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A NOTE ON THE COMPLETE INSTABILITY OF LINEAR NON-CONSERVATIVE UNDAMPED SYSTEMS

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Dedicated to the memory of Professor Aleksandar Bakša

Abstract. The note is concerned with the problem of determining the completely unstable linear non-conservative undamped (circulatory) dynamical systems. Several conditions that provide the complete instability for such systems are derived using the direct method of Lyapunov and the concept of controllability. The conditions are expressed directly via the matrices describing the dynamical system.

1. Introduction

Non-conservative undamped linear systems (circulatory systems) with n degrees of freedom are mostly expressed in the form

$$\ddot{q} + Kq + Pq = 0, \quad q \in \mathbb{R}^n, \tag{1.1}$$

where dot denotes time differentiation and the real $n \times n$ matrices $K = K^T$ and $P = -P^T$ correspond to potential (conservative) and non-conservative positional (circulatory) forces, respectively (see [1,2]. The skew-symmetric matrix P is called the circulatory matrix and K is the stiffness matrix, so $q^T Kq/2$ is the potential energy of the system. Such systems are important mathematical models in various areas of mechanics, physics and engineering (see [3]).

As equation (1.1) is linear, "stability of the system" is determined by the stability of its equilibrium state $(q, \dot{q}) = (0, 0)$. For many years, it has been well known that circulatory forces $-Pq$ can destabilize a stable equilibrium of purely potential (conservative) system, and that they can stabilize an unstable potential system [2,4]. Various results concerning the stability problem for circulatory systems can be found in [2–12].

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The *degree of instability* of the system (1.1), denoted by u, is the number of roots (counted with multiplicities) of the characteristic polynomial

$$\Delta(\lambda) = \det(\lambda^2 I + K + P), \tag{1.2}$$

in the open right complex half-plane. Here I is the identity matrix. This definition is in accordance with the definition of the degree of instability for arbitrary dynamical system proposed by Kozlov [13], which is a natural generalization of the Poincare degree of instability for conservative systems (see [1,2]). It is clear that $0 \leqslant u \leqslant n$, since the roots of (1.2) are located symmetrically with respect to the imaginary axis in the complex plane ($\Delta(\lambda) = \Delta(-\lambda)$). If $u = n$ (i.e., the polynomial (1.2) has no roots on the imaginary axis), the system is said to be *completely unstable*. Clearly, in this case the matrix $(K + P)$ must be *non-singular* and, consequently, the equilibrium position $q = 0$ is necessarily isolated.

The purpose of this note is to obtain conditions that ensure complete instability of the systems under consideration. A useful criterion in this direction (Theorem 3.1) is derived in Section 3. The derivation is based on the inertia theory for Lyapunov matrix equation. This theory is presented briefly in Section 2. In Section 4, it is shown that Theorem 3.1 generates a number of the conditions for complete instability expressed directly through the matrices K and P.

2. A brief overview of the inertia theory for Lyapunov equation

The *inertia* of a $m \times m$ real matrix A, denoted by $\mathrm{In}(A)$, is defined as the triplet $(\pi(A), \nu(A), \delta(A))$, where $\pi(A)$, $\nu(A)$, and $\delta(A)$ are, respectively, the number of eigenvalues of A with positive, negative, and zero real parts, counting multiplicities. Note that $\pi(A) + \nu(A) + \delta(A) = m$.

First, we recall a result of Lyapunov [14]: *If for any positive definite symmetric matrix $W (W > 0)$ there is a negative definite symmetric matrix $V (V < 0)$ satisfying*

$$AV + VA^T = W, \tag{2.1}$$

then $\mathrm{In}(A) = (0, m, 0)$. The matrix equation (2.1) is called Lyapunov equation. This classical result is a special case of the inertia theorem of Ostrowski and Schneider [15]: *If for any $W = W^T > 0$ there is a symmetric matrix V satisfying* (2.1), *then* $\mathrm{In}(A) = \mathrm{In}(V)$ *and* $\delta(A) = \delta(V) = 0$.

In the more general case when the matrix W is positive semi-definite ($\geqslant 0$), the triplets $\mathrm{In}(A)$ and $\mathrm{In}(V)$ do not generally coincide with each other. A useful result for this case involves the concept of controllability of the matrix pair (A, W).

Let B be a $m \times p$ matrix. The *controllability matrix* $C(A \mid B)$ of A and B is defined as the $m \times mp$ matrix

$$C(A \mid B) = (B, AB, A^2B, \ldots, A^{m-1}B).$$

The pair (A, B) is called *controllable*, if $\operatorname{rank} C(A \mid B) = m$.

The following result proved by Chen [16] plays an important role in forthcoming considerations.

THEOREM 2.1. *Let V be a symmetric matrix. If the matrix W given by* (2.1) *has the property $W \geqslant 0$ and the pair (A, W) is controllable, then $\delta(A) = \delta(V) = 0$ and $\mathrm{In}(A) = \mathrm{In}(V)$.*

3. The main result

Equation (1.1) is equivalent to the first order equation

$$\dot{x} = Ax,$$

with $x \in \mathbb{R}^n\{q\} \times \mathbb{R}^n\{\dot{q}\}$ and

$$A = \begin{pmatrix} 0 & I \\ \hat{A} & 0 \end{pmatrix}, \tag{3.1}$$

where I is the identity matrix of order n and $\hat{A} = -(K + P)$. The eigenvalues of A are the roots of characteristic polynomial (1.2) and, consequently, $u = \pi(A)$.

Let

$$V = \begin{pmatrix} 0 & M \\ M^T & 0 \end{pmatrix}, \quad M \in \mathbb{R}^{n\times n}. \tag{3.2}$$

LEMMA 3.1. $\pi(V) = \nu(V) = \operatorname{rank} M$.

PROOF. Let λ be a non-zero eigenvalue of V. Then there exists a non-zero vector $v = \binom{v_1}{v_2}$, $v_i \in \mathbb{R}^n$, such that $Vv = \lambda v$, or $Mv_2 = \lambda v_1$ and $M^T v_1 = \lambda v_2$. The last two equations yield $(M^T M)v_2 = \lambda^2 v_2$, and the result follows from $M^T M \geqslant 0$ and $\operatorname{rank} M^T M = \operatorname{rank} M$. □

Note that the matrix V is non-singular if and only if M is non-singular, and then $\pi(V) = \nu(V) = n$.

Now we put (3.1) and (3.2) in (2.1). Then we get

$$W = AV + VA^T = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix}, \tag{3.3}$$

where $W_1 = M + M^T$ and $W_2 = \hat{A}M + M^T\hat{A}^T$.

LEMMA 3.2. *The pair (A, W) is controllable, if and only if*

$$\operatorname{rank}(\hat{A}C(\hat{A} \mid W_1), C(\hat{A} \mid W_2)) = n.$$

PROOF. Because of the given block structure of (3.1) and (3.3), it is not difficult to see that the controllability matrix $C(A \mid W)$ can be transformed by the application of elementary column and row operations to the form:

$$\begin{pmatrix} C(\hat{A} \mid W_1) & C(\hat{A} \mid W_2) & 0 & 0 \\ 0 & 0 & \hat{A}C(\hat{A} \mid W_1) & C(\hat{A} \mid W_2) \end{pmatrix}.$$

Obviously, the pair (A, W) is controllable (i.e., $\operatorname{rank} C(A \mid W) = 2n$) if and only if $\operatorname{rank}(C(\hat{A} \mid W_1), C(\hat{A} \mid W_2)) = n$ and $\operatorname{rank}(\hat{A}C(\hat{A} \mid W_1), C(\hat{A} \mid W_2)) = n$, since the reduction of a matrix by elementary operations does not change its rank. Now, according to the Cayley–Hamilton theorem (see [14]), the matrix $\hat{A}^n W_1$ can be represented by a linear combination of the matrices $W_1, \hat{A}W_1, \dots, \hat{A}^{n-1}W_1$, and consequently

$$\operatorname{rank}(C(\hat{A} \mid W_1), \hat{A}^n W_1, C(\hat{A} \mid W_2)) = \operatorname{rank}(C(\hat{A} \mid W_1), C(\hat{A} \mid W_2)).$$

Also, it is easy to see that

$$(C(\hat{A} \mid W_1)\hat{A}^n W_1, C(\hat{A} \mid W_2)) = (W_1, \hat{A}C(\hat{A} \mid W_1), C(\hat{A} \mid W_2)),$$

and hence

$$\operatorname{rank}(C(\hat{A} \mid W_1), C(\hat{A} \mid W_2) \geqslant \operatorname{rank} \hat{A}C(\hat{A} \mid W_1), C(\hat{A} \mid W_2)),$$

and the result readily follows. □

REMARK 3.1. If either $\operatorname{rank} \hat{A}C(\hat{A} \mid W_1) = n$ or $\operatorname{rank} C(\hat{A} \mid W_2) = n$, then $\operatorname{rank} C(A \mid W) = 2n$.

THEOREM 3.1. *Let $M \in \mathbb{R}^{n\times n}$ and let $\hat{A} = -(K + P)$. The system (1.1) is completely unstable if the following three conditions hold:*

(a) $W_1 = M + M^T \geqslant 0$,
(b) $W_2 = \hat{A}M + M^T \hat{A}^T \geqslant 0$,
(c) $\operatorname{rank}(\hat{A}C(\hat{A} \mid W_1), C(\hat{A} \mid W_2)) = n$.

PROOF. Suppose that the conditions (a) and (b), and (c) are fulfilled. Then, in view of Lemma 3.2, the matrices A, V and W given by (3.1), (3.2) and (3.3), respectively, satisfy the conditions of Theorem 2.1, and hence $\pi(A) = \pi(V)$, $\nu(A) = \nu(V)$, and $\delta(A) = \delta(V) = 0$. Finally, according to Lemma 3.1, $\pi(V) = \nu(V) = n$, because $\delta(V) = 0$. □

REMARK 3.2. Obviously, this theorem implicitly requires that the matrix M must be non-singular.

REMARK 3.3. The controllability condition (c) of Theorem 3.1 is redundant if $W_2 > 0$. It is also redundant if $\hat{A}$ is non-singular and $W_1 > 0$.

Theorem 3.1 has some interesting consequences, which we present in the following section.

4. Some completely unstable systems

Assuming $P = 0$ in equation (1.1) we get the conservative system: $\ddot{q} + Kq = 0$. It is well known, and easily verified, that this system is completely unstable if and only if $K < 0$. The following assertion shows that a completely unstable conservative system remains completely unstable after the introduction of arbitrary non-conservative positional forces.

PROPOSITION 4.1. *If $\det(K + P) \neq 0$ and $K \leqslant 0$, then the system (1.1) is completely unstable.*

In particular, if $K = 0$ (purely non-conservative system) and $\det P \neq 0$ (then n is necessarily even), the system (1.1) is completely unstable.

PROOF. Let M be the identity matrix. Then $W_1 = 2I$ and $W_2 = -2K \geqslant 0$, because $K \leqslant 0$, i.e., the conditions (a) and (b) of Theorem 3.1 are satisfied. Also, by virtue of Remark 3.3, the condition (c) of Theorem 3.1 is satisfied, and the proposition follows. □

EXAMPLE 4.1. Let

$$K = \begin{pmatrix} -3 & 2 & 2 \\ 2 & -3 & -3 \\ 2 & -3 & -3 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{pmatrix}.$$

For this system, $\det(K+P) \neq 0$ and K is negative semi-definite, and hence complete instability follows from Proposition 4.1. The direct way to verify the validity of this result is to form the characteristic equation and obtain the eigenvalues of the system. Doing so results in $\lambda_{1,2} = \pm 1.126$, $\lambda_{3,4} = \pm 1.732$, $\lambda_{5,6} = \pm 2.175$, which is in agreement with the prediction of Proposition 4.1.

Now assuming that $M = P$ we get $W_1 = M + M^T = 0$ and $W_2 = \hat{A}M + M^T\hat{A}^T = -2P^2 + PK - KP$. Then the following proposition is a direct consequence of Theorem 3.1.

PROPOSITION 4.2. *The system* (1.1) *is completely unstable if the conditions*

$$W_2 = -2P^2 + PK - KP \geqslant 0, \tag{4.1}$$

and

$$\operatorname{rank} C((K+P) \mid W_2) = n \tag{4.2}$$

are satisfied.

Obviously, the condition (4.2) is redundant when the matrix W_2 in (4.1) is positive definite. We note also that, in view of Remark 3.2, the conditions (4.1) and (4.2) imply $\det P \neq 0$, and, in particular, n is even and $P^2 < 0$, since P is a $n \times n$ skew-symmetric matrix.

EXAMPLE 4.2. Consider the two degrees of freedom system and, without loss of generality, we assume that

$$K = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \quad \text{and} \quad P = p\begin{pmatrix} 0 & 1 \\ -1, & 0 \end{pmatrix}, \quad p \neq 0. \tag{4.3}$$

For this system, we have

$$W_2 = -2P^2 + PK - KP = p\begin{pmatrix} 2p & k_2 - k_1 \\ k_2 - k_1 & 2p \end{pmatrix},$$

and the condition (4.1) requires $d = 4p^2 - (k_2 - k_1)^2 \geqslant 0$. If $d > 0$ (i. e., W_2 is positive definite), then $\operatorname{rank} C((K+P) \mid W_2) = 2$. However, in the case $d = 0$, it is not difficult to see that $\operatorname{rank} C((K+P) \mid W_2) = 1$. Thus, according to Proposition 4.2, the system (1.1), (4.3) is completely unstable if $2|p| > |k_2 - k_1|$, which coincides with the necessary and sufficient condition of flutter instability for this system [8, 12].

COROLLARY 4.1. *If $KP = PK$, the system* (1.1) *is completely unstable if and only if either* $\det P \neq 0$ *or the restriction of the potential energy $q^T Kq/2$ on the subspace* $\operatorname{Ker} P$ *is negative definite.*

PROOF. Let $\det P \neq 0$. Then $-P^2 = P^T P > 0$ and, in view of Proposition 4.2, the system is completely unstable. Now we suppose that $\operatorname{rank} P = r < n$ (clearly, r is necessarily even). Let Q denotes an orthogonal matrix whose the last $n-r$ columns are a basis of the subspace $\operatorname{Ker} P$. Then Q transforms P into block-diagonal form

$$Q^T P Q = \begin{pmatrix} \hat{P} & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.4}$$

where $\hat{P} \in \mathbb{R}^{r\times r}$ and $\det \hat{P} \neq 0$. From $KP = PK$ we get

$$Q^T K Q = \begin{pmatrix} \hat{K} & 0 \\ 0 & \tilde{K} \end{pmatrix}, \tag{4.5}$$

where $\hat{K} \in \mathbb{R}^{r\times r}$ and $\tilde{K} \in \mathbb{R}^{(n-r)\times(n-r)}$. Putting $q = Qx$, $x \in \mathbb{R}^r\{y\} \times \mathbb{R}^{n-r}\{z\}$, premultiplying by Q^T, and using (4.4) and (4.5), the equation (1.1) is immediately reduced to the two decoupled equations

$$\ddot{y} + \hat{K} y + \hat{P} y = 0, \quad y \in \mathbb{R}^r, \tag{4.6}$$

and

$$\ddot{z} + \tilde{K} z = 0, \quad z \in \mathbb{R}^{n-r}. \tag{4.7}$$

Then, in view of Proposition 4.2, the subsystem described by (4.6) is completely unstable, because $\hat{K}\hat{P} = \hat{P}\hat{K}$ and $\det \hat{P} \neq 0$. On the other hand, the conservative subsystem (4.7) is completely unstable if and only if $\tilde{K} < 0$, i.e., $z^T \tilde{K} z = q^T K q \,|_{\operatorname{Ker} P} < 0$. □

Preserving only the hypothesis $KP = PK$, according to the above proof, it is clear that the degree of instability is not less than the rank of P. In the case $K = kI$, $k \in \mathbb{R}$, this conclusion is consistent with a classical result of Merkin [2] (also, see [3, 10] and the example in [17]), which states that the introduction of arbitrary circulatory forces into a stable conservative system with equal frequencies destroys the stability.

The next proposition supplements a result given in [9]. Also, it shows that the introduction sufficiently large non-degenerate circulatory forces ($\det P \neq 0$) in a stable conservative system of even degree of freedom destroys stability and makes the system completely unstable.

PROPOSITION 4.3. *If $K \geqslant 0$ and $P^2 + K^2 < 0$, then the system (1.1) is completely unstable.*

PROOF. Let $M = K + P$. Then, under the hypotheses of this proposition, we have $W_1 = 2K \geqslant 0$ and $W_2 = -2P^2 - 2K^2 > 0$, and the conditions (a) and (b) of Theorem 3.1 are satisfied. Also, by Remark 3.3 the condition (c) of this theorem is satisfied. □

The above propositions provide simple sufficient conditions for the complete instability directly in terms of the system matrices K and P. In order to obtain weaker conditions of the same type, it seems that when choosing the matrix M

in Theorem 3.1 one or more undetermined scalar parameters must be introduced. This expectation is illustrated by the following assertion.

PROPOSITION 4.4. *The system* (1.1) *is completely unstable if there exists a non-negative number* α *such that*

$$PK - KP - 2P^2 - \alpha K > 0.$$

PROOF. Let $M = \frac{\alpha}{2}I + P$, $\alpha \geqslant 0$. Then $W_1 = \alpha I \geqslant 0$ and, consequently, the conditions of Theorem 3.1 are satisfied when $W_2 = PK - KP - 2P^2 - \alpha K > 0$. □

EXAMPLE 4.3. We return to Example 4.2 and we choose $k_1 = 1$, $k_2 = -6$ and $p = 3$. In this case the first three propositions tell us nothing, since $2|p| < |k_2 - k_1|$ and K is indefinite. However, the matrix

$$PK - KP - 2P^2 - \alpha K = \begin{pmatrix} 18 - \alpha & -21 \\ -21 & 18 + 6\alpha \end{pmatrix}$$

is positive definite for $1.44 < \alpha < 13.56$, and hence complete instability follows from Proposition 4.4.

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О ПОТПУНОЈ НЕСТАБИЛНОСТИ ЛИНЕАРНИХ НЕКОНЗЕРВАТИВНИХ НЕПРИГУШЕНИХ СИСТЕМА

Резиме. Разматра се проблем одређивања потпуно нестабилних линеарних неконзервативних (циркулаторних) динамичких система. Помоћу Љапуновљевог директног метода изведено је неколико услова који обезбеђују потпуну нестабилност разматраних система. Ови услови су изражени директно преко описних матрица система.

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