

## ON DAMAGE TENSOR IN LINEAR ANISOTROPIC ELASTICITY

Jovo Jarić and Dragoslav Kuzmanović

*Dedicated to the memory of Aleksandar Bakša*

**ABSTRACT.** In this paper, the anisotropic linear damage mechanics is presented starting from the principle of strain equivalence. The authors have previously derived damage tensor components in terms of elastic parameters of undamaged (virgin) material in closed form solution. Here, making use of this paper, we derived elasticity tensor as a function of damage tensor also in closed form. The procedure we present here was applied for several crystal classes which are subjected to hexagonal, orthotropic, tetragonal, cubic and isotropic damage. As an example isotropic system is considered in order to present some possibility to evaluate its damage parameters.

### 1. Introduction

In continuum damage mechanics, usually a phenomenological approach is adopted. In this approach, the most important concept is that of the Representative Volume Element (RVE). The discontinuous and discrete elements of damage are not considered within the RVE; rather their combined effects are lumped together through the use of a macroscopic internal variable. In this way, the formulation may be derived consistently using sound mechanical and thermodynamic principles.

In most of the existing damage theories, the damaged elastic strain-stress (or stress-strain) response is formulated by using the notion of effective stress (strain) and the hypothesis of strain (stress) equivalence or stress-energy (strain-energy) equivalence [2, 3, 8–10, 13].

The damage variable (or tensor), based on the effective stress concept, represents average material degradation which reflects the various types of damage at the micro-scale level like nucleation and growth of voids, cracks, cavities, micro-cracks, and other microscopic defects.

In order to make the paper self-sufficient we present the main ideas of the principles of strain equivalence used by [3].

---

2010 *Mathematics Subject Classification:* 74R99.

*Key words and phrases:* damage, anisotropic elasticity, elasticity.

Let  $\boldsymbol{\sigma}$  be the second-rank Cauchy stress tensor and  $\tilde{\boldsymbol{\sigma}}$  be the corresponding effective stress tensor. The effective stress tensor  $\tilde{\boldsymbol{\sigma}}$  is the stress applied to a fictitious state of the material which is totally undamaged, i.e., all damage in this state has been removed. This fictitious state is assumed to be mechanically equivalent to the actual damage state of the material.

The effective stress  $\tilde{\boldsymbol{\sigma}}$  is the stress tensor to be applied to a virgin representative volume element in order to obtain the same elastic strain tensor,  $\boldsymbol{\varepsilon}^e$ , produced by applying the actual stress tensor  $\boldsymbol{\sigma}$ , to the damage volume element. Because the same elastic strain is considered in both damaged and undamaged materials, that strain is considered to be the equivalent strain.

By definition, often called the principle of strain equivalence, the actual stress and effective stress satisfy the equations:

$$(1.1) \quad \sigma_{ij} = \mathbb{C}_{ijkl} \varepsilon_{kl}^e,$$

$$(1.2) \quad \tilde{\sigma}_{ij} = \mathbb{E}_{ijkl} \varepsilon_{kl}^e,$$

where

$\mathbb{E}_{ijkl}$  is elastic modulus tensor of the virgin material,  
 $\mathbb{C}_{ijkl}$  is elasticity tensor of the damaged material.

In the virgin state, even in the most general case of anisotropy, there are only 21 independent elements of the fourth-order elastic modulus tensor  $\mathbb{E}$  as a result of general symmetry requirements

$$(1.3) \quad \mathbb{E}_{ijkl} = \mathbb{E}_{jikl} = \mathbb{E}_{ijlk} = \mathbb{E}_{klij},$$

where the first three result from the symmetry of the stress and strain tensors and the last one from the existence of a strain energy function. The symmetry of  $\mathbb{E}$  in equation (1.3), applied to  $\mathbb{C}$  as well, and dictates a maximum of 21 independent elements.

Following [1] it can be shown that

$$(1.4) \quad \sigma_{ij} = \mathbb{R}_{ijkl} \tilde{\sigma}_{kl},$$

where the fourth-order tensor  $\mathbb{R}$  possesses symmetry in successive pairs of indices.

It can be shown that  $\mathbb{R}$  can be written in the form

$$(1.5) \quad \mathbb{R} = \mathbb{I} - \mathbb{D},$$

where  $\mathbb{I}$  is the unit tensor, for the set of tensors with the symmetry of  $\mathbb{R}$ , given by

$$(1.6) \quad \mathbb{I}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

The fourth-order tensor  $\mathbb{D}$  is known as damage tensor.

From equations (1.1)–(1.5) [1] it is obtained

$$(1.7) \quad \mathbb{C} = \mathbb{E} - \mathbb{D}\mathbb{E},$$

or

$$(1.8) \quad \mathbb{C}_{ijkl} = \mathbb{E}_{ijkl} - \mathbb{D}_{ijpq}\mathbb{E}_{pqkl}.$$

We note that

$$\mathbb{D}_{ijkl} = \mathbb{D}_{jikl} = \mathbb{D}_{ijlk}$$

and, in general,  $\mathbb{D}_{ijkl} \neq \mathbb{D}_{klij}$ , i.e., the fourth-order tensor  $\mathbb{D}$  does not possess the major symmetry and therefore not the full symmetry of the elastic modulus tensors  $\mathbb{E}$  and  $\mathbb{C}$ . Thus, the damage tensor  $\mathbb{D}$  has at most 36 components.

But, the symmetries in the elements of  $\mathbb{C}$  and  $\mathbb{E}$  imply the following 15 constraint equations on the elements of  $\mathbb{D}$

$$(1.9) \quad \mathbb{D}_{ijpq}\mathbb{E}_{pqkl} - \mathbb{D}_{klpq}\mathbb{E}_{pqij} = 0.$$

Therefore, tensor  $\mathbb{D}$  can not possess more than 21 independent components.

Obviously, the number of independent elements of tensor  $\mathbb{D}$  and their values are determined by the value of tensors  $\mathbb{E}$  and  $\mathbb{C}$ . We investigate that assuming that  $\mathbb{E}$  is always given, and considering equation (1.8) as a linear equation with respect to  $\mathbb{C}$  and  $\mathbb{D}$ . Then, in order to determine, or equivalently to find the solution of (1.7), one of them has to be known.

**First.** Assume that we want to find  $\mathbb{C}$  for given  $\mathbb{D}$ . Then  $\mathbb{D}$  cannot be given arbitrarily in order to find  $\mathbb{C}$  since we assume that  $\mathbb{C}$  possesses major symmetry. In that case  $\mathbb{D}$  must satisfy conditions equation (1.9), or

$$(1.10) \quad \mathbb{D}\mathbb{E} = \mathbb{E}\mathbb{D}^T.$$

**Second.** We consider that  $\mathbb{E}$ ,  $\mathbb{C}$  are given and we want to find tensor  $\mathbb{D}$ . Since  $\mathbb{E}$  is positive definite there is always  $\mathbb{E}^{-1}$  such that  $\mathbb{E}\mathbb{E}^{-1} = \mathbb{I}$ . Thus

$$(1.11) \quad \mathbb{D} = \mathbb{I} - \mathbb{C}\mathbb{E}^{-1}.$$

PROPOSITION 1.1. *Damage tensor  $\mathbb{D}$ , given by equation (1.11), always satisfies the constraint equation (1.10).*

Starting with isotropic  $\mathbb{E}$  [1]

$$(1.12) \quad \mathbb{E} = \lambda\mathbf{I} \otimes \mathbf{I} + 2\mu\mathbb{I}$$

and general  $\mathbb{C}$ , considered damage tensor for the special case of the hypothesis of elastic strain equivalence. Here and further  $\lambda$  and  $\mu$  are Lamé constants, and  $\mathbf{I}$  is identity tensor of second order, i.e.,

$$\mathbf{I} = \mathbf{n}_i \otimes \mathbf{n}_i.$$

In this case, the following proposition holds

PROPOSITION 1.2. *The corresponding isotropy groups of tensors  $\mathbb{C}$  and  $\mathbb{D}$  are the same [6].*

The proofs of these propositions are given in [7].

REMARK 1.1. It is important to point out that Proposition 1.2 can be stated in a more general form when isotropic group  $\mathfrak{g}_{\mathbb{E}} \subset \mathcal{O}(3)$ , and  $\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{D}} \subset \mathfrak{g}_{\mathbb{E}}$ . In application, it is more restrictive since only  $\mathfrak{g}_{\mathbb{E}} = \mathcal{O}(3)$  contains all  $\mathfrak{g}$ , all isotropic groups  $\mathfrak{g}$  of crystals.

## 2. Elasticity tensor of damage

This work is practically a continuation of work [7] to which we refer.

The anisotropic linear damage mechanics is presented starting from the principle of strain equivalence. In the above paper, the damage tensor components are derived in terms of elastic parameters of undamaged (virgin) material in closed form solution. Here, making use of the results of the same paper, we derived elasticity tensor as a function of damage tensor also in closed form. The procedure is applied for several symmetries that are important for applications. Here we assumed that undamaged material is isotropic, i.e., tensor  $\mathbb{E}$  is given by (1.12). Then, from (1.7) we have

$$\mathbb{C} = \mathbb{E} - \lambda \mathbb{D} \mathbf{I} \otimes \mathbf{I} - 2\mu \mathbb{D},$$

and from (1.10)

$$(2.1) \quad \lambda (\mathbb{D} \mathbf{I} \otimes \mathbf{I})^a + 2\mu \mathbb{D}^a = 0$$

From the last two expressions we obtain that

$$(2.2) \quad \mathbb{C} = \mathbb{E} - \lambda (\mathbb{D} \mathbf{I} \otimes \mathbf{I})^s - 2\mu \mathbb{D}^s.$$

Here  $a$  and  $s$  represent skew-symmetric and symmetric part of tensors of fourth order, respectively.

The basic ideas are given below. Notations, mathematical preliminaries and basic terms are contained in [7].

To simplify notation we shall denote by  $d$  the parameters of damage tensor  $\mathbb{D}$ . The values of parameters, and their numbers, depend on crystal classes we investigate. Likewise elasticity damage tensor will be denoted by  $\mathbb{C}$  and its coefficient by  $\lambda$ . By the same reason, the values of coefficients, and their numbers, depend on crystal classes.

It is very important to note that the tensor of elasticity of damaged material can be determined in two ways. In both cases, we use the resulting expressions for damage tensors. The approaches are algebraically one and generally simple.

The logical question is: Are the expressions of anisotropic elasticity damage tensor the same when we derived it making use of these two approaches?

In the first case, using the expression the parameters of damage tensor, given as a functions of coefficients of corresponding elasticity damage tensor, we find the expressions of coefficients of elasticity damage tensor as functions of parameters of damage tensor. In the second case, we use the expression (1.7) or (1.8) to derive elasticity damage tensor. We shall call the second approach direct one. For some crystal classes we shall simultaneously apply both of them. The purpose to do this is twofold:

- a) to compare these approaches and
- b) to show that our results are consistent.

We shall start with hexagonal crystal class.

### 2.1. Hexagonal elasticity damage tensor. a) First approach

We consider the expression

$$(2.3) \quad 2\mu \mathbb{D}_{ijkl} = \left[ \frac{\lambda(3\lambda_1 + 2\lambda_2 + \lambda_4)}{3\lambda + 3\mu} - \lambda_1 \right] \delta_{ij} \delta_{kl} \\ + (\mu - \lambda_2)(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) - \lambda_3 n_{3i} n_{3j} n_{3k} n_{3l} \\ - \lambda_4 (n_{3i} n_{3j} \delta_{kl} + n_{3k} n_{3l} \delta_{ij}) - \lambda_5 (n_{3i} n_{3k} \delta_{jl} + n_{3j} n_{3l} \delta_{ik} \\ + n_{3i} n_{3l} \delta_{jk} + n_{3j} n_{3k} \delta_{il}) + \frac{\lambda(\lambda_3 + 3\lambda_4 + 4\lambda_5)}{3\lambda + 2\mu} n_{3i} n_{3j} \delta_{kl}.$$

which is derived under the known assumption

$$\mathbb{C}_{ijkl} = \lambda_1 \delta_{ij} \delta_{kl} + \lambda_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \lambda_3 n_{3i} n_{3j} n_{3k} n_{3l} + \lambda_4 (n_{3i} n_{3j} \delta_{kl} \\ + n_{3k} n_{3l} \delta_{ij}) + \lambda_5 (n_{3i} n_{3k} \delta_{jl} + n_{3j} n_{3l} \delta_{ik} + n_{3i} n_{3l} \delta_{jk} + n_{3j} n_{3k} \delta_{il}),$$

(see [7, equations 30 and 32]).

Making use of the following notations

$$(2.4) \quad 2\mu d_1 = \frac{\lambda(3\lambda_1 + 2\lambda_2 + \lambda_4)}{3\lambda + 3\mu} - \lambda_1, \\ 2\mu d_2 = \mu - \lambda_2 \\ 2\mu d_3 = -\lambda_3 \\ 2\mu d_4 = -\lambda_4 \\ 2\mu d_5 = -\lambda_5 \\ 2\mu d_6 = -\frac{\lambda(d_3 + 3d_4 + 4d_5)}{3\lambda + 2\mu}$$

we obtain

$$(2.5) \quad \lambda_1 = -(3\lambda + 2\mu)d_1 + \lambda(1 - 2d_2 - d_4) = \lambda(1 - 3d_1 - 2d_2 - d_4) - 2\mu d_1 \\ \lambda_2 = (1 - 2d_2)\mu \\ \lambda_3 = -2\mu d_3 \\ \lambda_4 = -2\mu d_4 \\ \lambda_5 = -2\mu d_5.$$

From relation (2.4)<sub>6</sub> it is obvious that  $d_6$  is not an independent parameter. Therefore, damage tensor  $\mathbb{D}_{ijkl}$  has five independent parameters as elasticity tensor  $\mathbb{C}_{ijkl}$ , as it should be [4]. This relation a consequence of conditions in [7, equation 15].

b) Second approach. Expression (2.3) (see also [1]) suggests that tensor  $\mathbb{D}$  is given by

$$(2.6) \quad \mathbb{D}_{ijkl} = d_1 \delta_{ij} \delta_{kl} + d_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + d_3 n_{3i} n_{3j} n_{3k} n_{3l} \\ + d_4 (n_{3i} n_{3j} \delta_{kl} + n_{3k} n_{3l} \delta_{ij}) \\ + d_5 (n_{3i} n_{3k} \delta_{jl} + n_{3j} n_{3l} \delta_{ik} + n_{3i} n_{3l} \delta_{jk} + n_{3j} n_{3k} \delta_{il}) \\ + d_6 n_{3i} n_{3j} \delta_{kl}.$$

In order to determine tensor  $\mathbb{C}$ , we have to use (1.8), i.e.,

$$\mathbb{C}_{ijkl} = \mathbb{E}_{ijkl} - \mathbb{D}_{ijpq} \mathbb{E}_{pqkl}.$$

Having in mind that

$$(2.7) \quad \mathbb{E}_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

we have that

$$\mathbb{C}_{ijkl} = \mathbb{E}_{ijkl} - \mathbb{D}_{ijpq} \mathbb{E}_{pqkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) - \lambda \mathbb{D}_{ijpp} \delta_{kl} - 2\mu \mathbb{D}_{ijkl}$$

Further, from (2.6) and

$$\mathbb{D}_{ijpp} = (3d_1 + 2d_2 + d_4) \delta_{ij} + (d_3 + 3d_4 + 4d_5 + 3d_6) n_{3i} n_{3j},$$

we obtain, taking into account the symmetric parts of  $\mathbb{D}_{ijpp} \delta_{kl}$  and  $\mathbb{D}_{ijkl}$ ,

$$\begin{aligned} \mathbb{C}_{ijkl} = & [\lambda(1 - 3d_1 - 2d_2 - d_4) - 2\mu] \delta_{ij} \delta_{kl} \\ & + \mu(1 - 2d_2) (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \\ & - [\lambda(d_3 + 3d_4 + 4d_5 + 3d_6) + 2\mu d_6] n_{3i} n_{3j} \delta_{kl} \\ & - 2\mu d_3 n_{3i} n_{3j} n_{3k} n_{3l} - 2\mu d_4 (n_{3i} n_{3j} \delta_{kl} + n_{3k} n_{3l} \delta_{ij}) \\ & - 2\mu d_5 (n_{3i} n_{3k} \delta_{jl} + n_{3j} n_{3l} \delta_{ik} + n_{3i} n_{3l} \delta_{jk} + n_{3j} n_{3k} \delta_{il}). \end{aligned}$$

Now, obviously the coefficients of elasticity damage tensor, for hexagonal class, are given by

$$\begin{aligned} \lambda_1 &= \lambda(1 - 3d_1 - 2d_2 - d_4) - 2\mu d_1 \\ \lambda_2 &= \mu(1 - 2d_2) \\ \lambda_3 &= -2\mu d_3 \\ \lambda_4 &= -2\mu d_4 \\ \lambda_5 &= -2\mu d_5, \end{aligned}$$

and

$$\lambda(d_3 + 3d_4 + 4d_5 + 3d_6) + 2\mu d_6 = 0$$

which are identical with (2.5) and (2.4)<sub>6</sub> given in implicit form.

Comparing these two approaches we conclude that they are consistent, but in b) approach we have obtained the coefficients of elasticity tensor directly. This is the reason why we call b) approach direct one. Having in mind that these two approaches are consistent in general, further on we shall apply the direct approach to other crystal classes investigate here.

**2.2. Orthotropic damage.** In this case, from [7, equation 41], can be written in a simple form

$$(2.8) \quad \begin{aligned} \mathbb{D}_{ijkl} = & d_1 \delta_{ij} \delta_{kl} + d_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + d_3 n_{1i} n_{1j} n_{1k} n_{1l} + \\ & + d_4 n_{2i} n_{2j} n_{2k} n_{2l} + d_5 (n_{1i} n_{1j} \delta_{kl} + n_{1k} n_{1l} \delta_{ij}) \\ & + d_6 (n_{2i} n_{2j} \delta_{kl} + n_{2k} n_{2l} \delta_{ij}) + d_7 (n_{1i} n_{1j} n_{2k} n_{2l} + n_{1k} n_{1l} n_{2i} n_{2j}) \\ & + d_8 (n_{1i} n_{1k} \delta_{jl} + n_{1i} n_{1l} \delta_{jk} + n_{1j} n_{1k} \delta_{il} + n_{1j} n_{1l} \delta_{ik}) \\ & + d_9 (n_{2i} n_{2k} \delta_{jl} + n_{2i} n_{2l} \delta_{jk} + n_{2j} n_{2k} \delta_{il} + n_{2j} n_{2l} \delta_{ik}) \\ & + d_{10} n_{1i} n_{1j} \delta_{kl} + d_{11} n_{2i} n_{2j} \delta_{kl}, \end{aligned}$$

which is more convenient for the calculation.

In the same way, as for the hexagonal system, tensor  $\mathbb{C}$  can be determined from (2.2). Making use of (2.7) and (2.8), after simple, but lengthy calculation, we obtained:

$$\begin{aligned}\mathbb{C}_{ijkl} = & [\lambda(1 - 3d_1 - 2d_2 - d_5 - d_6) - 2\mu d_1]\delta_{ij}\delta_{kl} \\ & + \mu(1 - 2d_2)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - 2\mu d_3 n_{1i}n_{1j}n_{1k}n_{1l} \\ & - 2\mu d_4 n_{2i}n_{2j}n_{2k}n_{2l} - 2\mu d_5 (n_{1i}n_{1j}\delta_{kl} + n_{1k}n_{1l}\delta_{ij}) \\ & - 2\mu d_6 (n_{2i}n_{2j}\delta_{kl} + n_{2k}n_{2l}\delta_{ij}) - 2\mu d_7 (n_{1i}n_{1j}n_{2k}n_{2l} + n_{1k}n_{1l}n_{2i}n_{2j}) \\ & - 2\mu d_8 (n_{1i}n_{1k}\delta_{jl} + n_{1i}n_{1l}\delta_{jk} + n_{1j}n_{1k}\delta_{il} + n_{1j}n_{1l}\delta_{ik}) \\ & - 2\mu d_9 (n_{2i}n_{2k}\delta_{jl} + n_{2i}n_{2l}\delta_{jk} + n_{2j}n_{2k}\delta_{il} + n_{2j}n_{2l}\delta_{ik}).\end{aligned}$$

From this expression it is obvious that coefficients of elasticity tensor are:

$$\begin{aligned}\lambda_1 &= \lambda(1 - 3d_1 - 2d_2 - d_5 - d_6) - 2\mu d_1, \\ \lambda_2 &= \mu(1 - 2d_2), \quad \lambda_3 = -2\mu d_3, \quad \lambda_4 = -2\mu d_4, \quad \lambda_5 = -2\mu d_5, \\ \lambda_6 &= -2\mu d_6, \quad \lambda_7 = -2\mu d_7, \quad \lambda_8 = -2\mu d_8, \quad \lambda_9 = -2\mu d_9.\end{aligned}$$

In order to complete our calculation, next we consider (2.1). Then the following relations are obtained:

$$\begin{aligned}\lambda(d_3 + 3d_5 + d_7 + 4d_8) + (3\lambda + 2\mu)d_{10} &= 0, \\ \lambda(d_4 + 3d_6 + d_7 + 4d_9) + (3\lambda + 2\mu)d_{11} &= 0,\end{aligned}$$

which represent compatibility conditions which have to be satisfied by damage tensor  $\mathbb{D}$ .

**2.3. Tetragonal system.** This system is defined by class  $4, \bar{4}, 4/m$ , and crystallographic directions  $\mathbf{n}_i$ ,  $i = 1, 2, 3$ , are orthonormal. One of them (say)  $\mathbf{n}_3$  is a four-fold axis of rotation. Then  $\mathbf{n}_1 \rightarrow \mathbf{n}_2$  and  $\mathbf{n}_2 \rightarrow -\mathbf{n}_1$ . Invariance of these changes results in the following form of  $\mathbb{D}$  given in Voigt notation

$$\mathbb{D} = \begin{pmatrix} D_{1111} & D_{1122} & D_{1133} & 0 & 0 & 0 \\ D_{1122} & D_{1111} & D_{1133} & 0 & 0 & 0 \\ D_{3311} & D_{3311} & D_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{2323} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{1212} \end{pmatrix}$$

For further calculation we shall write damage tensor in explicit form, which is not given in our paper (see [7]), in more detail. Then

$$\begin{aligned}(2.9) \quad \mathbb{D}_{ijkl} = & d_1(n_{1i}n_{1j}n_{1k}n_{1l} + n_{2i}n_{2j}n_{2k}n_{2l}) + d_2 n_{3i}n_{3j}n_{3k}n_{3l} \\ & + d_3(n_{1i}n_{1j}n_{2k}n_{2l} + n_{2i}n_{2j}n_{1k}n_{1l}) \\ & + d_4(n_{1i}n_{1j}n_{3k}n_{3l} + n_{2i}n_{2j}n_{3k}n_{3l}) + d_5(n_{3i}n_{3j}n_{1k}n_{1l} + n_{3i}n_{3j}n_{2k}n_{2l}) \\ & + d_6(n_{2i}n_{3j}n_{2k}n_{3l} + n_{3i}n_{2j}n_{2k}n_{3l} + n_{2i}n_{3j}n_{3k}n_{2l} + n_{3i}n_{2j}n_{3k}n_{2l} \\ & \quad + n_{1i}n_{3j}n_{1k}n_{3l} + n_{3i}n_{1j}n_{1k}n_{3l} + n_{1i}n_{3j}n_{3k}n_{1l} + n_{3i}n_{1j}n_{3k}n_{1l}) \\ & + d_7(n_{1i}n_{2j}n_{1k}n_{2l} + n_{2i}n_{1j}n_{1k}n_{2l} + n_{1i}n_{2j}n_{2k}n_{1l} + n_{2i}n_{1j}n_{2k}n_{1l})\end{aligned}$$

where

$$\begin{aligned} d_1 &= \mathbb{D}_{1111}, & d_2 &= \mathbb{D}_{3333}, & d_3 &= \mathbb{D}_{1122}, & d_4 &= \mathbb{D}_{1133}, \\ d_5 &= \mathbb{D}_{3311}, & d_6 &= \mathbb{D}_{2323}, & d_7 &= \mathbb{D}_{1212} \end{aligned}$$

In this case we have to calculate (2.1) and (2.2) for  $\mathbb{D}$ , given by (2.9). It is easy to see that

$$(\mathbb{D}\mathbf{I})_{ijpp} = (d_1 + d_3 + d_4)(n_{1i}n_{1j} + n_{2i}n_{2j}) + (d_2 + 2d_5)n_{3i}n_{3j},$$

and from this

$$\begin{aligned} (2.10) \quad (\mathbb{D}\mathbf{I} \otimes \mathbf{I})_{ijkl} &= (d_1 + d_3 + d_4) \times (n_{1i}n_{1j}n_{1k}n_{1l} + n_{1i}n_{1j}n_{2k}n_{2l} \\ &\quad + n_{1i}n_{1j}n_{3k}n_{3l} + n_{2i}n_{2j}n_{1k}n_{1l} \\ &\quad + n_{2i}n_{2j}n_{2k}n_{2l} + n_{2i}n_{2j}n_{3k}n_{3l}) \\ &\quad + (d_2 + 2d_5)(n_{3i}n_{3j}n_{1k}n_{1l} + n_{3i}n_{3j}n_{2k}n_{2l} + n_{3i}n_{3j}n_{3k}n_{3l}). \end{aligned}$$

After simple calculation we find that

$$\begin{aligned} (2.11) \quad \lambda(\mathbb{D}\mathbf{I} \otimes \mathbf{I})_{ijkl}^a &= \frac{1}{2}\lambda(d_1 + d_3 + d_4 - d_2 - 2d_5) \\ &\quad \times (n_{1i}n_{1j}n_{3k}n_{3l} + n_{2i}n_{2j}n_{3k}n_{3l} \\ &\quad - n_{3i}n_{3j}n_{1k}n_{1l} - n_{3i}n_{3j}n_{2k}n_{2l}) \end{aligned}$$

and

$$(2.12) \quad 2\mu\mathbb{D}_{ijkl}^a = \frac{1}{2}2\mu(d_4 - d_5) \times (n_{1i}n_{1j}n_{3k}n_{3l} + n_{2i}n_{2j}n_{3k}n_{3l} \\ - n_{3i}n_{3j}n_{1k}n_{1l} - n_{3i}n_{3j}n_{2k}n_{2l}).$$

Therefore, from (2.11), (2.12) and (2.1), i.e.,

$$\lambda(\mathbb{D}\mathbf{I} \otimes \mathbf{I})^a + 2\mu\mathbb{D}^a = 0,$$

it follows that

$$(2.13) \quad 2(\lambda + \mu)d_5 = \lambda(d_1 + d_3 - d_2) + (\lambda + 2\mu)d_4.$$

Hence, the number of independent parameters of tensor  $\mathbb{D}$  is six.

Next we calculate

$$(2.14) \quad \mathbb{C} = \mathbb{E} - \lambda(\mathbb{D}\mathbf{I} \otimes \mathbf{I})^s - 2\mu\mathbb{D}^s,$$

or, in componential form,

$$\mathbb{C}_{ijkl} = \mathbb{E}_{ijkl} - \lambda(\mathbb{D}\mathbf{I} \otimes \mathbf{I})_{ijkl}^s - 2\mu\mathbb{D}_{ijkl}^s.$$

Now, from (2.10) and (2.9), we find that

$$\begin{aligned} \lambda(\mathbb{D}\mathbf{I} \otimes \mathbf{I})_{ijkl}^s &= \lambda(d_1 + d_3 + d_4)(n_{1i}n_{1j}n_{1k}n_{1l} + n_{1i}n_{1j}n_{2k}n_{2l} + n_{2i}n_{2j}n_{1k}n_{1l} + \\ &\quad + n_{2i}n_{2j}n_{2k}n_{2l}) + \lambda(d_2 + 2d_5)n_{3i}n_{3j}n_{3k}n_{3l} + \\ &\quad + \lambda(d_1 + d_3 + d_4 + d_2 + 2d_5) \times (n_{1i}n_{1j}n_{3k}n_{3l} + n_{2i}n_{2j}n_{3k}n_{3l}), \end{aligned}$$

and

$$\begin{aligned} 2\mu\mathbb{D}_{ijkl}^s &= 2\mu d_1(n_{1i}n_{1j}n_{1k}n_{1l} + n_{2i}n_{2j}n_{2k}n_{2l}) + 2\mu d_2 n_{3i}n_{3j}n_{3k}n_{3l} \\ &\quad + 2\mu d_3(n_{1i}n_{1j}n_{2k}n_{2l} + n_{2i}n_{2j}n_{1k}n_{1l}) \\ &\quad + 2\mu(d_4 + d_5)(n_{1i}n_{1j}n_{3k}n_{3l} + n_{2i}n_{2j}n_{3k}n_{3l})^s \end{aligned}$$



$$\begin{aligned}
& + 2\mu d_6(n_{2i}n_{3j}n_{2k}n_{3l} + n_{3i}n_{2j}n_{2k}n_{3l} + n_{2i}n_{3j}n_{3k}n_{2l} + n_{3i}n_{2j}n_{3k}n_{2l} \\
& \quad + n_{1i}n_{3j}n_{1k}n_{3l} + n_{3i}n_{1j}n_{1k}n_{3l} + n_{1i}n_{3j}n_{3k}n_{1l} + n_{3i}n_{1j}n_{3k}n_{1l}) \\
& + 2\mu d_7(n_{1i}n_{2j}n_{1k}n_{2l} + n_{2i}n_{1j}n_{1k}n_{2l} + n_{1i}n_{2j}n_{2k}n_{1l} + n_{2i}n_{1j}n_{2k}n_{1l}).
\end{aligned}$$

Substituting these relations and

$$\begin{aligned}
\mathbb{E}_{ijkl} & = (\lambda + 2\mu)(n_{1i}n_{1j}n_{1k}n_{1l} + n_{2i}n_{2j}n_{2k}n_{2l}) + (\lambda + 2\mu)(n_{3i}n_{3j}n_{3k}n_{3l}) \\
& \quad + \lambda(n_{1i}n_{1j}n_{2k}n_{2l} + n_{2i}n_{2j}n_{1k}n_{1l}) + 2\lambda(n_{1i}n_{1j}n_{3k}n_{3l} + n_{3i}n_{3j}n_{2k}n_{2l})^s \\
& \quad + \mu(n_{2i}n_{3j}n_{2k}n_{3l} + n_{3i}n_{2j}n_{2k}n_{3l} + n_{2i}n_{3j}n_{3k}n_{2l} + n_{3i}n_{2j}n_{3k}n_{2l} \\
& \quad \quad + n_{1i}n_{3j}n_{1k}n_{3l} + n_{3i}n_{1j}n_{1k}n_{3l} + n_{1i}n_{3j}n_{3k}n_{1l} + n_{3i}n_{1j}n_{3k}n_{1l}) \\
& \quad + \mu(n_{1i}n_{2j}n_{1k}n_{2l} + n_{2i}n_{1j}n_{1k}n_{2l} + n_{1i}n_{2j}n_{2k}n_{1l} + n_{2i}n_{1j}n_{2k}n_{1l})
\end{aligned}$$

in (2.14), we obtain

$$\begin{aligned}
\mathbb{C}_{ijkl} & = \mathbb{E}_{ijkl} - [\lambda(d_1 + d_3 + d_4) + 2\mu d_1](n_{1i}n_{1j}n_{1k}n_{1l} + n_{2i}n_{2j}n_{2k}n_{2l}) \\
& \quad - [\lambda(d_2 + 2d_5) + 2\mu d_2](n_{3i}n_{3j}n_{3k}n_{3l}) - [\lambda(d_1 + d_3 + d_4) + 2\mu d_3] \\
& \quad \times (n_{1i}n_{1j}n_{2k}n_{2l} + n_{2i}n_{2j}n_{1k}n_{1l}) \\
& \quad - [\lambda(d_1 + d_3 + d_4 + d_2 + 2d_5) + 2\mu(d_4 + d_5)] \\
& \quad \times (n_{1i}n_{1j}n_{3k}n_{3l} + n_{2i}n_{2j}n_{3k}n_{3l})^s \\
& \quad - 2\mu d_6(n_{2i}n_{3j}n_{2k}n_{3l} + n_{3i}n_{2j}n_{2k}n_{3l} + n_{2i}n_{3j}n_{3k}n_{2l} + n_{3i}n_{2j}n_{3k}n_{2l} \\
& \quad \quad + n_{1i}n_{3j}n_{1k}n_{3l} + n_{3i}n_{1j}n_{1k}n_{3l} + n_{1i}n_{3j}n_{3k}n_{1l} + n_{3i}n_{1j}n_{3k}n_{1l}) \\
& \quad - 2\mu d_7(n_{1i}n_{2j}n_{1k}n_{2l} + n_{2i}n_{1j}n_{1k}n_{2l} + n_{1i}n_{2j}n_{2k}n_{1l} + n_{2i}n_{1j}n_{2k}n_{1l}).
\end{aligned}$$

After some calculation we obtain

$$\begin{aligned}
\mathbb{C}_{ijkl} & = [\lambda + 2\mu - \lambda(d_3 + d_4) - (\lambda + 2\mu)d_1] \times (n_{1i}n_{1j}n_{1k}n_{1l} + n_{2i}n_{2j}n_{2k}n_{2l}) \\
& \quad + [\lambda + 2\mu - (\lambda + 2\mu)d_2 - 2\lambda d_5](n_{3i}n_{3j}n_{3k}n_{3l}) \\
& \quad + [\lambda - (\lambda + 2\mu)d_3 - \lambda(d_1 + d_4)] \times (n_{1i}n_{1j}n_{2k}n_{2l} + n_{2i}n_{2j}n_{1k}n_{1l}) \\
& \quad + [\lambda - \lambda(d_1 + d_3) - (\lambda + 2\mu)d_4] \times (n_{1i}n_{1j}n_{3k}n_{3l} + n_{3i}n_{3j}n_{2k}n_{2l}) \\
& \quad + \mu(1 - 2d_6)(n_{2i}n_{3j}n_{2k}n_{3l} + n_{3i}n_{2j}n_{2k}n_{3l} + n_{2i}n_{3j}n_{3k}n_{2l} + n_{3i}n_{2j}n_{3k}n_{2l} \\
& \quad \quad + n_{1i}n_{3j}n_{1k}n_{3l} + n_{3i}n_{1j}n_{1k}n_{3l} + n_{1i}n_{3j}n_{3k}n_{1l} + n_{3i}n_{1j}n_{3k}n_{1l}) \\
& \quad + \mu(1 - 2d_7)(n_{1i}n_{2j}n_{1k}n_{2l} + n_{2i}n_{1j}n_{1k}n_{2l} + n_{1i}n_{2j}n_{2k}n_{1l} + n_{2i}n_{1j}n_{2k}n_{1l}).
\end{aligned}$$

From the last expression we obtain the values  $\lambda$  of elastic coefficients of elasticity damage tensor in the following forms:

$$\begin{aligned}
\lambda_1 & = (\lambda + 2\mu) - \lambda(d_1 + d_3 + d_4) - 2\mu d_1, & \lambda_2 & = \lambda - \lambda(d_1 + d_3 + d_4) - 2\mu d_3, \\
\lambda_3 & = (\lambda + 2\mu) - \lambda(d_2 + 2d_5) - 2\mu d_2, & \lambda_4 & = \lambda - \lambda(d_1 + d_3) - (\lambda + 2\mu)d_4, \\
\lambda_5 & = \mu(1 - 2d_6), & \lambda_6 & = \mu(1 - 2d_7).
\end{aligned}$$

Note that in the further calculation we have to use compatibility condition (2.13) in  $\lambda_3$ .

Finally, we write tensor  $\mathbb{C}$  in the Voigt notation

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_4 & 0 & 0 & 0 \\ \lambda_2 & \lambda_1 & \lambda_4 & 0 & 0 & 0 \\ \lambda_4 & \lambda_4 & \lambda_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_6 \end{pmatrix}.$$

**2.4. Cubic damage.** In this case, (see [7, equation 35]), we have

$$\begin{aligned} \mathbb{D}_{ijkl} &= d_1 \delta_{ij} \delta_{kl} + d_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \\ &\quad + d_3 (n_{1i} n_{1j} n_{1k} n_{1l} + n_{2i} n_{2j} n_{2k} n_{2l} + n_{3i} n_{3j} n_{3k} n_{3l}). \end{aligned}$$

In the same way as before we obtain

$$\begin{aligned} \mathbb{C}_{ijkl} &= \mathbb{E}_{ijkl} - \lambda \mathbb{D}_{ijpp} \delta_{kl} - 2\mu \mathbb{D}_{ijkl} \\ &= [\lambda(1 - 3d_1 - 2d_2 - 3d_3) - 2\mu d_1] \delta_{ij} \delta_{kl} + \mu(1 - 2d_2) (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \\ &\quad - 2\mu d_3 (n_{1i} n_{1j} n_{1k} n_{1l} + n_{2i} n_{2j} n_{2k} n_{2l} + n_{3i} n_{3j} n_{3k} n_{3l}), \end{aligned}$$

so that

$$\lambda_1 = \lambda(1 - 3d_1 - 2d_2 - 3d_3) - 2\mu d_1, \quad \lambda_2 = \mu(1 - 2d_2), \quad \lambda_3 = -2\mu d_3.$$

Notice that compatibility conditions (1.9) are here satisfied identically.

**2.5. Isotropic damage.** This is the simplest case and it is obtained from cubic damage when we take  $d_3 = 0$ . Then

$$\mathbb{C}_{ijkl} = \lambda_1 \delta_{ij} \delta_{kl} + \lambda_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),$$

where

$$(2.15) \quad \lambda_1 = \lambda(1 - 3d_1 - 2d_2) - 2\mu d_1, \quad \lambda_2 = \mu(1 - 2d_2).$$

### 3. An evaluation of damage parameters for isotropic damage

There remains also the quantitative evaluation of damage parameters from actual tests. This is an endeavor that can be done by the present work by means of the physically meaningful damage parameters that are given in terms of the basic elements of  $\mathbb{D}$ .

We shall demonstrate it for isotropic systems. It is known (see [11, p. 550, eq. 8.4.10]), for other crystal systems see [12] that the velocities of wave propagation for isotropic elastic material (undamage material) have the following expressions

$$\begin{aligned} v_L &= \left( \frac{\lambda + 2\mu}{\rho} \right)^{1/2}, \quad - \text{longitudinal velocity} \\ v_T &= \left( \frac{\mu}{\rho} \right)^{1/2}, \quad - \text{transversal velocity.} \end{aligned}$$

We now apply it to isotropic damage materials. In this case the corresponding quantities are

$$\lambda \rightarrow \lambda_1 = \lambda(1 - 3d_1 - 2d_2) - 2\mu d_1 = \rho_d (v_{dL}^2 - 2v_{dT}^2),$$

$$\mu \rightarrow \lambda_2 = \mu(1 - 2d_2) = \varrho_d v_{dT}^2.$$

Also, we denoted by subindex  $d$  the corresponding density and velocities of isotropic damaged materials, i.e.,

$$\varrho \rightarrow \varrho_d, \quad v_L \rightarrow v_{dL}, \quad v_T \rightarrow v_{dT}.$$

Then

$$\lambda_1 = \varrho_d(v_{dL}^2 - 2v_{dT}^2), \quad \lambda_2 = \varrho_d v_{dT}^2,$$

and hence

$$d_1 = \frac{\varrho_d}{3\lambda + 2\mu} \left( v_{dL}^2 - \frac{\lambda + 2\mu}{\mu} v_{dT}^2 \right), \quad d_2 = \frac{1}{2} \left( 1 - \frac{\varrho_d}{\mu} v_{dT}^2 \right).$$

In principle, the same approach can be applied to the other crystal classes.

It is also possible to estimate values of damage parameters  $d_1$  and  $d_2$  making use of the expression for Poisson's ratio  $\nu$  in classical linear elasticity, i.e.,

$$\nu = \frac{\lambda}{2(\lambda + \mu)}$$

(see [11, p.294, eq. 6.2.41]). Then

$$-1 < \nu < 1/2.$$

As the above, identifying the corresponding quantities for linear isotropic damage materials, we have

$$\nu \rightarrow \nu_d = \frac{\lambda_1}{2(\lambda_1 + \lambda_2)},$$

and

$$-1 < \nu_d = \frac{\lambda_1}{2(\lambda_1 + \lambda_2)} < 1/2.$$

Further,

$$\lambda_1 + \lambda_2 = \lambda - (3\lambda + 2\mu)d_1 - 2\lambda d_2.$$

The above inequality is satisfied when

$$\begin{aligned} 2\lambda_1 < 2(\lambda_1 + \lambda_2) &\Rightarrow \lambda_2 > 0, \\ \lambda_2 = \mu(1 - 2d_2) > 0 &\Rightarrow d_2 < \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} -2(\lambda_1 + \lambda_2) < \lambda_1 &\Rightarrow 3\lambda_1 + 2\lambda_2 > 0, \\ 3\lambda_1 + 2\lambda_3 = 3\lambda(1 - 3d_1 - 2d_2) + 2\mu(1 - 3d_1 - 2d_2) > 0, \\ 3d_1 + 2d_2 < 1. \end{aligned}$$

The graph of these two inequalities

$$d_2 < \frac{1}{2}, \quad 3d_1 + 2d_2 < 1$$

is given in Fig. 1.

In Fig. 1 the region of possible values of parameters  $d_1$  and  $d_2$  is given, i.e.,

$$\begin{aligned} 0 &\leq d_1 < 1/3, \\ 0 &< d_2 < 1/2. \end{aligned}$$

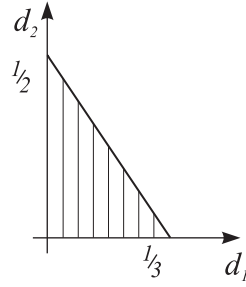


FIGURE 1.

#### 4. Discussion

Let us assume that homogeneous isotropic (virgin) material is subjected to an isotropic damage such that Lamé's coefficients of damage and virgin materials are proportional, i.e.,

$$\lambda_1 = k\lambda, \quad \lambda_2 = k\mu,$$

where  $k \neq 0$  is the coefficient of proportionality. But from (2.15) we have

$$\begin{aligned} \lambda_1 &= k\lambda = \lambda(1 - 3d_1 - 2d_2) - 2\mu d_1 = \lambda(1 - 2d_2) - (3\lambda + 2\mu)d_1, \\ \lambda_2 &= k\mu = \mu(1 - 2d_2), \end{aligned}$$

which are satisfied when  $d_1 = 0$  and  $d_2 = (1 - k)/2$ . We may go further to estimate possible values of  $k$  having in mind that  $0 < d_2 \leq 1/2$ . Then it follows that  $0 \leq k < 1$ .

Therefore, we conclude that we need only one damage parameter to explain such particular kind of damage. An example of such damage can be illustrated by homogeneous elastic ball which preserves its shape after isotropic elastic damage. This suggests a correlation damage parameter  $d_2$  with the change in the volume of material damage.

More generally, we may state the following

PROPOSITION 4.1.

$$d_1 = 0 \quad \text{if and only if} \quad \frac{\lambda_1}{\lambda_2} = \frac{\lambda}{\mu}.$$

The proof is straightforward.

At this point, concerning the number of damage parameters we need to explain isotropic damage of isotropic virgin material, in general, it is worth to refer to [5, chapter 12.5] (A Model with Two Damage Quantities. The Unilateral Phenomenon) (see p.(321)) where he wrote: "... When damage is produced, microcracks appear in the zones where extensions exist. When changing the sign of the principal deformations by changing the sign of the loading, these microcracks close. On the macroscopic level, the initial stiffness is then recovered: it is the unilateral phenomenon... To take it into account, the model is completed by introducing two damage quantities instead of a single one... they are  $\beta_t$  for extension and  $\beta_c$  for contraction. ..."

In the same chapter his plot of the possible values of these parameters (see [5, fig. 12.2, p. 322]) in some sense resembles our Fig. 1.

## 5. Conclusion

In the present paper the anisotropic elasticity damage mechanics is investigated within the framework of the classical theory of elasticity.

Starting from the principle of strain equivalence, elasticity tensor components are derived in terms of damage parameters in close form.

We show that the tensor of elasticity of damaged material can be determined in two ways. In both cases, we use the resulting expressions for damage tensors. In the case of hexagonal elasticity damage tensor we used both approaches in order to compare them. We show that the derived results are equivalent.

The approaches are algebraically the same and generally simple.

The procedure we present here was applied to several crystal classes which are subjected to hexagonal, orthotropic, tetragonal, cubic and isotropic damage. We underline that this procedure can be applied to all crystal systems.

Finally, we propose a possible procedure to determine damage parameters of isotropic damage, and evaluate them.

**Acknowledgement.** We are grateful to the anonymous referee for very valuable comments and suggestions that help us in improving the paper.

## References

1. A. Cauvin, R. Testa, *Damage mechanics: Basic variables in continuum theories*, Int. J. Solids Struct. **36**(5) (1999), 747–761.
2. C. Chow, J. Wang, *An anisotropic theory of elasticity for continuum damage mechanics*, Int. J. Fract. **33** (1987), 3–16.
3. J. Cordebois, F. Sidoroff, *Damage Induced Elastic Anisotropy*, Colloque Euromech **115**, Villard de Lans, 1979.
4. C.S. Cowin, *Continuum Mechanics of Anisotropic Materials*, Springer-Verlag, New York, 2013.
5. M. Frémond, *Non-Smooth Thermomechanics*, Springer-Verlag, Berlin, Heidelberg, 2002.
6. M.E. Gurtin, *The Linear Theory of Elasticity*, Handbuch der Physik **6a2**, Berlin, Springer-Verlag, 1972.
7. J.P. Jarić, D.S. Kuzmanović, D. Šumarac, *On anisotropic elasticity damage mechanics*, International Journal of Damage Mechanics **22**(7) (2013), 1023–1038.
8. J.W. Ju, *Isotropic and anisotropic damage variables in continuum damage mechanics*, Journal of Engineering Mechanics **116**(12) (1990), 2764–2770.
9. J. Lemaitre, J.-L. Chaboche, *Mechanics of Solid Materials*, Cambridge University Press, 1990.
10. J. Lemaitre, *A Course on Damage Mechanics*, Springer-Verlag, Berlin, Heidelberg, 1992.
11. L.E. Malvern, *Introduction to the Mechanics of a Continuous Medium*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1969.
12. Michael A. Slawinski, *Waves and Rays in Elastic Continua*, World Scientific Publishing Co. Pte. Ltd., Singapore, 2010.
13. G.Z. Voyiadjis, M.A. Yousef, P.I. Kattan, *New tensors for anisotropic damage in continuum damage mechanics*, Journal of Engineering Materials and Tehnology, **134**(2) (2012), 021015, 7 pages.

## О ТЕНЗОРУ ОШТЕЋЕЊА У ЛИНЕАРНОЈ АНИЗОТРОПНОЈ ЕЛАСТИЧНОСТИ

РЕЗИМЕ. У овом раду разматра се анизоотропна линеарна механика оштећења користећи принцип еквивалентне деформације. У претходном раду аутори су одредили компоненте тензора оштећења у затвореном облику преко чланова еластичних параметара неоштећеног материјала. Користећи резултате претходног рада, изводи се еластични тензор такође у затвореном облику, као функција тензора оштећења. Поступак, који се овде излаже, примењује се на кристалним класама које су подвргнуте хексагоналном, ортотропном, тетрагоналном, кубном и изотропном оштећењу. Посебно се разматра изотропан систем као пример указивања на неке могућности израчунавања парамететара његовог оштећења.

University of Belgrade  
Faculty of Mathematics  
Belgrade  
Serbia  
jovojaric@yahoo.com

(Received 06.03.2017.)  
(Revised 15.11.2017.)  
(Available online 07.12.2017.)

University of Belgrade  
Faculty of Transport and Traffic Engineering  
Belgrade  
Serbia  
d.kuzmanovic@sf.bg.ac.rs