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THE ROUTH THEOREM FOR MECHANICAL SYSTEMS WITH UNKNOWN FIRST INTEGRALS

Alexander V. Karapetyan and Alexander S. Kuleshov

Abstract. In this paper we discuss problems of stability of stationary motions of conservative and dissipative mechanical systems with first integrals. General results are illustrated by the problem of motion of a rotationally symmetric rigid body on a perfectly rough plane.

Application of the Routh–Salvadori theorem and its generalizations [1–4] for investigation of stability of stationary motions of mechanical systems with first integrals $U_0 = c_0, U_1 = c_1, \dots, U_k = c_k$ is reduced to study the type of stationary value of U_0 (here U_0 can be also a nonincreasing along system trajectories function) for fixed values of $U_1, \dots, U_k$. The effective method of such investigation is proposed in [5]. This method does not take into account equations of motion of the considered system however it is supposed that all first integrals are known explicitly. On the other hand using results by I. M. Mindlin and G. K. Pozharitskii [6] it is possible to distinguish the systems [7] for which the stability analysis does not require the explicit form of all first integrals $U_1 = c_1, \dots, U_k = c_k$, except $U_0 = c_0$.

Let equations of motion of a mechanical system have the following form (here $\mathtt{T}$ means transposition):

$$\frac{d}{dt}\Big(\frac{\partial K}{\partial \dot{\boldsymbol{q}}}\Big) = \frac{\partial K}{\partial \boldsymbol{q}} + \boldsymbol{G}\dot{\boldsymbol{q}} - \frac{\partial W}{\partial \boldsymbol{q}} - \boldsymbol{\Gamma}^{\mathtt{T}}\frac{\partial W}{\partial \boldsymbol{p}}, \qquad \dot{\boldsymbol{p}} = \boldsymbol{\Gamma}\dot{\boldsymbol{q}}. \tag{1}$$

Here

$$\boldsymbol{q} = (q_1, \dots, q_m)^{\mathtt{T}}, \qquad \boldsymbol{p} = (p_1, \dots, p_k)^{\mathtt{T}}, \qquad W = W(\boldsymbol{q}, \boldsymbol{p}),$$
$$K = \frac{1}{2}\dot{\boldsymbol{q}}^{\mathtt{T}}\boldsymbol{A}(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \forall \dot{\boldsymbol{q}} \neq 0,$$
$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}(\boldsymbol{q}, \boldsymbol{p}), \qquad \boldsymbol{G} = \boldsymbol{G}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{p}), \qquad \boldsymbol{G}^{\mathtt{T}} = -\boldsymbol{G}.$$

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We will assume that the positive definite $(m \times m)$ matrix $\boldsymbol{A}(\boldsymbol{q})$ and the scalar function $W(\boldsymbol{q}, \boldsymbol{p})$ are two times differentiable functions of their arguments. Similarly, the $(m \times m)$ matrix $\boldsymbol{G}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{p})$ and $(k \times m)$ matrix $\boldsymbol{\Gamma}(\boldsymbol{q}, \boldsymbol{p})$ are (one time) differentiable functions of their arguments.

In particular, equations (1) can describe the motion of mechanical systems with quasicyclic coordinates (in this case $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ are position variables and velocities respectively and $\boldsymbol{p}$ are generalized impulses or quasivelocities of quasicyclic coordinates).

Obviously, equations (1) possess the generalized energy integral

$$U_0(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{p}) = K + W = c_0. \tag{2}$$

If the matrix $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}(\boldsymbol{q}, \boldsymbol{p})$ is zero-matrix, then equations (1) coincide with equations of conservative mechanical systems with cyclic coordinates described in Routh variables [1] and possess k cyclic integrals

$$U_1 = p_1 = c_1, \ldots, U_k = p_k = c_k. \tag{3}$$

In this case sets of stationary points of the energy integral (2) on fixed levels of first integrals (3) correspond to the stationary motions of the form

$$\boldsymbol{q} = \boldsymbol{q}^0, \quad \dot{\boldsymbol{q}} = 0, \quad \boldsymbol{p} = \boldsymbol{p}^0 \tag{4}$$

where $\boldsymbol{p}^0$ are arbitrary constants and $\boldsymbol{q}^0$ are constants determined from the following equations

$$\frac{\partial W}{\partial \boldsymbol{q}} = 0. \tag{5}$$

Stationary motions (4) form the family S_0 of dimension more or equal than the number of cyclic coordinates. For such $(\boldsymbol{\Gamma} = 0)$ systems the Routh theorem can be formulated as follows [1,2].

THEOREM 1. *If the function W (the amended potential) has a strict minimum at the point $(\boldsymbol{q}^0, \boldsymbol{p}^0)$ for the fixed values of integrals (3), then corresponding stationary motion (4) is stable.*

Note that conditions of Theorem 1 are trivially fulfilled if all the eigenvalues of the matrix $(\partial^2 W/\partial \boldsymbol{q}^2)$ are positive at $(\boldsymbol{q}^0, \boldsymbol{p}^0)$.

In the general case $(\boldsymbol{\Gamma} \neq 0)$ system (1) also has the stationary motions (4), but in this case $\boldsymbol{q}^0$ and $\boldsymbol{p}^0$ are determined from the equations

$$\frac{DW}{D\boldsymbol{q}} = 0 \quad \Big(\frac{D}{D\boldsymbol{q}} \overset{\text{def}}{=} \frac{\partial}{\partial \boldsymbol{q}} + \boldsymbol{\Gamma}^{\mathrm{T}} \frac{\partial}{\partial \boldsymbol{p}}\Big). \tag{6}$$

Generally speaking these equations do not coincide with equations (5). Obviously, as in the case $\boldsymbol{\Gamma} = 0$ stationary motions (4) form a family of dimension more or equal than the number of quasicyclic coordinates because for the determination of $k + m$ unknown constants $\boldsymbol{q}^0$ and $\boldsymbol{p}^0$ we have k equations (6). Again we will denote this family by S_0.

The following theorem provides the sufficient conditions for stability of these motions [7].

THEOREM 2. *If all the eigenvalues of the matrix*

$$\Big(\frac{D^2W}{D\boldsymbol{q}^2}\Big) \tag{7}$$

are positive at $(\boldsymbol{q}^0, \boldsymbol{p}^0)$ *and in some neighborhood of this point the conditions*

$$\frac{D\gamma_{\alpha i}}{Dq_j} = \frac{D\gamma_{\alpha j}}{Dq_i}, \quad (i,\, j = 1, \dots, m;\ \alpha = 1, \dots, k) \tag{8}$$

are fulfilled (here $\gamma_{\alpha i} = \gamma_{\alpha i}(\boldsymbol{q}, \boldsymbol{p})$ *– elements of the matrix* $\boldsymbol{\Gamma}, \alpha = 1, \dots, k,$ $i = 1, \dots, m$*) then stationary motion* (4) *of the system* (1) *is stable.*

PROOF. Note that under conditions (8) the system $k \times m$ partial differential equations

$$\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{q}} = \boldsymbol{\Gamma} \tag{9}$$

with respect to k unknown functions $\boldsymbol{p}(\boldsymbol{q})$ is completely integrable in some neighborhood O_δ of the point $(\boldsymbol{q}^0, \boldsymbol{p}^0)$ (here $\delta > 0$ is a small constant). Therefore in this neighborhood there is exists the family of solutions of the system (9) of the form

$$\boldsymbol{p} = \boldsymbol{F}(\boldsymbol{q}, \boldsymbol{c}), \tag{10}$$

depending on k arbitrary constants $\boldsymbol{c} = (c_1, \dots, c_k)^{\mathrm{T}}$. Equations (10) are solvable with respect to these constants. Therefore system (1), in addition to the energy integral, possesses k first integrals of the form

$$U_1(\boldsymbol{q}, \boldsymbol{p}) = c_1, \dots, U_k(\boldsymbol{q}, \boldsymbol{p}) = c_k \tag{11}$$

and according to the definition of these integrals we have

$$U_i(\boldsymbol{q}, \boldsymbol{F}(\boldsymbol{q}, \boldsymbol{c})) \equiv c_i, \quad i = 1, \dots, k.$$

Note that under condition (8) the matrix (7) is symmetric. If all the eigenvalues of this matrix are positive at the point $(\boldsymbol{q}^0, \boldsymbol{p}^0)$, then the function $W(\boldsymbol{q}, \boldsymbol{F}(\boldsymbol{q}, \boldsymbol{c}))$ takes a strict minimum at $\boldsymbol{q} = \boldsymbol{q}^0$, $\boldsymbol{c} \equiv \boldsymbol{c}^0 \equiv \boldsymbol{U}(\boldsymbol{q}^0, \boldsymbol{p}^0)$, $\boldsymbol{U} = (U_1, \dots U_k)$. Therefore function $W(\boldsymbol{q}, \boldsymbol{p})$ and integral (2) (since $K > 0\ \forall \dot{\boldsymbol{q}} \neq 0$) takes a strict minimum for the fixed values of integrals (11) on unperturbed motion and this motion is stable according to the Routh theorem [1, 2]. □

Obviously both in the case $\boldsymbol{\Gamma} = 0$ and in the case $\boldsymbol{\Gamma} \neq 0$ the following theorem is valid.

THEOREM 3. *If*

$$\Big(\frac{D^2W}{D\boldsymbol{q}^2}\Big) < 0$$

at the point $(\boldsymbol{q}^0, \boldsymbol{p}^0)$ *then stationary motion* (4) *is unstable.*

REMARK 1. Application of Theorem 2 for the investigation of stability of stationary motions (4) of the system (1) is connected with the investigation of eigenvalues of matrix (7) and requires the knowledge of function $W(\boldsymbol{q}, \boldsymbol{p})$ and matrix $\boldsymbol{\Gamma}(\boldsymbol{q}, \boldsymbol{p})$ only; the explicit form of first integrals (11) (and also matrices $\boldsymbol{A}(\boldsymbol{q})$ and

$\boldsymbol{G} = \boldsymbol{G}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{p})$) is not required. Note also that condition (6) for search the stationary values of integral (2) on fixed levels of integrals (11) does not require the knowledge of explicit form of these integrals. Moreover, equations (6) determine stationary motions (4) of system (1) even in the case, when these integrals are absent (i.e., when equations (8) are failed).

REMARK 2. Conditions (8) are trivially fulfilled if $\boldsymbol{\Gamma} = 0$ (in this case we can write explicit form (3) of integrals (11)) or when for $\boldsymbol{\Gamma} \neq 0$ we have $\dim \boldsymbol{q} = 1$ (in this case, generally speaking, it is not possible to write the explicit form of integrals (11)).

REMARK 3. For the fixed values of constants $\boldsymbol{p}^0$ both equations (5) and equations (6) (with respect to $\boldsymbol{q}$) can have not only one solution $\boldsymbol{q}^0$ but, generally speaking, other solutions $\boldsymbol{q}^1, \boldsymbol{q}^2, \ldots$. These solutions also depend on the constants $\boldsymbol{p}$, so the stationary motions $\boldsymbol{q}^1(\boldsymbol{p}), \boldsymbol{q}^2(\boldsymbol{p}), \ldots$ form the families $S_1, S_2, \ldots$. Thus the set S of all stationary motions of the systems represents the union of families $S_0, S_1, S_2, \ldots$.

If the considered mechanical system is subjected to (except the potential forces) dissipative forces $\boldsymbol{Q} = \boldsymbol{Q}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ corresponding to the position variables $\boldsymbol{q}$, then its equations of motion can be written as follows

$$\begin{gathered} \frac{d}{dt}\Big(\frac{\partial K}{\partial \dot{\boldsymbol{q}}}\Big) = \frac{\partial K}{\partial \boldsymbol{q}} + \boldsymbol{G}\dot{\boldsymbol{q}} - \frac{\partial W}{\partial \boldsymbol{q}} - \boldsymbol{\Gamma}^{\mathrm{T}} \frac{\partial W}{\partial \boldsymbol{p}} + \boldsymbol{Q}, \\ \dot{\boldsymbol{p}} = \boldsymbol{\Gamma}\dot{\boldsymbol{q}}. \end{gathered} \tag{12}$$

In this case according to the definition of dissipative forces we have

$$\frac{d}{dt}(K + W) = (\boldsymbol{Q} \cdot \dot{\boldsymbol{q}}) \leqslant 0; \quad \boldsymbol{Q}(\boldsymbol{q}, 0) \equiv 0$$

and, instead of the generalized energy integral (2), the system (12) possesses a nonincreasing along system trajectories function

$$U_0 \equiv K + W \leqslant c_0.$$

Obviously, Theorems 1 and 2 are again valid for the stationary motions (4) of the system (12) and $(\boldsymbol{q}^0, \boldsymbol{p}^0)$ are again satisfied to equations (5) (for $\boldsymbol{\Gamma} = 0$) or equations (6) (for $\boldsymbol{\Gamma} \neq 0$). Moreover, if

$$(\boldsymbol{Q} \cdot \dot{\boldsymbol{q}}) \neq 0, \quad \forall\, \dot{\boldsymbol{q}} \neq 0 \tag{13}$$

then the following theorems are valid.

THEOREM 4. *If for $\boldsymbol{\Gamma} = 0$ function W takes a local strict minimum at $(\boldsymbol{q}^0, \boldsymbol{p}^0)$ for fixed values $\boldsymbol{p}^0 = \boldsymbol{c}^0$ of integrals (3), this point is isolated from other stationary points of W (if these points exist at all), then the corresponding to these parameters $\boldsymbol{q}^0$, $\boldsymbol{p}^0$ stationary motion (4) of the system (12) is stable and under condition (13) every perturbed motion sufficiently close to the unperturbed tends asymptotically as $t \to +\infty$ to some stationary motion (4) of the family S_0; in particular, if the constants $\boldsymbol{p}^0 = \boldsymbol{c}^0$ of integrals (3) remain unperturbed, then the unperturbed motion is asymptotically stable.*

THEOREM 5. *If for* $\mathbf{\Gamma} = 0$ *function* W *takes a stationary value at* $(\boldsymbol{q}^0, \boldsymbol{p}^0)$*, that is not even nonstrict minimum for fixed values* $\boldsymbol{p}^0 = \boldsymbol{c}^0$ *of integrals* (3)*, this point is isolated from other stationary points of* W *(if these points exist at all), then under condition* (13) *the stationary motion corresponding to* $\boldsymbol{q} = \boldsymbol{q}^0$*,* $\boldsymbol{p} = \boldsymbol{p}^0$ *is unstable.*

Note that the first condition of Theorem 4 [Theorem 5] are trivially fulfilled if all the eigenvalues of the matrix $(\partial^2 W/\partial \boldsymbol{q}^2)$ are positive at $(\boldsymbol{q}^0, \boldsymbol{p}^0)$ [the matrix $(\partial^2 W/\partial \boldsymbol{q}^2)$ has negative eigenvalues at $(\boldsymbol{q}^0, \boldsymbol{p}^0)$].

THEOREM 6. *If all the eigenvalues of the matrix* (7) *are positive at* $(\boldsymbol{q}^0, \boldsymbol{p}^0)$ *and in some neighborhood of this point the conditions* (8) *are fulfilled, then the stationary motion* (4) *of the system* (12) *is stable and under condition* (13) *every perturbed motion sufficiently close to the unperturbed tends asymptotically as* $t \to +\infty$ *to some stationary motion* (4) *of the family* S_0*; in particular, if variables* $\boldsymbol{p}$ *remain unperturbed, then the unperturbed motion is asymptotically stable.*

THEOREM 7. *If the matrix* (7) *has a negative eigenvalues at* $(\boldsymbol{q}^0, \boldsymbol{p}^0)$ *and in some neighborhood of this point the conditions* (8) *are fulfilled, then under condition* (13) *the stationary motion* (4) *of the system* (12) *is unstable.*

Theorems 4–7 follow from the results discussed in [3, 4, 7]

EXAMPLE 1. Let us consider the problem of motion of a rotationally symmetric rigid body on a fixed perfectly rough horizontal plane. Suppose that the center of mass G of the body is situated on the symmetry axis $G\zeta$, and moments of inertia about principal axes of inertia $G\xi$ and $G\eta$ perpendicular to $G\zeta$ are equal to each other. The body moves in presence of the homogeneous gravity field. Denote by M the contact point of the body with the supporting plane.

Let $Oxyz$ be the fixed coordinate frame with the origin in the supporting plane Oxy and the Oz axis directed upwards. Denote the angle between the symmetry axis of the body and the vertical by θ, the angle between the meridian $M\zeta$ of the body and a certain fixed meridian plane by β and the angle between the horizontal tangent MQ of the meridian $M\zeta$ and the Ox axis by α. The position of the body is completely determined by the angles α, β and θ and the x and y coordinates of the point M.

Let us specify now the position of the coordinate system $G\xi\eta\zeta$. Suppose that the $G\xi$ axis is always situated in the plane of vertical meridian $M\zeta$ while the $G\eta$ axis is perpendicular to this plane (Figure 1). In this case the coordinate system $G\xi\eta\zeta$ moves both in the space and in the body. Denote the components of velocity $\mathbf{v}$ of the center of mass G in the coordinate system $G\xi\eta\zeta$ by v_ξ, v_η, v_ζ and the components of the angular velocity vector $\boldsymbol{\omega}$ of the body and the angular velocity $\mathbf{\Omega}$ of the trihedron $G\xi\eta\zeta$ by ω_ξ, ω_η, ω_ζ and Ω_ξ, Ω_η, Ω_ζ respectively. Then we have the following obvious equation for the component ω_η:

$$\omega_\eta = -\frac{d\theta}{dt} = -\dot{\theta}.$$

Let m be the mass of the body, A_1 be its moment of inertia about axes $G\xi$ and $G\eta$, and A_3 be its moment of inertia about the symmetry axis. Note that the

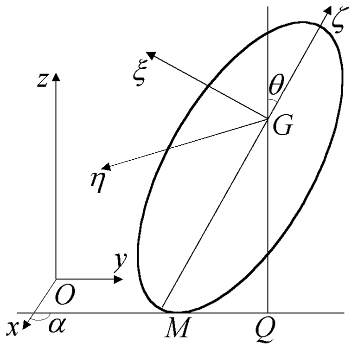

FIGURE 1.

distance GQ of the center of mass over the plane Oxy is a function of angle θ, i.e., $GQ = f(\theta)$ [8]. Denote by ξ, η, ζ the coordinates of the point of contact M of the body with the supporting plane in the coordinate system $G\xi\eta\zeta$. Then $\eta = 0$ and

$$\xi = -f(\theta)\sin\theta - f'(\theta)\cos\theta, \quad \zeta = -f(\theta)\cos\theta + f'(\theta)\sin\theta, \tag{14}$$

where $(\,)'$ is a derivative of function $f(\theta)$ with respect to θ [8]. Thus we can completely characterizes the shape of the moving body using the function $f(\theta)$. Since the $G\zeta$ axis is fixed in the body, then $\Omega_\xi = \omega_\xi$, $\Omega_\eta = \omega_\eta = -\dot\theta$. The plane $G\xi\zeta$ is always vertical, i.e., the projection of the angular velocity $\mathbf{\Omega}$ of the axes $G\xi\eta\zeta$ on $\overrightarrow{MQ}$ equals to zero, therefore $\Omega_\zeta = \Omega_\xi \cot\theta$. Since the body moves without sliding then

$$v_\xi - \dot\theta\zeta = 0, \quad v_\eta + \omega_\zeta\xi - \omega_\xi\zeta = 0, \quad v_\zeta + \dot\theta\xi = 0$$

and for three unknown functions θ, ω_ξ and ω_ζ we have closed system of equations [8]

$$\begin{aligned}
&(A_1 + m\xi^2 + m\zeta^2)\ddot\theta = -\,mgf'(\theta) - (A_3\omega_\zeta - A_1\omega_\xi\cot\theta)\omega_\xi + \\
&\qquad\qquad + m\omega_\xi(\zeta\cot\theta + \xi)(\omega_\xi\zeta - \omega_\zeta\xi) - m\dot\theta^2(\xi\xi' + \zeta\zeta'), \\
&\dot\omega_\xi = \Big(-\frac{\cos\theta}{\sin\theta} - \frac{A_3 m\zeta(\xi + \zeta')}{\Delta}\Big)\omega_\xi\dot\theta + \frac{A_3(A_3 + m\xi^2 + m\xi'\zeta)}{\Delta}\omega_\zeta\dot\theta, \\
&\dot\omega_\zeta = \frac{A_1 m\xi(\xi + \zeta')}{\Delta}\omega_\xi\dot\theta + \frac{m\xi(A_3\zeta - A_1\xi')}{\Delta}\omega_\zeta\dot\theta, \\
&\Delta = A_1A_3 + A_1m\xi^2 + A_3m\zeta^2.
\end{aligned} \tag{15}$$

If we introduce the following notations

$$\begin{aligned}
&\omega_\xi = p_1, \quad \omega_\zeta = p_2, \quad K = \tfrac{1}{2}(A_1 + m\xi^2 + m\zeta^2)\dot\theta^2, \\
&W = \frac{A_1}{2}p_1^2 + \frac{A_3}{2}p_2^2 + \frac{m}{2}(p_1\zeta - p_2\xi)^2 + mgf, \\
&\Gamma_1 = \Big(-\frac{\cos\theta}{\sin\theta} - \frac{A_3 m\zeta(\xi + \zeta')}{\Delta}\Big)p_1 + \frac{A_3(A_3 + m\xi^2 + m\xi'\zeta)}{\Delta}p_2, \\
&\Gamma_2 = \frac{A_1 m\xi(\xi + \zeta')}{\Delta}p_1 + \frac{m\xi(A_3\zeta - A_1\xi')}{\Delta}p_2,
\end{aligned}$$

then the system (15) can be rewritten in the form

$$\frac{d}{dt}\Big(\frac{\partial K}{\partial \dot\theta}\Big) = \frac{\partial K}{\partial \theta} - \frac{\partial W}{\partial \theta} - \Gamma_1 \frac{\partial W}{\partial p_1} - \Gamma_2 \frac{\partial W}{\partial p_2},$$
$$\dot p_1 = \Gamma_1 \dot\theta, \qquad \dot p_2 = \Gamma_2 \dot\theta.$$

Thus, the system (15), describing motion of a heavy rigid rotationally symmetric body on a perfectly rough horizontal plane has a form of the system (1). Since in this case we have only one position variable θ and $\Gamma_1 \neq 0$, $\Gamma_2 \neq 0$, then all the conditions of Theorem 2 are valid for the system (15). Stationary motions of the body of the form (4)

$$\theta = \theta_0, \quad \dot\theta = 0, \quad p_1 = p_1^0, \quad p_2 = p_2^0 \tag{16}$$

are determined from the equation

$$\frac{DW}{D\theta} = \frac{\partial W}{\partial \theta} + \Gamma_1 \frac{\partial W}{\partial p_1} + \Gamma_2 \frac{\partial W}{\partial p_2} = 0. \tag{17}$$

and form a two-dimensional family.

In the explicit form equation (17) may be written as follows

$$mgf' + Dp_1p_2 - Cp_1^2 \cot\theta = 0,$$
$$C = A_1 - \frac{m\zeta}{\cos\theta} f, \quad D = A_3 - \frac{m\xi}{\sin\theta} f.$$

Analysis of the sign of expression

$$\frac{D^2W}{D\theta^2}$$

on the stationary motion (16) provides the sufficient condition of stability of this motion with respect to θ, $\dot\theta$, p_1 and p_2 in the form

$$mgf'' + (Dp_2 - 2Cp_1 \cot\theta)\Gamma_1 - C'p_1^2 \cot\theta + \frac{C}{\sin^2\theta} p_1^2 + D'p_1p_2 + Dp_1\Gamma_2 > 0. \tag{18}$$

It is possible to prove [7], when the condition (18) is not valid, the stationary motion (16) is unstable.

According to Theorem 5 the obtained results retain their validity when the considered system is subjected to the dissipative force $Q(\theta, \dot\theta)$, corresponding to the position variable θ. Moreover if the condition

$$Q(\theta, \dot\theta)\dot\theta < 0, \ \forall\, \dot\theta \neq 0$$

is valid then under condition (18) the stationary motions (16) are asymptotically stable with respect to variables

$$\dot\theta, \quad P = mgf' + Dp_1p_2 - C\cot\theta p_1^2.$$

Example 2. Let us consider a particular case of the previous example, when the rotationally symmetric body, moving on a fixed perfectly rough plane, is a circular disk of a radius a [9–12]. Let m be the mass of a disk, $A_1 = kma^2$ and $A_3 = 2kma^2$ are its moments of inertia. In a case of a homogeneous disk we have

$k = 1/4$ and in a case of a hoop we have $k = 1/2$. Then $GQ = a\sin\theta$ and according to (14) we get

$$\xi = -a, \quad \zeta = 0.$$

System (15) will take the form

$$(19)\quad \begin{aligned} &(k+1)a\ddot{\theta} = -\mathrm{g}\cos\theta + ka\omega_\xi^2\cot\theta - (2k+1)a\omega_\xi\omega_\zeta,\\ &\dot{\omega_\xi} = \Big(-\frac{\cos\theta}{\sin\theta}\omega_\xi + 2\omega_\zeta\Big)\dot{\theta}, \qquad \dot{\omega_\zeta} = \frac{\omega_\xi}{(2k+1)}\dot{\theta}. \end{aligned}$$

If we denote again $\omega_\xi = p_1$ and $\omega_\zeta = p_2$, then the stationary motions (16) of a disk are determined from the equation

$$(20)\quad kap_1^2\cot\theta - (2k+1)ap_1p_2 - \mathrm{g}\cos\theta = 0.$$

Note that in the considered problem system (19) can be solved with respect to $\omega_\xi = p_1$ and $\omega_\zeta = p_2$. The corresponding solution has the form:

$$(21)\quad \begin{aligned} p_1 &= \sin\theta\Big(\mathrm{c}_1F\Big(\alpha+1,\beta+1,2;\sin^2\frac{\theta}{2}\Big) - \mathrm{c}_2F\Big(\alpha+1,\beta+1,2;\cos^2\frac{\theta}{2}\Big)\Big)\\ &= \sin\theta(\mathrm{c}_1v_1 - \mathrm{c}_2v_2),\\ p_2 &= \mathrm{c}_1F\Big(\alpha,\beta,1;\sin^2\frac{\theta}{2}\Big) + \mathrm{c}_2F\Big(\alpha,\beta,1;\cos^2\frac{\theta}{2}\Big) = \mathrm{c}_1u_1 + \mathrm{c}_2u_2. \end{aligned}$$

Here c_1 and c_2 are arbitrary constants and $F(\alpha,\beta,1;z)$ is the Gauss hypergeometric function with the parameters α and β satisfying the equation

$$s^2 - s + \frac{2}{2k+1} = 0.$$

Substituting expressions (21) for p_1 and p_2 to equation (20) and introducing the dimensionless constants $x_i = \mathrm{c}_i\sqrt{a/\mathrm{g}}$, $i = 1,2$ we rewrite (20) in the dimensionless form

$$(22)\quad \sum_{i,j=1}^{2} a_{i\,j}x_ix_j - \cos\theta = 0,$$

$$a_{i\,j} = a_{j\,i} = ((k+1/2)((-1)^iu_jv_i + (-1)^ju_iv_j) + (-1)^{i+j}kv_iv_j\cos\theta)\sin\theta.$$

In the space of x_1, x_2 and θ equation (22) defines a surface. Figures 2 to 5 show cross sections of this surface by the planes $x_2 = lx_1$ for different l and $k = 1/4$ as in the case of a homogeneous disk. Note that the similar cross sections have been constructed in [9].

It is easy to see that for each fixed θ equation (22) defines a second order curve. By analyzing its invariants we proved that for $\theta \neq \pi/2$ this curve is a hyperbola and for $\theta = \pi/2$ it is a pair of straight lines. These straight lines are defined by the equations $x_1 = x_2$ and $x_1 = -x_2$ and correspond to the two single parametric subfamilies of stationary motions of a disk of the form

$$(23)\quad \theta = \frac{\pi}{2}, \quad \dot{\theta} = 0, \quad p_2 = 2u_*\mathrm{c}_1 = \Omega, \quad p_1 = 0; \quad u_* = F\Big(\alpha,\beta,1;\frac{1}{2}\Big)$$

$$(24)\quad \theta = \frac{\pi}{2}, \quad \dot{\theta} = 0, \quad p_1 = 2v_*\mathrm{c}_1 = \omega, \quad p_2 = 0; \quad v_* = F\Big(\alpha+1,\beta+1,2;\frac{1}{2}\Big).$$

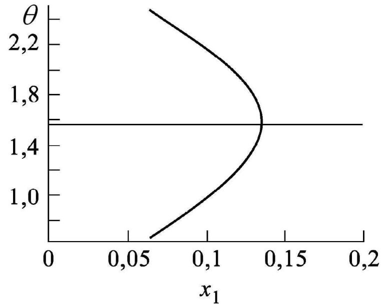

FIGURE 2. $x_2 = x_1$

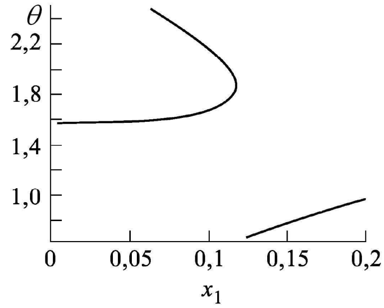

FIGURE 3. $x_2 = 0.5x_1$

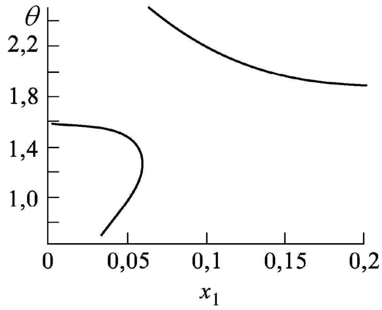

FIGURE 4. $x_2 = 2x_1$

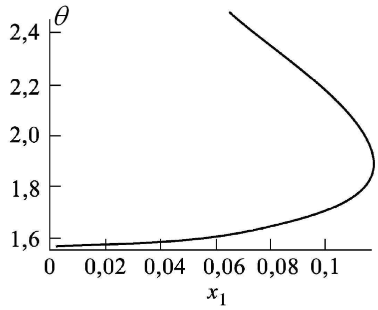

FIGURE 5. $x_2 = 0$

These subfamilies correspond to uniform rolling of a vertically placed disk along a straight line (23) and to uniform rotation of a disk about its vertical diameter (24). The stationary motion (23) is stable [unstable] for

$$\Omega^2 > \Omega_0^2 = \frac{g}{2a(2k+1)} \quad [\Omega^2 < \Omega_0^2],$$

while the stationary motion (24) is stable [unstable] for

$$\omega^2 > \omega_0^2 = \frac{g}{a(k+1)} \quad [\omega^2 < \omega_0^2]$$

(for more details see [9, 10]).

Condition (18) for stability of stationary motions of a disk can be written in the dimensionless form as follows

$$\sum_{i,j=1}^{2} b_{i\,j} x_i x_j - \sin\theta \geqslant 0, \tag{25}$$

$$b_{i\,j} = b_{j\,i} = 2(2k+1)u_i u_j + (3k+1/2)((-1)^i u_j v_i + (-1)^j u_i v_j)\cos\theta + \\ + (-1)^{i+j}((k+1)\sin^2\theta + 3k\cos^2\theta)v_i v_j.$$

For each fixed θ the boundary of region of stability is also a second order curve. By analyzing its invariants we proved that for $k > 1/\sqrt{3} - 1/2$ this curve is an ellipse with the origin at $x_1 = 0$, $x_2 = 0$. The stable region is outside this ellipse and the unstable region is within it.

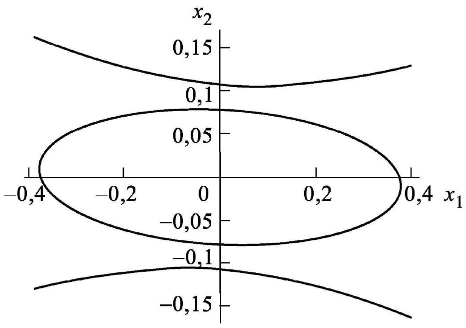

FIGURE 6. The hyperbola and the ellipse for $\theta = \pi/3$.

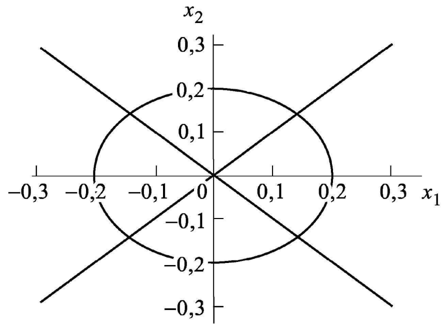

FIGURE 7. The hyperbola and the ellipse for $\theta = \pi/2$.

Thus we can give a geometric interpretation for conditions of existence and stability of stationary motions of a disk [11,12]. Obviously, the stationary motions of a disk corresponding to the points of the hyperbola lying outside the ellipse are stable (Figures 6 and 7). If for a fixed $\theta = \theta_0$ the hyperbola and the ellipse do not intersect then the stationary motions corresponding to θ_0 are stable independently of x_1 and x_2 (Figures 6 and 7).

Conditions of existence (22) and stability (25) of stationary motions (16) of a disk have been analyzed in [11,12]. In particular, it was shown [11,12] that the stationary motions (16) of a disk are stable (independently of x_1 and x_2) for any θ satisfying the condition

$$\cos^2\theta > \cos^2\theta_* = \frac{2(2k+1)\left[4k+3-\sqrt{6(2k+1)(k+1)}\right]}{(2k+3)^2+3(2k+1)^2}.$$

In particular, for a homogeneous disk ($k = 1/4$) we have

$$\cos^2\theta > \frac{25-9\sqrt{5}}{38} \approx 0.102, \quad \theta_* \approx 1.2457$$

For a hoop ($k = 1/2$) we have

$$\cos^2\theta > \frac{5-3\sqrt{2}}{7} \approx 0.108, \quad \theta_* \approx 1.2356.$$

For other values of θ stationary motions (16) will be stable if the absolute value of x_1 exceeds a certain critical value. The explicit expression of this critical value is very complicated and we omit it here. The results obtained in [11,12] are in completely agreement with bifurcation diagrams presented in [9,10] and here.

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РУТОВА ТЕОРЕМА ЗА МЕХАНИЧКЕ СИСТЕМЕ СА НЕПОЗНАТИМ ПРВИМ ИНТЕГРАЛИМА

Резиме. У овом раду разматрамо проблеме стабилности стационарног кретања конзервативних и дисипативних механичких система са првим интегралима. Општи резултати су илустровани проблемом котрљања без клизања тешког ротационо симетричног крутог тела по хоризонталној равни.

Department of Mechanics and Mathematics
Lomonosov Moscow State University
Moscow
Russia
avkarapetyan@yandex.ru

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Department of Mechanics and Mathematics
Lomonosov Moscow State University
Moscow
Russia
kuleshov@mech.math.msu.su