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COMPRESSIBLE FLOW THROUGH SOLAR CHIMNEYS WITH VARIABLE CROSS SECTION - AN EXACT SOLUTION

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Abstract. Buoyancy driven, adiabatic and compressible flow in relatively high solar chimneys is treated in the paper analytically by using one-dimensional model of flow. General equations written suitably in a non-dimensional form are used for a qualitative discussion pertaining to the mutual effects of gravity, viscosity and varying cross section of the chimney. It is shown that in case of low Mach number flow these equations possess exact solutions obtainable by ordinary mathematical methods for any given chimney shape. Also shown, and demonstrated on an example, is the procedure of evaluation of the chimney shape that satisfies a condition imposed beforehand upon the flow. For better insight into the role of various parameters the solutions are presented in the form of power series expansions.

1. Introduction

It is well known that there are several ways to extract energy from the Sun. One of them, to which much attention has been paid in literature recently, is the production of energy in Solar chimney power plants (SCPP). A SCPP consists of three main parts: collector, turbine(s) and chimney. Collector has a transparent roof so that the ambient air is heated by solar radiation and flows through the plant toward the chimney. The flow is driven by buoyancy and is strongly affected by the chimney height. On its way out of the plant air passes through one or several turbines and transfers its energy to them. The first plant of this type was a pilot one and built in Manzanares, Spain, in 1982. Its operational characteristics described in detail in Haaf et al. [4] and Haaf [3] were quite encouraging, so that nowadays new SCPP projects are being proposed in several countries, in Australia, Spain, Namibia, China, USA. They all belong to so-called large scale projects, which means their typical dimensions are: chimney diameter and height 160 m and 1500 m, respectively, the collector diameter measured in kilometers, and the expected power of 200 MW. For more details concerned with SCPP technology,

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with all benefits, but also with some drawbacks, see excellent reviewing papers by Zhou and coworkers [9].

The air flow in SCCP belongs to the class of low Mach number flow. In a large scale SCPP the flow is exposed to relatively large temperature and pressure variations, so that compressibility of the air has to be taken into account. Reliable calculation of the flow parameters in all constitutive parts of SCPP is crucially important for its proper functionality. It is no wonder then that a large number of papers has been published in related scientific literature recently. The most important of them are cited in the aforementioned reviewing paper [9]. Among them we will here mention just a few, directly related to our subject matter. In [7] and [5] compressible flow in vertical tall chimneys of variable cross-section is treated by one-dimensional (1-D) flow model. At that, the influence of wall friction, loss coefficients and possible drag of obstructions upon flow characteristics is also included into the analysis. Characteristics of the flow are expressed in terms of Mach number, and particular attention has been paid to the evaluation of pressure drop in the chimney and the effect of variable cross-section ("flaring" effect). In [1] a comprehensive study of 1-D flow in all constitutive parts of a SCPP is presented, i.e., at the inlet into the collector, collector itself, turbines, collector-to-chimney transition section and the chimney with constant cross-section. This complex problem is solved numerically. The obtained results enable reliable evaluation of flow characteristics in the SCPP, and, based on that, enable the reliable estimation of some global SCPP characteristics, like mass flow rate, maximum power extracted by turbines, economically acceptable chimney heights, etc.

In this paper we confine ourselves to the flow in the chimney of variable cross-section only. The flow is compressible, adiabatic and viscous, and driven by buoyancy. As in [1] we use Mach number and Froude number as nondimensional dependent variables instead of velocity and temperature. We show that the system of nonlinear differential equations of the first order, that govern this flow, possesses exact solutions which can be found by ordinary mathematical operations for an arbitrary shape of the chimney. What is more, if additional condition is imposed on the chimney flow, the chimney shape that satisfies this condition can be readily found. We demonstrate that on the example of a chimney in which velocity is equal in all of its cross-sections.

A brief remark on the role of analytically exact solutions! Analytically exact solutions, if exists, used to be the most important source of information about a problem, if not the only one. With the advancement of computer technology, however, it must be admitted that they have lost much of their importance. But still, exact solutions serve as a very useful tool to check the accuracy of several approximate methods used, like asymptotic, numerical and empirical ones, and even of experimental work, Also, we think that the role of exact solutions in revealing the influence of the parameters that govern a problem is simply invaluable.

2. Problem statement and governing equations

We treat the problem depicted in Fig. 1. Compressible adiabatic gas flow takes place up a vertical, tall chimney with variable cross between stations 1-1 and 2-2 placed on the heights H_1 and H_2, respectively, above the ground. Diameter of an arbitrary cross-section is denoted by $D(z)$, z being the chimney axis pointing upwards. The flow is exposed to the effects of gravity, viscosity and chimney cross-section variations.

Equations describing such a flow within 1-D flow theory in this, or in a little different form, can be found in several textbooks, see for example [8, 10] and [2]. Written in differential form they read:

- equation of continuity

$$\frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}v}{v} + \frac{\mathrm{d}A}{A} = 0 \tag{2.1}$$

- momentum equation

$$\frac{\mathrm{d}p}{\rho} + v\,\mathrm{d}v + g\mathrm{d}z + \frac{\tau_w \mathcal{O}\,\mathrm{d}z}{\rho A} = 0 \tag{2.2}$$

- equation of state for an ideal gas

$$\frac{\mathrm{d}p}{p} - \frac{\mathrm{d}\rho}{\rho} - \frac{\mathrm{d}T}{T} = 0. \tag{2.3}$$

- energy equation

$$\underbrace{c_p \frac{\mathrm{d}T}{T} - R\frac{\mathrm{d}p}{p}}_{\mathrm{d}s} = \frac{\tau_w \mathcal{O}\,\mathrm{d}z}{\rho A T} \tag{2.4}$$

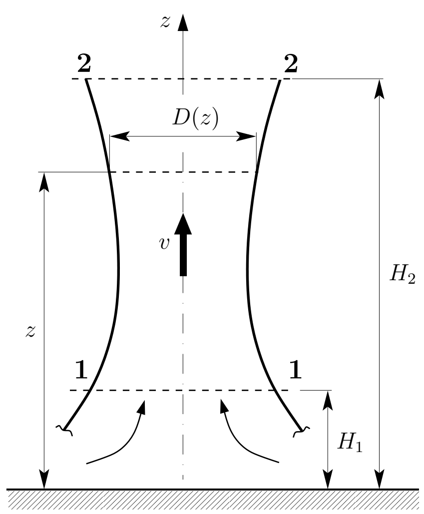

FIGURE 1. Physical model for one-dimensional flow

In here, ρ, v, p and T are basic flow quantities: density, velocity, pressure and temperature respectively, $A = \pi D^2/4$-cross-sectional area, $\mathcal{O} = \pi D$-circumference of an arbitrary cross-section, g-acceleration due to gravity, c_p-specific heat at constant pressure, R-gas constant, s-entropy and $\tau_w = f\frac{1}{2}\rho v^2$-wall shear stress ($f$-friction coefficient). As well known, friction coefficient depends on the Reynolds number and on the relative roughness of the wall. At relatively high Reynolds numbers f asymptoticly tends to a constant value. In this case of flow Reynolds numbers attain as a rule very high values, of the order of 10^8 [6], so we will consider f as a constant in what follows.

For convenience we will introduce the Froude number: $F = v/\sqrt{gD}$ and the Mach number: $M = v/c$, where $c = \sqrt{\gamma p/\rho} = \sqrt{\gamma RT}$ is the speed of sound, into equations (2.2) and (2.4). We get respectively:

$$\frac{\mathrm{d}p}{p} + \gamma M^2\frac{\mathrm{d}v}{v} + M^2\Big(1 + \frac{2}{\lambda F^2}\Big)\frac{\gamma\lambda}{2D}\mathrm{d}z = 0 \tag{2.2'}$$

$$c_p\frac{\mathrm{d}T}{T} - R\frac{\mathrm{d}p}{p} = RM^2\frac{\gamma\lambda}{2D}\mathrm{d}z, \tag{2.4'}$$

where $\lambda = 4f$. In order to get some qualitative estimations concerned with the behavior of flow in the chimney, we will treat equations (2.1), (2.2'), (2.3) and (2.4') as a system of algebraic equations for the "unknown" logarithmic derivatives of basic physical quantities. Solving for them, we get:

$$(1 - M^2)\frac{\mathrm{d}v}{v} = M^2\Big(1 + \frac{2}{\gamma\lambda F^2}\Big)\frac{\gamma\lambda}{2D}\mathrm{d}z - \frac{\mathrm{d}A}{A} \tag{2.5}$$

$$(1 - M^2)\frac{\mathrm{d}\rho}{\rho} = -M^2\Big(1 + \frac{2}{\gamma\lambda F^2}\Big)\frac{\gamma\lambda}{2D}\mathrm{d}z + M^2\frac{\mathrm{d}A}{A} \tag{2.6}$$

$$(1 - M^2)\frac{\mathrm{d}p}{p} = -M^2\Big[1 + (\gamma-1)M^2 + \frac{2}{\lambda F^2}\Big]\frac{\gamma\lambda}{2D}\mathrm{d}z + \gamma M^2\frac{\mathrm{d}A}{A} \tag{2.7}$$

$$(1 - M^2)\frac{\mathrm{d}T}{T} = -(\gamma-1)M^2\Big(M^2 + \frac{2}{\gamma\lambda F^2}\Big)\frac{\gamma\lambda}{2D}\mathrm{d}z + (\gamma-1)M^2\frac{\mathrm{d}A}{A}. \tag{2.8}$$

We also write down the same type of relations for some derived physical quantities, like M, F, total temperature T_0, total pressure p_0 and entropy s. They are deduced by taking logarithmic derivatives of their definitions, like

$$\frac{\mathrm{d}F}{F} = \frac{\mathrm{d}v}{v} - \frac{1}{2}\frac{\mathrm{d}(D)}{D}, \qquad \frac{\mathrm{d}M}{M} = \frac{\mathrm{d}v}{v} - \frac{1}{2}\frac{\mathrm{d}T}{T}, \text{ etc.}$$

and using equations (2.5)–(2.8). Taking into the account that $\frac{\mathrm{d}A}{A} = 2\frac{\mathrm{d}(D)}{D}$, we obtain:

$$(1 - M^2)\frac{\mathrm{d}M}{M} = M^2\Big(1 + \frac{\gamma-1}{2}M^2 + \frac{\gamma+1}{\gamma\lambda F^2}\Big)\frac{\gamma\lambda}{2D}\mathrm{d}z - \Big(1 + \frac{\gamma-1}{2}M^2\Big)\frac{\mathrm{d}A}{A} \tag{2.9}$$

$$(1 - M^2)\frac{\mathrm{d}F}{F} = M^2\Big(1 + \frac{2}{\lambda\gamma F^2}\Big)\frac{\gamma\lambda}{2D}\mathrm{d}z - \frac{5 - M^2}{4}\frac{\mathrm{d}A}{A} \tag{2.10}$$

$$\frac{\mathrm{d}T_0}{T_0} = -(\gamma-1)\frac{M^2}{1 + \frac{\gamma-1}{2}M^2}\frac{2}{\lambda\gamma F^2}\frac{\gamma\lambda}{2D}\mathrm{d}z \tag{2.11}$$

$$\frac{\mathrm{d}p_0}{p_0} = -M^2\left(1 + \frac{1}{1+\frac{\gamma-1}{2}M^2}\frac{2}{\lambda F^2}\right)\frac{\gamma\lambda}{2D}\mathrm{d}z \tag{2.12}$$

$$\mathrm{d}s = RM^2\frac{\gamma\lambda}{2D}\mathrm{d}z \tag{2.13}$$

Equations (2.5)–(2.13) serve for a qualitative analysis of the flow in the chimney. Obviously, effects of friction and gravity are superimposed in (2.5)–(2.10). If the cross-section of the chimney contracts in the direction of the flow, the effect of the contraction is also superimposed to these two, with the exception of Froude number behavior for $M^2 > 5$! Thus, in subsonic flow in contracting chimney v, M and F increase in the flow direction, while p, ρ and T decrease. If the cross-section of the chimney extends in the flow direction, the effect of this extension is to partly or fully cancel the effects of friction and gravity. This property of the equations (2.5)–(2.10) can be very usefully exploited in practice, as will be shown later in the paper. Supersonic flow in solar chimney is of no practical interest so far, and will not be discussed further.

Independently of whether $M < 1$ or $M > 1$, T_0 and p_0 decrease in the flow direction, T_0 being not affected by the friction, while s increases. These variations of T_0, p_0 and s are quite in accordance with our general knowledge of gas dynamics phenomena [2, 10].

3. Low Mach number flow

The flow in a solar chimney is characterized by relatively low velocities and high temperatures, so that Mach number attains the values which are less than one in all cross-sections of the chimney (for typical numerical values of various physical quantities in the chimney, see [6]). Thus, we will now neglect all terms proportional to M^2 with respect to 1 in the derived equations. For convenience, we will also introduce a nondimensional independent variable Z as:

$$Z = \frac{z - H_1}{\Delta H}, \quad \Delta H = H_2 - H_1; \qquad 0 \leqslant Z \leqslant 1.$$

Equations (2.9) and (2.10) become:

$$\frac{\mathrm{d}(M^2)}{\mathrm{d}Z} = (M^2)^2\left(1 + \frac{\gamma+1}{\gamma\lambda F^2}\right)\gamma\lambda\frac{\Delta H}{D} - \frac{2M^2}{A}\frac{\mathrm{d}A}{\mathrm{d}Z}, \tag{3.1}$$

$$\frac{\mathrm{d}(F^2)}{\mathrm{d}Z} = F^2M^2\left(1 + \frac{2}{\gamma\lambda F^2}\right)\gamma\lambda\frac{\Delta H}{D} - \frac{5F^2}{2A}\frac{\mathrm{d}A}{\mathrm{d}Z} \tag{3.2}$$

and will serve as our new basic equations for the analysis to follow. For given chimney shape $A(Z)$ and chimney height ΔH, equations (3.1) and (3.2) are to be solved for prescribed, "initial" values of M and F in the cross-section 1-1:

$$Z = 0: \quad M = M_1 \quad \text{and} \quad F = F_1. \tag{3.3}$$

One more physical boundary condition has to be satisfied, that the pressure at the exit cross-section of the chimney 2-2 equals the ambient pressure p_{amb} at the height H_2 above the ground:

$$Z = 1: \quad P = P_{\mathrm{amb}}, \tag{3.4}$$

where P stands for the nondimensional pressure p/p_a (p_a-atmospheric pressure). Assuming conditions of the Standard atmosphere, ambient pressure reads:

$$P_{\text{amb}} = \left(1 - \frac{kH_2}{T_a}\right)^{\frac{g}{kR}},$$

where $k = 6.5\,\text{K/km}$ is the temperature lapse rate in Standard atmosphere and T_a is atmospheric temperature. Since in large SCPP $H_1 \ll H_2$, we will for convenience write ambient pressure as:

$$P_{\text{amb}} \approx \left(1 - \frac{k\Delta H}{T_a}\right)^{\frac{g}{kR}}. \tag{3.5}$$

We show further that, in order to satisfy the exit boundary condition (3.5), it is not necessary to solve differential equation (2.7). Instead, we may use the expression for the mass flow rate: $\dot{m} = \rho v A$. By the help of the equation of state for an ideal gas, it can be transformed into a nodimensional form:

$$\dot{M} = \tilde{D}^{3/2}\frac{PM^2}{F}, \tag{3.6}$$

where $\tilde{D} = D/D_1$ is nondimensional chimney diameter and $\dot{M} = \frac{\dot{m}\sqrt{gD_1}}{\gamma p_a D_1^2 \pi/4}$ is nondimensional mass flow rate. Application of the condition (3.4) now enables the evaluation of $\dot{M}$:

$$\dot{M} = \left(\frac{D_2}{D_1}\right)^{3/2}\frac{P_{\text{amb}}M_2^2}{F_2}, \tag{3.7}$$

where M_2 and F_2 are Mach and Froude number, respectively, at the exit cross-section of the chimney, which are evaluated by solving equations (3.1) and (3.2). Since $\dot{M}$ is constant according to continuity equation, pressure can now be found in every cross-section of the chimney. Obviously, the solution of equations (3.1) and (3.2), with boundary conditions (3.3), is the keystone of the problem under consideration.

4. Exact solutions

We show in Appendix how the first integral (A.5) to the problem defined by (A.1)–(A.3) is found. Returning to physical denotations: $x = M^2$, $y = F^2$, etc., we write

$$\frac{F^2}{M^2} = \frac{F_1^2}{M_1^2}\frac{D_1}{D} - (\gamma - 1)\frac{\Delta H}{D}Z. \tag{4.1}$$

We further show that, by using definitions of Froude and Mach number, equation (4.1) reduces to very simple form in which variable diameter of the chimney is eliminated:

$$c_p T_1 + gH_1 = c_p T + gz. \tag{4.2}$$

One can easily identify equation (4.2) as the low Mach number version of the more general energy equation in algebraic form, stated for the station 1-1 and a arbitrary station on the height z above the ground

$$c_pT_1 + \frac{v_1^2}{2} + gH_1 = c_pT + \frac{v^2}{2} + gz. \tag{4.3}$$

It is noteworthy that (4.2) and (4.3) hold for the flow affected by viscosity also, although viscosity is not explicitly present in these equations! This can be explained in the following manner. In addition to terms already present in (4.2) and (4.3), these equations should contain just one more term in our case-the work per unit mass of the viscous forces. However, this work is zero because of the no-slip condition that has to be fulfilled at the wall. Thus, (4.2) is an exact solution of the problem considered herein, revealing the (adiabatic) temperature lapse rate in the chimney to be: $g/c_p = 9.77\,\mathrm{K/km}$, which is notably larger than in the surrounded Standard atmosphere.

One more integral of the governing equations for $D = D_1 =$ const. is derived in Appendix, equation (A.6). Written in physical denotations it reads:

$$\frac{F^2}{M^2} = \frac{F_1^2}{M_1^2}\left(\frac{1+\frac{2}{\gamma\lambda F^2}}{1+\frac{2}{\gamma\lambda F_1^2}}\right)^{\frac{\gamma-1}{2}}. \tag{4.4}$$

Equations (4.1) and (4.4) complete the solution of this problem.

For a chimney of arbitrary shape, an analytic solution can be also found. It is determined by (A.8), where:

$$\begin{aligned} a &= \frac{\gamma\lambda\Delta H}{D(Z)}, & m &= \Big(C_1 - \frac{\gamma-1}{\gamma\lambda}Z\Big)a(Z),\\ C_1 &= \frac{F_1^2}{M_1^2}\frac{D_1}{\gamma\lambda\Delta H}, & C_3 &= \frac{a_1^4}{M_1^2}\Big(a_1\frac{F_1^2}{M_1^2}\Big)^{\frac{\gamma+1}{\gamma-1}}, \end{aligned} \tag{4.5}$$

and:

$$M^2 = \frac{a^4}{u(Z)}, \qquad F^2 = \frac{a^5}{u(Z)}\Big(C_1 - \frac{\gamma-1}{\gamma\lambda}Z\Big). \tag{4.6}$$

It can be shown that this solution of the general problem reduces in the special case $D = D_1 =$ const. to the previously derived solution described by (4.1) and (4.4), as expected.

As noticed in the Introduction, we may impose a special requirement upon the flow in the chimney and ask for the chimney shape that satisfies this requirement, i.e., to treat the equations (2.5)–(2.10) as if $\mathrm{d}A/A$ was an unknown variable. To demonstrate this possibility we will ask for the shape of chimney in which the velocity is equal in all of its cross-sections: $v = v_1 =$ const. Since velocity increases in the chimney of constant cross-section, as well known, keeping the velocity constant will lessen the exit velocity, and thus will reduce the exit loss and improve the efficiency of SCPP.

For $\mathrm{d}v = 0$, we get from (2.5):

$$\frac{\mathrm{d}A}{A} = M^2\Big(1+\frac{2}{\gamma\lambda F^2}\Big)\frac{\gamma\lambda}{2D}\,\mathrm{d}z. \tag{4.7}$$

In here:

$$M^2 = \frac{v^2}{\gamma RT} = \frac{v_1^2}{\gamma RT_1}\frac{T_1}{T} = \frac{M_1^2}{\tilde{T}} \qquad \text{and} \qquad F^2 = \frac{v_1^2}{gD} = \frac{v_1^2}{gD_1}\frac{D_1}{D} = \frac{F_1^2}{\tilde{D}}$$

where: $\frac{T}{T_1} = \tilde{T}$ and $\frac{D}{D_1} = \tilde{D}$. Having in mind that: $\frac{\mathrm{d}A}{A} = 2\frac{\mathrm{d}\tilde{D}}{\tilde{D}}$ and $z = H_1 + \Delta HZ$, equation (4.7) becomes:

$$\tilde{T}\frac{\mathrm{d}(\tilde{D})}{\mathrm{d}Z} = M_1^2\Big(1 + \frac{2\tilde{D}}{\gamma\lambda F_1^2}\Big)\frac{\gamma\lambda}{4}\frac{\Delta H}{D_1}. \tag{4.8}$$

Nodimensional temperature $\tilde{T}$ in an arbitrary cross-section of the chimney is determined by equation (4.2) to be:

$$\tilde{T} = 1 - \frac{g\Delta H}{c_pT_1}Z.$$

Thus, equation (4.8) is a linear equation with varying coefficients that determines the shape of the chimney. The solution satisfying boundary condition: $\tilde{D}(0) = 1$ reads:

$$\tilde{D} = \Big(1 + \frac{\gamma\lambda F_1^2}{2}\Big)\Big(1 - \frac{g\Delta H}{c_pT_1}Z\Big)^{-\frac{1}{2(\gamma-1)}} - \frac{\gamma\lambda F_1^2}{2}. \tag{4.9}$$

For the test case stated in [6] ($\Delta H = 1500\,\mathrm{m}$, $D_1 = 160\,\mathrm{m}$, $v_1 = 12.82\,\mathrm{m/s}$, $T_1 = 323\,\mathrm{K}$, $\lambda = 4f = 0.003384$) we get from here: $D_2 = 166.272\,\mathrm{m}$. If the friction is neglected, the exit diameter becomes slightly lower: $D = 166.256\,\mathrm{m}$.

Since $\frac{g\Delta H}{c_pT_1}Z < 1$, equation (4.9) can be expanded into binomial series:

$$\tilde{D} = 1 + \frac{1}{2}\Big(1 + \frac{\gamma\lambda F_1^2}{2}\Big)\frac{g\Delta H}{\gamma RT_1}Z + \frac{2\gamma - 1}{2}\Big(1 + \frac{\gamma\lambda F_1^2}{2}\Big)\Big(\frac{g\Delta H}{\gamma RT_1}\Big)^2 Z^2 + \text{h. o. t.}, \tag{4.10}$$

where h. o. t. stands for higher order terms. Clearly, diameter of the chimney should be flared in order to keep velocity constant as clearly deduced in [6].

Note the appearance of a new similarity parameter in (4.10): $g\Delta H/\gamma RT_1$. It is the ratio of the squares of the half of the initial velocity of a vertical projectile necessary to reach the height ΔH above the ground, moving without friction, and the speed of sound at the station 1-1.

5. Series expansions

In order to present the obtained results in a more suitable form and to get more insight into the role played by different parameters, we will now expand the solutions into binomial series, like the one shown by (4.10). We will do this first for a chimney of constant cross-section: $D = D_1 = \text{const}$. For this case, the first integral (4.1), taking into account the definitions of M_1 and F_1, can be written as:

$$\frac{(F/F_1)^2}{(M/M_1)^2} = 1 - \frac{g\Delta H}{c_pT_1} \tag{5.1}$$

If this is inserted into (4.4), a single equation for F is obtained, and F routinely into the following series expanded:

$$\frac{F^2}{F_1^2} = 1 + (2 + \gamma\lambda F_1^2)\frac{g\Delta H}{\gamma R T_1} Z + \mathrm{h.\,o.\,t.} \tag{5.2}$$

The corresponding expansion for M is obtained from equation (5.1):

$$\frac{M^2}{M_1^2} = 1 + (\gamma + 1 + \gamma\lambda F_1^2)\frac{g\Delta H}{\gamma R T_1} Z + \mathrm{h.\,o.\,t.} \tag{5.3}$$

For $Z = 1$ the values for the Mach number M_2 and the Froude number F_2 at the exit cross-section are obtained from (5.1) and (5.2), and then the nondimensional mass flow rate can be obtained from (3.7) for $D_2 = D_1$. For that purpose the ambient pressure (3.5) has to be expanded into a series also. We get:

$$P_{\mathrm{amb}} = 1 - \gamma\frac{T_1}{T_a}\frac{g\Delta H}{\gamma R T_1} + \mathrm{h.\,o.\,t.} \tag{5.4}$$

Then:

$$\dot{M} = \frac{M_1^2}{F_1}\Big[1 - \gamma\Big(\frac{T_1}{T_a} - 1 - \frac{\lambda}{2}F_1^2\Big)\frac{g\Delta H}{\gamma R T_1} + \mathrm{h.\,o.\,t.}\Big]. \tag{5.5}$$

Since in this case: $\dot{M} = \frac{P_1 M_1^2}{F_1}$, in accordance with the continuity equation, the expansion for pressure P_1 is easily deduced from equation (5.5). Then, we can also evaluate the nondimensional pressure drop in the chimney to be:

$$P_1 - P_{\mathrm{amb}} = \gamma\Big(1 + \frac{\lambda}{2}F_1^2\Big)\frac{g\Delta H}{\gamma R T_1} + \mathrm{h.\,o.\,t.} \tag{5.6}$$

In dimensional form it looks like:

$$\frac{p_1}{p_a}(p_1 - p_{\mathrm{amb}}) = \rho_1 g\Delta H + \lambda\frac{\Delta H}{D_1}\frac{1}{2}\rho_1^2 v_1^2 + \mathrm{h.\,o.\,t.}$$

For the already mentioned test case we get from here: $P_1 - P_{\mathrm{amb}} = 0.10596$. Thus, the pressure drop in the chimney is 10.596 % of the atmospheric pressure. With neglected viscosity is slightly lower: 10.577 %.

For a chimney of variable cross-section the series representation of the solution is derived in Appendix. At that, and due to simplicity, it is developed by solving equation (A.7) by means of power series, and not by using the exact analytic solution (A.8). Such a procedure suffices for our purposes. The solution represented by (A.10), if converted into physical denotations: $x = M^2$, $y = F^2$, etc., reads:

$$\frac{F^2}{F_1^2} = 1 + \Big[(2 + \gamma\lambda F_1^2)\frac{g\Delta H}{\gamma R T_1} - 5\dot{\tilde{D}}_0\Big] Z + \mathrm{h.\,o.\,t.} \tag{5.7}$$

$$\frac{M^2}{M_1^2} = 1 + \Big[(\gamma + 1 + \gamma\lambda F_1^2)\frac{g\Delta H}{\gamma R T_1} - 4\dot{\tilde{D}}_0\Big] Z + \mathrm{h.\,o.\,t.} \tag{5.8}$$

Both, (5.7) and (5.8) represent slight, but reasonable generalizations of (5.2) and (5.3), respectively, to the solution for the flow in a chimney of constant cross-section. If we have in mind the special case of a chimney in which velocities in every cross-section are equal, elaborated at the end of Section 4, we can find $\dot{\tilde{D}}_0$

for this chimney by putting $F^2 = F_1^2/\tilde{D}$ in (5.7), or $M^2 = M_1^2/\tilde{T}$ in (5.8). In both cases we get:

$$\dot{\tilde{D}}_0 = \frac{1}{2}\Big(1 + \frac{\gamma\lambda F_1^2}{2}\Big)\frac{g\Delta H}{\gamma R T_1}.$$

As expected, this is exactly the coefficient in front of Z in equation (4.10).

Finally, nondimensional mass flow rate is obtained by using equation (3.6) as:

$$\begin{aligned}\dot{M} &= \Big(\frac{D_2}{D_1}\Big)^{3/2}\frac{P_{\text{amb}}M_2^2}{F_2} = \\ &= \Big(\frac{D_2}{D_1}\Big)^{3/2}\frac{M_1^2}{F_1}\Big[1 - \gamma\Big(\frac{T_1}{T_a} - 1 - \frac{\lambda}{2}F_1^2\Big)\frac{g\Delta H}{\gamma R T_1} - \frac{3}{2}\dot{\tilde{D}}_0\Big] + \text{h. o. t.}\end{aligned}$$

Expression for the nondimensional pressure P_1 emerges from here as $P_1 = \frac{F_1}{M_1^2}\dot{M}$, so that the pressure drop in the chimney can be also found. At that, if it is supposed that diameter of the chimney varies slowly with height on the scale of $g\Delta H/\gamma R T_1$, as infered from (4.10), it can be readily shown that the same expression as for the chimney of constant cross-section (5.6) is obtained. Of course this conclusion holds for the first approximation only.

6. Conclusions

Briefly, we may draw the following conclusions:

- 1-D model of flow can be successfully applied for studying free convecting, compressible, viscous flow in relatively high solar chimneys with variable cross-section.
- What is more, it is shown that for low Mach number flow in the chimney governing equations posses analytically exact solutions which can be found by conventional mathematical methods for any given chimney shape.
- It is also shown that, if a specific condition is imposed upon the chimney flow, the shape of the chimney satisfying this condition can be found. This is demonstrated on the example of a chimney in which velocities are equal in all of its cross-sections.
- Finally, all solutions are expanded into binomial series and first terms of these series are evaluated. These simple analytic expressions, that offer very good approximation to the problem considered, yield very useful insight into the role of different parameters affecting the chimney flow.

Appendix A. Analytical solution of differential equations

1. We first write equations (3.1) and (3.2) with conditions (3.3) into a form which is more suitable for mathematical operations to follow. We make following substitutions: $M^2 = x$, $F^2 = y$, $Z = t$, $\frac{\gamma\lambda\Delta H}{D} = a$, $M_1^2 = x_1$, $F_1^2 = y_1$ and $\frac{\mathrm{d}()}{\mathrm{d}t} = \dot{()}$, and get:

$$\dot{x} = ax^2\Big(1 + \frac{\gamma + 1}{\gamma\lambda y}\Big) + 4x\frac{\dot{a}}{a} \tag{A.1}$$

$$\dot{y} = axy\Big(1 + \frac{2}{\gamma\lambda y}\Big) + 5y\frac{\dot{a}}{a} \tag{A.2}$$

$$t = 0: \quad x = x_1, \ \ y = y_1. \tag{A.3}$$

Dividing (A.1) and (A.2) with x and y respectively, and subtracting we obtain

$$\frac{\dot{y}}{y} - \frac{\dot{x}}{x} = -\frac{\gamma - 1}{\gamma\lambda} a \frac{x}{y} + \frac{\dot{a}}{a}.$$

Introducing $\frac{y}{x} = m$, the equation for $m(t)$ becomes a linear one, with non-constant coefficients:

$$\dot{m} - \frac{\dot{a}}{a} m = -\frac{\gamma - 1}{\gamma\lambda} a$$

General solution of this equation can be routinely found to be:

$$m = \Big(C_1 - \frac{\gamma - 1}{\gamma\lambda} t\Big) a, \tag{A.4}$$

where C_1 is a constant of integration. Conditions (A.3) give: $C_1 = \frac{y_1}{x_1}\frac{D_1}{\gamma\lambda\Delta H}$, so that finally the solution of the problem is:

$$\frac{y}{x} = \frac{y_1}{x_1}\frac{D_1}{D} - (\gamma - 1)\frac{\Delta H}{D} t \tag{A.5}$$

This is the first integral of the problem under consideration.

2. One more integral is needed in order to complete the solution of the problem. It can be particularly simply obtained in case of a chimney with constant cross-section: $D = D_1 =$ const. and $a = a_1 = \frac{\gamma\lambda\Delta H}{D_1} =$ const. Dividing (A.1) with (A.2) in this case, we get:

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{x\big(1 + \frac{\gamma+1}{\gamma\lambda y}\big)}{y\big(1 + \frac{2}{\gamma\lambda y}\big)}.$$

Variables are separated in this equation, so the solution is obtained by quadratures, to be:

$$x = C_2 y^{\frac{\gamma+1}{2}} \Big(y + \frac{2}{\gamma\lambda}\Big)^{-\frac{\gamma-1}{2}},$$

or, when C_2 is determined from (A.3):

$$\frac{x}{x_1} = \frac{y}{y_1}\left(\frac{1 + \frac{2}{\gamma\lambda y_1}}{1 + \frac{2}{\gamma\lambda y}}\right)^{\frac{\gamma-1}{2}}. \tag{A.6}$$

Integrals (A.5) and (A.6) fully determine the solution of the problem.

3. Analytical solution for a chimney with variable cross-section can also be obtained. If y is eliminated by $y = mx$ from (A.1), this equation becomes a single nonlinear equation with non-constant coefficients for $x(t)$:

$$\dot{x} = ax^2 + \frac{\gamma + 1}{\gamma\lambda}\frac{a}{m} x + 4\frac{\dot{a}}{a} x,$$

where a/m depends on t only and is determined by (A.4). Substitution $x = a^4/u(t)$ converts this equation into a linear, non-homogeneous one:

$$\dot{u} + \frac{\gamma+1}{\gamma\lambda}\frac{a}{m}u = -a^5, \tag{A.7}$$

and the solution of this equation is routinely obtained to be:

$$u = \Big(\frac{m}{a}\Big)^{\frac{\gamma+1}{\gamma-1}}\Big[C_3 - \int_0^t a^5\Big(\frac{m}{a}\Big)^{-\frac{\gamma+1}{\gamma-1}}\mathrm{d}q\Big], \tag{A.8}$$

where C_3 is the constant of integration. Application of boundary condition:

$$t = 0: \quad u = \frac{a_1^4}{x_1}$$

yields:

$$C_3 = \frac{a_1^4}{x_1}\Big(\frac{x_1}{y_1}\Big)^{\frac{\gamma+1}{\gamma-1}} a_1^{\frac{\gamma+1}{\gamma-1}}.$$

Thus, for given shape of the chimney, $u(t)$ is determined by (A.8), and then

$$x = \frac{a^4}{u}, \quad y = mx = \frac{a^5}{u}\Big(C_1 - \frac{\gamma-1}{\gamma\lambda}t\Big). \tag{A.9}$$

This is the solution of the general problem.

For practical applications this solution might be a little to complicated. As an alternative we now derive a solution of (A.7)–(A.9) in a form of power series in t. For that purpose we first expand chimney diameter in such series:

$$\tilde{D} = 1 + \dot{\tilde{D}}_0 t + \frac{1}{2}\ddot{\tilde{D}}_0 t^2 + \text{h.o.t.}$$

where

$$\dot{\tilde{D}}_0 = \frac{\mathrm{d}\tilde{D}}{\mathrm{d}t}\Big|_{t=0}, \quad \ddot{\tilde{D}}_0 = \frac{\mathrm{d}^2\tilde{D}}{\mathrm{d}t^2}\Big|_{t=0}, \quad \text{etc.}$$

Then

$$a = \frac{\gamma\lambda\Delta H}{D} = \frac{a_1}{\tilde{D}} = a_1\big(1 - \dot{\tilde{D}}_0 t + \text{h.o.t.}\big), \quad a_1 = \frac{\gamma\lambda\Delta H}{D_1}, \quad \text{etc.}$$

We look for a solution of (A.7) in the form:

$$u = u_0 + u_1 t + u_2 t^2 + \text{h.o.t.},$$

insert this into (A.7) and get by applying the boundary condition and equating terms of like powers of t on both sides of the equation:

$$u_0 = \frac{a_1^4}{x_1}, \quad u_1 = -a_1^5\Big(1 + \frac{\gamma+1}{\gamma\lambda y_1}\Big), \quad \text{etc.}$$

Thus:

$$u = \frac{a_1^4}{x_1} - a_1^5\Big(1 + \frac{\gamma+1}{\gamma\lambda y_1}\Big)t + \text{h.o.t.}$$

The solution of the problem in the form of the series expansion is now obtained from (A.9) to be:

$$\text{(A.10)}\quad \begin{aligned} \frac{x}{x_1} &= 1 + \Big[a_1 x_1\Big(1 + \frac{\gamma+1}{\gamma\lambda y_1}\Big) - 4\dot{\tilde{D}}_0\Big]t + \text{h. o. t.} \\ \frac{y}{y_1} &= 1 + \Big[a_1 x_1\Big(1 + \frac{2}{\gamma\lambda y_1}\Big) - 5\dot{\tilde{D}}_0\Big]t + \text{h. o. t.} \end{aligned}$$

References

1. A. S. Ćoćić, V. D. Djordjević, *One-dimensional analysis of compressible flow in solar chimney power plants*, Solar Energy **135** (2016), 810–820.
2. V. D. Djordjević, *Dynamics of one-dimensional fluid flow*, 4th ed., University of Belgrade, Faculty of Mechanical Engineering, 2004.
3. W. Haaf, *Solar chimneys part II: Preliminary test results from Manzanares pilot plant*, International Journal of Solar Energy **2** (1984), 141–161.
4. W. Haaf, K. Friedrich, G. Mayr, J. Schlaich, *Solar chimneys part I: Principle and construction of the pilot plant in Manzanares*, International Journal of Solar Energy **2**(1) (1983), 3–20.
5. T. W. von Backström, *Calculations of pressure and density in solar power plant chimneys*, ASME Journal of solar energy engineering **125** (2003), 127–129.
6. T. W. von Backström, A. Bernhardt, A. J. Gannon, *Pressure drop in solar power plant chimneys*, ASME Journal of Solar Energy Engineering **125** (2003), 165–169.
7. T. W. von Backström, A. J. Gannon, *Compressible flow through solar power plant chimneys*, ASME Journal of solar energy engineering **122** (2000), 138–145.
8. F. M. White, *Fluid Mechanics*, 5th ed., McGraw Hill, 2003.
9. X. Zhou, F. Wang, M.R. Ochieng, *A review of solar chimney power techology*, Renewable and Sustainable Energy Reviews **14** (2010), 2315–2338.
10. M. J. Zucrow, J. D. Hoffman, *Gas Dynamics*, Willey, 1976.

СТИШЉИВО СТРУЈАЊЕ КРОЗ СОЛАРНИ ТОРАЊ ПРОМЕЊИВОГ ПОПРЕЧНОГ ПРЕСЕКА - ТАЧНО РЕШЕЊЕ

Резиме. У овом раду се разматра аналитичко решење за случај једнодимензијског узгонског, адијабатског и стишљивог струјања у релативно високим соларним торњевима. Основне једначине написане у погодном бездимензијском облику се користе за квалитативну дискусију ефеката гравитације, вискозности (трења) и промене попречног пресека торња. Показано је да у случају спорих струјања, са малим вредностима Маховог броја, једначине имају тачна решења. До њих се долази стандарним математичким поступцима, и то за било који облик торња. Такође је, на погодно изабраном примеру, представљена процедура за одређивање облика торња тако да одговарајући услов који се намеће као карактеристика струјања буде задовољен. Ради бољег увида у улогу одговарајућих параметара, решење је такође представљено и у виду степеног реда.

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