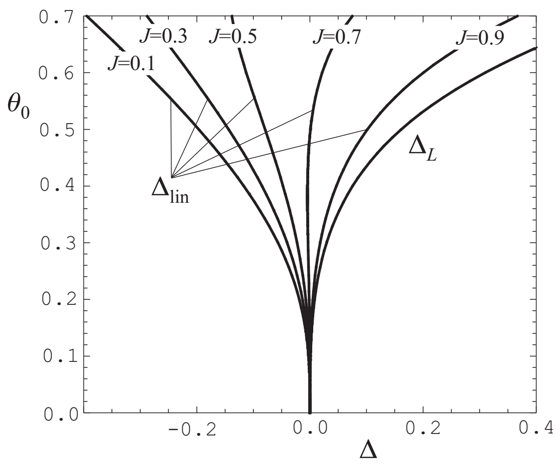

FIGURE 5. Critical curves by linear theory (Δ_{lin}) versus critical curve by Lyapunov method Δ_L.

Inequality (6.4) is the only one which imposes genuine restrictions on the values of Δ and θ_0 using linear stability theory. It can be shown that inequality (6.4) predicts larger region of stability in (Δ, θ_0) plane for each value of J, $0 < J < 1$, than the one obtained by Lyapunov method (5.5). However, simple calculation shows that

$$\Delta > \Delta_L(\theta_0) \geqslant \Delta_{\text{lin}}(\theta_0, J), \tag{6.5}$$

where Δ_L, given by (5.5), is the stability bound obtained by Lyapunov method, and $\Delta_L(0) = \Delta_{\text{lin}}(0, J)$. Moreover, pointwise convergence of linear stability bound $\Delta_{\text{lin}}(\theta_0, J)$ to non-linear (Lyapunov) one can be easily proved by the straightforward calculation

$$\lim_{J\to 1} |\Delta_{\text{lin}}(\theta_0, J) - \Delta_L(\theta_0)| = 0. \tag{6.6}$$

This result is supported by the graphs given in Fig. 5.

Conclusions which may be drawn from this comparative study of linear and non-linear stability are the following:

(1) inequalities (a) and (b) from linear stability analysis, do not impose any restrictions to the values of deviation angle Δ and nutation angle θ_0;
(2) inequality (c) from linear stability analysis leads to the stability bound (6.4); it is less restrictive than the non-linear bound (5.5) for each $J \in (0, 1)$ (see inequality (6.5)); however, it tends to (5.5) when $J \to 1$

Although Lyapunov method provides more restrictive stability bound than the linear theory, it is superior to the latter one since linear theory yields only marginal stability in this case. Also, it is a universal one since does not depend on J.

7. Conclusions

In this study we analyzed the stability of synchronous stationary motion of the magnetic top. It was motivated by the fact that existing stability results were based upon conditions of marginal stability of linear stability analysis. The problem was treated in dimensionless form and principal results of previous studies, given by Eq. (4.7), were recovered and generalized. Main contribution of this study is the non-linear stability analysis of the problem by means of Lyapunov method. The new result, inequality (5.5), came out from this analysis, determining the admissible values of the deviation angle Δ and nutation angle θ_0 for which the stationary synchronous motion is stable. This result was compared with linear stability analysis without small angle approximation (2.18). It was shown that Lyapunov method provides universal stability criterion, established by inequality (6.5), independent of inertial properties of the top.

The results mentioned above provide a solid basis for further studies. First, the lack non-linear stability results motivated the application of Lyapunov method. However, there is possibility to apply the Lyapunov-Malkin theorem which avoids the introduction of the Lyapunov function. Second, the motion of magnetic top is analyzed under rather restrictive assumption that the center of mass is at rest. Taking into account its motion and analyzing the stability problem in its completeness is the problem for possible future studies. Finally, all the present studies were concerned with stability analysis. Mechanisms of stability loss remained out of their scope. Apart from purely mechanical dissipation, there are other more subtle influences which could change magnetic properties of the top. Consequently, stability bounds could be altered and stationary motion could become unstable in the course of time. This may also be the fruitful field for prospective studies.

Acknowledgements. The preparation of this paper was supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia, within the project "Mechanics of nonlinear and dissipative systems—contemporary models, analysis and applications" (Project No. ON174016). Part of this study was worked out by Sanja Djaković and Milena Savić within their graduation theses.

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О СТАБИЛНОСТИ ЛЕВИТРОНА - ПОНОВО

Резиме. У овом раду се разматрају неки проблеми стабилности стационарног кретања лебдеће намагнетисане чигре у хомогеном магнетном пољу. Стабилност синхроног кретања је анализирана коришћењем упрошћеног модела у ком је игнорисано лебдеће кретање центра маса. Границе стабилности су одређене применом директног метода Љапунова. Конкретно, показано је да, за дати угао Δ између момента магнетног дипола и осе симетрије чигре, постоји интервал стационарних вредности угла нутације θ_0 за које је стационарно синхроно кретање стабилно.

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(Received 07.11.2017.)
(Revised 01.12.2017.)
(Available online 07.12.2017.)