

ON GEOMETRIZATION OF MOTION OF SOME NONHOLONOMIC SYSTEMS

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In this note we consider mechanical systems with smooth linear nonholonomic constraints which do not depend on time. For a special case of Chaplygin systems, when the motion of the system can be described as a closed system of differential equations in local coordinates of the reduced space of the same dimension as the dimension of the constraint distribution, we define linear connections for which the equations of motion take the form of equations of geodesic lines. In the case of inertial motion, or motion under influence of potential forces, we give explicit expressions for coefficients of linear connections, in a form much simpler than those given for a general case in [4].

Let us consider mechanical system \mathcal{M} with local coordinates q^i ($i = 1, \dots, n$). It is well known that the configuration space V_n of \mathcal{M} is the Riemannian space with the metric defined by the expression

$$(1) \quad ds^2 = 2T dt^2,$$

where $T = \frac{1}{2} a_{ij}(q^1, \dots, q^n) \dot{q}^i \dot{q}^j$ ($i, j = 1, \dots, n$) is the kinetic energy.

Suppose that a motion is subject to the linear scleronomous nonholonomic constraints

$$(2) \quad \varphi_{(\alpha)i} \dot{q}^i = 0, \quad \varphi_{(\alpha)i} = \varphi_{(\alpha)i}(q^1, \dots, q^n) \quad (\alpha = 1, \dots, k),$$

for which we assume that are independent and nonintegrable. The virtual displacements δq^i of motion satisfy the equations

$$(3) \quad \varphi_{(\alpha)i} \delta q^i = 0,$$

i.e. they belong to an m -dimensional nonholonomic manifold V_n^m ($m = n - k$). Here, the nonholonomic manifold V_n^m on V_n is defined as usual: a vector (η^i)

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belongs to V_n^m at a given point of V_n if and only if it satisfies the relations

$$(4) \quad \varphi_{(\alpha)i} \eta^i = 0$$

for every α at the given point. We can consider $(\varphi_{(\alpha)i})$ ($i = 1, \dots, n$), for every fixed α , as the components of a covector (i.e., one-form) on V_n . Namely, under the change of local coordinates

$$\bar{q}^i = \bar{q}^i(q^1, \dots, q^n),$$

the constraints take the form

$$\bar{\varphi}_{(\alpha)j} \bar{q}^j = 0,$$

where $(\varphi_{(\alpha)i})$ is transformed by the following rule

$$\bar{\varphi}_{(\alpha)j} = \varphi_{(\alpha)i} \frac{\partial q^i}{\partial \bar{q}^j}.$$

At every point of the space V_n , the covectors $(\varphi_{(\alpha)i})$ span a k -dimensional space orthogonal to the nonholonomic manifold V_n^m at that point. By V_n^k we denote the associate collection of all k -dimensional spaces. Under the assumption that the nonholonomic constraints are ideal, the reaction forces (R_i) ($i = 1, \dots, n$) belong to V_n^k since

$$R_i \delta q^i = 0.$$

1. The equations of motion of the considered nonholonomic problem are (e.g, see [5])¹

$$(5) \quad \frac{\delta}{\delta t}(a_{\mu j} \dot{q}^j) + \varphi_{\mu}^{\mu'} \frac{\delta}{\delta t}(a_{\mu' j} \dot{q}^j) = Q_{\mu} + \varphi_{\mu}^{\mu'} Q_{\mu'},$$

$$(j = 1, \dots, n; \quad \mu = 1, \dots, m; \quad \mu' = m + 1, \dots, n)$$

where Q_i ($i = 1, \dots, n$) are the generalised forces of the system \mathcal{M} and the constraints (2) are written in the form

$$\dot{q}^{\mu'} = \varphi_{\mu}^{\mu'} \dot{q}^{\mu}.$$

(From now on small Latin letters denote the indices $1, \dots, n$, Greek letters denote the indices $1, \dots, m = n - k$, and Greek letters with the prime denote the indices $m + 1, \dots, n$.)

We will transform the equations (5) into a form that is more appropriate for further study. For that purpose, we substitute in the kinetic energy the velocities $\dot{q}^{\mu'}$ by the expressions given by nonholonomic constraints. As a result we get

$$(6) \quad 2\tilde{T} = b_{\mu\nu} \dot{q}^{\mu} \dot{q}^{\nu}, \quad b_{\mu\nu} = b_{\mu\nu}(q^1, \dots, q^m),$$

where

$$(7) \quad b_{\mu\nu} = (a_{\mu\nu} + a_{\mu'\nu} \varphi_{\mu}^{\mu'} + a_{\mu\mu'} \varphi_{\nu}^{\mu'} + a_{\mu'\nu'} \varphi_{\mu}^{\mu'} \varphi_{\nu}^{\nu'})_{(\mu,\nu)}$$

(Here by (μ, ν) we denoted the symmetry in indices μ and ν .)

¹Translator's note. Here $\frac{\delta}{\delta t}$ denotes the total (Bianchi) derivative of the Levi-Civita connection of the metric (1).

Now, the equations (5) can be written as

$$(8) \quad \ddot{q}^\tau + \frac{1}{2}b^{\tau\mu} \left(\frac{\partial b_{\mu\nu}}{\partial q^\sigma} + \frac{\partial b_{\sigma\mu}}{\partial q^\nu} - \frac{\partial b_{\nu\sigma}}{\partial q^\mu} + \frac{\partial b_{\mu\nu}}{\partial q^{\sigma'}} \varphi_\sigma^{\sigma'} + \frac{\partial b_{\sigma\mu}}{\partial q^{\nu'}} \varphi_\nu^{\nu'} - \frac{\partial b_{\nu\sigma}}{\partial q^{\mu'}} \varphi_\mu^{\mu'} \right) \dot{q}^\nu \dot{q}^\sigma \\ = b^{\tau\mu} (a_{\nu\mu'} + a_{\nu'\mu'} \varphi_\nu^{\nu'}) \gamma_{\mu\sigma}^{\mu'} \dot{q}^\sigma \dot{q}^\nu + b^{\tau\mu} (Q_\mu + Q_{\mu'} \varphi_\mu^{\mu'}),$$

where $b^{\tau\mu}$ are defined by

$$(9) \quad b_{\sigma\tau} b^{\tau\mu} = \delta_\sigma^\mu$$

and the Hamel coefficients are given by

$$(10) \quad \gamma_{\mu\sigma}^{\mu'} = \frac{\partial \varphi_\mu^{\mu'}}{\partial q^\sigma} - \frac{\partial \varphi_\sigma^{\mu'}}{\partial q^\mu} + \frac{\partial \varphi_\mu^{\mu'}}{\partial q^{\sigma'}} \varphi_\sigma^{\sigma'} - \frac{\partial \varphi_\sigma^{\mu'}}{\partial q^{\sigma'}} \varphi_\mu^{\sigma'}.$$

Let us note that the equation (8) can be obtained by expanding the Voronets equations as well.

If both, the components a_{ij} of the kinetic energy and the coefficients $\varphi_\mu^{\mu'}$ defining the constraints do not depend on $q^{\mu'}$, then also the coefficients $b_{\mu\nu}$ do not depend on $q^{\mu'}$ (so-called Chaplygin systems). In that case we can take a coordinate m -space $L \subset V_n$ with the Riemannian metric defined by²

$$d\sigma^2 = 2\tilde{T} dt^2 = b_{\mu\nu} dq^\mu dq^\nu.$$

Note that L is not a Riemannian subspace of V_n , since the metric $d\sigma^2$ differs from the Riemannian metric induced from ds^2 .

If in addition the generalized forces Q_i do not depend on coordinates $q^{\mu'}$, we can consider the equations (8) as a closed system on L . Then, if one integrates the equations and finds a trajectory $q^\mu(t)$, the motion of the nonholonomic problem can be obtained by solving the additional non-autonomous equations determined by the constraints. Therefore the problem is reduced to the problem of solving and analysis of the equations (8) on the space L . From now on we consider this reduced problem. Let us also note that under the assumption that the system is of the Chaplygin type, the equations (8) and expressions (10) simplify since all derivatives in $q^{\mu'}$ vanish.

After introducing the Christoffel symbols of the Levi-Civita connection associated to the metric $d\sigma^2$

$$[\mu\nu, \sigma] = \left(\frac{\partial b_{\nu\sigma}}{\partial q^\mu} + \frac{\partial b_{\sigma\mu}}{\partial q^\nu} - \frac{\partial b_{\mu\nu}}{\partial q^\sigma} \right); \quad \left\{ \begin{array}{c} \tau \\ \mu\nu \end{array} \right\} = b^{\tau\sigma} [\mu\nu, \sigma],$$

the equations (8) become

$$(11) \quad \ddot{q}^\tau + \left\{ \begin{array}{c} \tau \\ \sigma\nu \end{array} \right\} \dot{q}^\sigma \dot{q}^\nu = \Theta_{\nu\sigma}^\tau \dot{q}^\sigma \dot{q}^\nu + P^\tau,$$

where, in order to simplify the notation, we set

$$(12) \quad \Theta_{\nu\sigma}^\tau = b^{\tau\mu} (a_{\nu\mu'} + a_{\nu'\mu'} \varphi_\nu^{\nu'}) \gamma_{\mu\sigma}^{\mu'}$$

²Translator's note. In the original manuscript it is written V_n^m instead of L in a few places. This obvious typo is corrected in the translation.

and

$$(13) \quad P^\tau = (Q_\mu + Q_{\mu'} \varphi_\mu^{\mu'}) b^{\tau\mu}.$$

It is obvious that the equations (11), for $P^\tau = 0$, are not the equations of the geodesic lines of the metric $d\sigma^2$, except for the case $\gamma_{\mu\sigma}^{\mu'} = 0$ that implies integrability of the constraints. Our aim is to describe linear connections on L for which the equations (11) represent the differential equations of the geodesic lines.

2. Firstly we consider the case $P^\tau = 0$, $\tau = 1, \dots, m$, when we have motion under the inertia

$$(11') \quad \ddot{q}^\tau + \left\{ \begin{matrix} \tau \\ \sigma\nu \end{matrix} \right\} \dot{q}^\sigma \dot{q}^\nu = \Theta_{\nu\sigma}^\tau \dot{q}^\sigma \dot{q}^\nu.$$

Let $L(\Gamma)$ be the space L endowed with a linear connection with the Christoffel symbols $\Gamma_{\mu\nu}^\tau$ and assume that the above equations have the form

$$(14) \quad \ddot{q}^\tau + \Gamma_{\sigma\nu}^\tau \dot{q}^\sigma \dot{q}^\nu = \alpha \dot{q}^\tau,$$

where $\alpha = \alpha(t)$ is a scalar function. By subtracting (14) and (11') we get

$$(15) \quad \left(\Gamma_{\sigma\nu}^\tau - \left\{ \begin{matrix} \tau \\ \sigma\nu \end{matrix} \right\} \right) \dot{q}^\sigma \dot{q}^\nu = \alpha \dot{q}^\tau - \Theta_{\nu\sigma}^\tau \dot{q}^\sigma \dot{q}^\nu.$$

Until now we did not specify the function α and we can take it to be a linear one with respect to the velocities

$$\alpha = \alpha_\nu \dot{q}^\nu,$$

where α_ν are components of a covector. Then, the equations (15) imply

$$\left(\Gamma_{\sigma\nu}^\tau - \left\{ \begin{matrix} \tau \\ \sigma\nu \end{matrix} \right\} - \alpha_\nu \delta_\sigma^\tau + \Theta_{\nu\sigma}^\tau \right) \dot{q}^\sigma \dot{q}^\nu = 0$$

and, because the above relations hold for every \dot{q}^ν , we get

$$(16) \quad \Gamma_{\sigma\nu}^\tau + \Gamma_{\nu\sigma}^\tau - 2 \left\{ \begin{matrix} \tau \\ \sigma\nu \end{matrix} \right\} - \alpha_\nu \delta_\sigma^\tau - \alpha_\sigma \delta_\nu^\tau + \Theta_{\nu\sigma}^\tau + \Theta_{\sigma\nu}^\tau = 0.$$

The system (16) has $\frac{1}{2}m^2(1+m)$ equations for $m^3 + m$ unknown variables: $\Gamma_{\sigma\nu}^\tau$ and α_ν . Since we are interested only in the geodesics equations, without loss of generality we can assume that the space $L(\Gamma)$ is torsion free. Thus we assume that the coefficients of the connection $\Gamma_{\sigma\nu}^\tau$ are symmetric in the lower indices (which does not affect the equations of geodesic lines). Then the number of unknown variables reduces to $\frac{1}{2}m^2(m+1) + m$ and it is still larger than the number of equations. Therefore, from (16) we can determine $\Gamma_{\sigma\nu}^\tau$ as functions of α_ν :

$$(17) \quad \Gamma_{\sigma\nu}^\tau = \left\{ \begin{matrix} \tau \\ \sigma\nu \end{matrix} \right\} + \frac{1}{2}(\alpha_\nu \delta_\sigma^\tau + \alpha_\sigma \delta_\nu^\tau) - \frac{1}{2}(\Theta_{\nu\sigma}^\tau + \Theta_{\sigma\nu}^\tau).$$

It remains to find the coefficients α_ν . The scalar function α in the geodesics equations (14) depends on the parametrisation of the geodesic lines. Let us take

the time t as the canonical parameter. Then $\alpha = \alpha_\nu \dot{q}^\nu = 0$, i.e., $\alpha_\nu = 0$ for all ν , and we obtain the coefficients of the connection

$$(18) \quad \Gamma_{\sigma\nu}^\tau = \left\{ \begin{array}{c} \tau \\ \sigma\nu \end{array} \right\} - \frac{1}{2} b^{\tau\mu} \left((a_{\nu\mu'} + a_{\nu'\mu'} \varphi_\nu^{\nu'}) \gamma_{\mu\sigma}^{\mu'} + (a_{\sigma\mu'} + a_{\nu'\mu'} \varphi_\sigma^{\nu'}) \gamma_{\mu\nu}^{\mu'} \right).$$

Next, note that the above Christoffel symbols define a linear connection. Indeed, let us consider a general coordinate transformation of the form

$$(19) \quad \begin{aligned} \bar{q}^\sigma &= \bar{q}^\sigma(q^1, \dots, q^m), \\ \bar{q}^{\sigma'} &= q^{\sigma'}, \end{aligned}$$

which leaves L and the form of equations of nonholonomic constraints invariant:

$$\dot{q}^{\mu'} = \varphi_{\mu'}^{\mu'} \dot{q}^\mu \quad \Longrightarrow \quad \dot{\bar{q}}^{\mu'} = \bar{\varphi}_{\mu'}^{\mu'} \dot{\bar{q}}^\mu.$$

One can easily prove that the coefficients (18) with respect to the change of local coordinates (19) transform as coefficients of a linear connection:

$$\bar{\Gamma}_{\beta\gamma}^\alpha = \Gamma_{\sigma\nu}^\tau \frac{\partial \bar{q}^\alpha}{\partial q^\tau} \frac{\partial q^\sigma}{\partial \bar{q}^\beta} \frac{\partial q^\nu}{\partial \bar{q}^\gamma} + \frac{\partial^2 q^\sigma}{\partial \bar{q}^\beta \partial \bar{q}^\gamma} \frac{\partial \bar{q}^\alpha}{\partial q^\sigma}.$$

Finally, we can conclude: differential equations of motion of the considered system \mathcal{M} under the inertia are the equations of the geodesic lines on the space with the linear connection $L(\Gamma)$,

$$\frac{\delta^\Gamma \dot{q}^\tau}{\delta t} = \ddot{q}^\tau + \Gamma_{\sigma\mu}^\tau \dot{q}^\sigma \dot{q}^\mu = 0,$$

where the coefficients of the connection are given by (18).

Note that, since

$$\nabla_\sigma^\Gamma b_{\mu\nu} = \frac{1}{2} (a_{\nu\mu'} + a_{\nu'\mu'} \varphi_\nu^{\nu'}) \gamma_{\mu\sigma}^{\mu'} + \frac{1}{2} (a_{\nu\mu'} + a_{\nu'\mu'} \varphi_\nu^{\nu'}) \gamma_{\mu\sigma}^{\mu'},$$

the space $L(\Gamma)$ is not a metric space, defined by the condition $\nabla_\sigma^\Gamma b_{\mu\nu} = 0$, and it is not a semi-metric space, defined by the condition $\nabla_\sigma^\Gamma b_{\mu\nu} = p_\sigma b_{\mu\nu}$, with respect to the metric tensor $b_{\mu\nu}$. On the other hand, the kinetic energy \tilde{T} is conserved along the geodesic lines in $L(\Gamma)$:

$$\begin{aligned} \frac{d\tilde{T}}{dt} &= \frac{\delta^\Gamma \tilde{T}}{\delta t} = \frac{1}{2} \frac{\delta^\Gamma}{\delta t} (b_{\mu\nu} \dot{q}^\mu \dot{q}^\nu) \\ &= \frac{1}{2} \frac{\delta^\Gamma b_{\mu\nu}}{\delta t} \dot{q}^\mu \dot{q}^\nu + b_{\mu\nu} \dot{q}^\mu \frac{\delta^\Gamma \dot{q}^\nu}{\delta t} = \frac{1}{2} (a_{\nu\mu'} + a_{\nu'\mu'} \varphi_\nu^{\nu'}) \gamma_{\mu\sigma}^{\mu'} \dot{q}^\sigma \dot{q}^\mu \dot{q}^\nu = 0. \end{aligned}$$

3. Now we consider the motion of the system \mathcal{M} under the influence of generalised forces

$$Q_i = Q_i(q^1, \dots, q^m)$$

with a potential³

$$V = V(q^1, \dots, q^n).$$

³Translator's note. Here the author implicitly assumed that the partial derivatives $\partial V / \partial q^i$ depend only on q^1, \dots, q^m .

Again, our aim is to find a linear connection $\Gamma_{\sigma\nu}^\tau$, such that the equations of motion

$$(11'') \quad \ddot{q}^\tau + \left\{ \begin{matrix} \tau \\ \sigma\nu \end{matrix} \right\} \dot{q}^\sigma \dot{q}^\nu = \Theta_{\nu\sigma}^\tau \dot{q}^\sigma \dot{q}^\nu - \left(\frac{\partial V}{\partial q^\mu} + \frac{\partial V}{\partial q^{\mu'}} \varphi_\mu^{\mu'} \right) b^{\tau\mu}$$

are the equations of the geodesics lines on $L(\Gamma)$. Analogously to the previous considerations, we get that the Christoffel symbols $\Gamma_{\sigma\nu}^\tau$ satisfy the following identities

$$\begin{aligned} \left(\left\{ \begin{matrix} \tau \\ \sigma\nu \end{matrix} \right\} - \Gamma_{\sigma\nu}^\tau \right) \dot{q}^\sigma \dot{q}^\nu &= (\Theta_{\nu\sigma}^\tau - \alpha_\nu \delta_\sigma^\tau) \dot{q}^\sigma \dot{q}^\nu + P^\tau, \\ P^\tau &= -\frac{\partial V}{\partial q^\mu} b^{\tau\mu} - \frac{\partial V}{\partial q^{\mu'}} \varphi_\mu^{\mu'} b^{\tau\mu}. \end{aligned}$$

If we impose the energy integral (which is well known for the considered non-holonomic problem)

$$\tilde{T} + V = h = \text{const.},$$

the above identities become

$$\left(\left\{ \begin{matrix} \tau \\ \sigma\nu \end{matrix} \right\} - \Gamma_{\sigma\nu}^\tau - \Theta_{\nu\sigma}^\tau + \alpha_\nu \delta_\sigma^\tau - \frac{P^\tau b_{\sigma\nu}}{2(h-V)} \right) \dot{q}^\sigma \dot{q}^\nu.$$

Under the natural assumption that the connection is torsion free, i.e., that coefficients $\Gamma_{\sigma\nu}^\tau$ are symmetric in the lower indices, one gets

$$(20) \quad \Gamma_{\sigma\nu}^\tau = \left\{ \begin{matrix} \tau \\ \sigma\nu \end{matrix} \right\} - \frac{1}{2} (\Theta_{\nu\sigma}^\tau + \Theta_{\sigma\nu}^\tau) + \frac{1}{2} (\alpha_\nu \delta_\sigma^\tau + \alpha_\sigma \delta_\nu^\tau) - \frac{P^\tau b_{\sigma\nu}}{2(h-V)}.$$

By taking the time t as the canonical parameter, we obtain

$$(21) \quad \Gamma_{\sigma\nu}^\tau = \left\{ \begin{matrix} \tau \\ \sigma\nu \end{matrix} \right\} - \frac{1}{2} (\Theta_{\nu\sigma}^\tau + \Theta_{\sigma\nu}^\tau) - \frac{P^\tau b_{\sigma\nu}}{2(h-V)}.$$

Note that the difference of the connection coefficients (20) and (21) equals to the symmetric tensor $\alpha_{(\sigma} \delta_{\nu)}^\tau$.⁴

Apart from the above solution of the problem, we are looking for $L(\Lambda)$, such that the equations of motion of the system with the given energy h are equations of the geodesic lines on $L(\Lambda)$, but now with the canonical parameter σ being the length of a geodesic with respect to the metric $b_{\mu\nu}$. Since

$$\begin{aligned} \dot{q}^\tau &= \frac{dq^\tau}{dt} = \frac{dq^\tau}{d\sigma} \frac{d\sigma}{dt} = \sqrt{2\tilde{T}} \frac{dq^\tau}{d\sigma}, \\ \ddot{q}^\tau &= 2\tilde{T} \frac{d^2 q^\tau}{d\sigma^2} + P_\mu \frac{dq^\mu}{d\sigma} \frac{dq^\tau}{d\sigma}, \quad P_\mu = b_{\mu\tau} P^\tau, \end{aligned}$$

the equations of the geodesic lines (14) (for $\alpha = \alpha_\nu \dot{q}^\nu$), after reparametrisation $d\sigma = \sqrt{2\tilde{T}} dt$, transform to

$$\frac{d^2 q^\tau}{d\sigma^2} + \left(\Gamma_{\sigma\nu}^\tau + \frac{P_\mu}{2\tilde{T}} \delta_\nu^\tau - \alpha_\mu \delta_\nu^\tau \right) \frac{dq^\mu}{d\sigma} \frac{dq^\nu}{d\sigma} = 0.$$

⁴Translator's note. Here, in order for $\Gamma_{\sigma\nu}^\tau$ to be functions on L , the potential V should depend only on q^1, \dots, q^m , which is pointed out below.

If we take α_μ in the following form

$$\alpha_\mu = \frac{P_\mu}{2\tilde{T}} \quad \Longrightarrow \quad \alpha = \frac{1}{2\tilde{T}} \frac{d\tilde{T}}{dt} = \frac{\frac{d^2\sigma}{dt^2}}{\frac{d\sigma}{dt}},$$

the geodesic equations take the form

$$\frac{d^2q^\tau}{d\sigma^2} + \Lambda_{\mu\nu}^\tau \frac{dq^\mu}{d\sigma} \frac{dq^\nu}{d\sigma} = 0,$$

where the Christoffel symbols $\Lambda_{\sigma\nu}^\tau$ are given by

$$\Lambda_{\sigma\nu}^\tau = \left\{ \begin{matrix} \tau \\ \sigma\nu \end{matrix} \right\} - \frac{1}{2}(\Theta_{\nu\sigma}^\tau + \Theta_{\sigma\nu}^\tau) + \frac{1}{4(h-V)}(P_\sigma\delta_\nu^\tau + P_\nu\delta_\sigma^\tau) - \frac{P^\tau b_{\sigma\nu}}{2(h-V)}.$$

Further, for the potentials V that depend only on q^1, \dots, q^m , by an analogy with conservative holonomic systems we can use the action line element $d\Omega^2$ (e.g., see [1]) defined by the relation

$$d\Omega^2 = (h-V)d\sigma^2 = g_{\mu\nu}dq^\mu dq^\nu.$$

Similarly to the derivations given in [1], the equations of motion with the given energy h of the considered nonholonomic system can be written in the form⁵

$$\frac{d^2q^\tau}{d\Omega^2} + \left(\left\{ \begin{matrix} \tau \\ \sigma\nu \end{matrix} \right\}_g - \Theta_{\nu\sigma}^\tau \right) \frac{dq^\sigma}{d\Omega} \frac{dq^\nu}{d\Omega} = 0.$$

They are equations of the geodesic lines

$$\frac{d^2q^\tau}{d\Omega^2} + \Lambda_{\sigma\nu}^\tau \frac{dq^\sigma}{d\Omega} \frac{dq^\nu}{d\Omega} = 0,$$

for the torsion-free connection with the coefficients given by

$$\Lambda_{\sigma\nu}^\tau = \left\{ \begin{matrix} \tau \\ \sigma\nu \end{matrix} \right\}_g - \frac{1}{2}(\Theta_{\sigma\nu}^\tau + \Theta_{\nu\sigma}^\tau).$$

At the end, we note again that all connections are constructed on the reduced m -dimensional space L .

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⁵Translator's note. Here $\left\{ \begin{matrix} \tau \\ \sigma\nu \end{matrix} \right\}_g$ are the Christoffel symbols of the Levi-Civita connection associated to the metric $d\Omega^2$.

О ГЕОМЕТРИЗАЦИЈИ КРЕТАЊА НЕКИХ НЕХОЛОНОМНИХ СИСТЕМА

РЕЗИМЕ. Овде се разматрају механички системи чије кретање ограничавају глатке склерономне нехолономне везе, линеарне у односу на генералисане брзине. За специјални случај, кад је кретање система могуће описати диференцијалним једначинама са реалним променљивим чији је број једнак броју једначина кретања (тз. Чаплигинови системи), налази се простор са линеарном повезаношћу у коме су једначине кретања изражене у облику диференцијалних једначина геодезијских линија.

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