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SMALL OSCILLATIONS OF AN IDEAL LIQUID CONTAINED IN A VESSEL CLOSED BY AN ELASTIC CIRCULAR PLATE, IN UNIFORM ROTATION

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Abstract. The problem of the small oscillations of an ideal liquid contained in a vessel in uniform rotation has been studied by Kopachevskii and Krein in the case of an entirely rigid vessel [3]. We propose here, a generalization of this model by considering the case of a vessel closed by an elastic circular plate.

In this context, the linearized equations of motion of the system plate-liquid are derived. Functional analysis is used to obtain a variational equation of the small amplitude vibrations of the coupled system around its equilibrium position, and then two operatorial equations in a suitable Hilbert space are presented and analyzed. We show that the spectrum of the system is real and consists of a countable set of eigenvalues and an essential continuous spectrum filling an interval. Finally the existence and uniqueness theorem for the solution of the associated evolution problem is proved by means the semi-groups theory.

1. Introduction

When an elastic disk rotates uniformly in its own plane, in relative equilibrium, the principal tensions are along and perpendicular to the radius vector. If a is the radius, ρ_0 the density and ω_0 the angular velocity of rotation, their values at the distance r from the centre are [1,4,5]:

$$\sigma_{rr} \overset{\text{def}}{=} P = A(a^2 - r^2)\rho_0\omega_0^2; \qquad \sigma_{\theta\theta} \overset{\text{def}}{=} Q = (Aa^2 - Br^2)\rho_0\omega_0^2$$

with

$$A = \frac{3+\sigma}{8}; \qquad B = \frac{1+3\sigma}{8},$$

σ denoting Poisson's ratio.

Since $0 < \sigma < \frac{1}{2}$, we have $A > B$ and $P \geqslant 0$, $Q > 0$. These values of P and Q are unaffected by a small displacement w normal to the plane of the disk.

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In the following, we are going to study the transverse vibrations of the disk.

The normal displacement w is a function of the polar coordinates (r, θ) and of the time t: $w(r, \theta, t)$.

The potential energy of the centrifugal forces is [4, 5]:

$$V_2 = \iint \frac{h}{2}\left[P\Big(\frac{\partial w}{\partial r}\Big)^2 + \frac{Q}{r^2}\Big(\frac{\partial w}{\partial \theta}\Big)^2\right] r \,\mathrm{d}r\,\mathrm{d}\theta,$$

where h is the thickness of the disk and the integration extends to the area of the disk.

The potential energy of flexure is [4]:

$$V_1 = \frac{Eh^3}{24(1-\sigma^2)} \iint \Bigg\{\Big(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\Big)^2 - 2(1-\sigma)\left[\frac{\partial^2 w}{\partial r^2}\Big(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\Big) - \left[\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial w}{\partial r}\Big)^2\right]\right]\Bigg\} r\,\mathrm{d}r\,\mathrm{d}\theta,$$

where E is Young's modulus.

The kinetic energy of the plate is

$$\mathscr{C} = \frac{\rho_0 h}{2} \iint \Big(\frac{\partial w}{\partial t}\Big)^2 r\,\mathrm{d}r\,\mathrm{d}\theta$$

We denote by $q(r, \theta, t)$ the density of normal external forces acting on the disk.

Using Hamilton's principle

$$\int_{t_1}^{t_2}\left[\delta\mathscr{C} - \delta V_1 - \delta V_2 + \iint q\delta w r\,\mathrm{d}r\,\mathrm{d}\theta\right]\mathrm{d}t = 0,$$

we obtain the equation of the transverse vibrations of the plate [1, 4]:

$$\rho_0 h \frac{\partial^2 w}{\partial t^2} = \frac{h}{r}\frac{\partial}{\partial r}\Big(Pr\frac{\partial w}{\partial r}\Big) + \frac{hQ}{r^2}\frac{\partial^2 w}{\partial \theta^2} - D\Delta\Delta w + q,$$

where $D = \frac{Eh^3}{24(1-\sigma^2)}$ is the flexural rigidity.

If the boundary of the plate is clamped, we must take

$$w = 0, \quad \frac{\partial w}{\partial r} = 0 \quad \text{for} \quad r = a.$$

2. Position of the problem and equations of motion

We consider an axisymmetrical container closed by a circular plate Γ (density ρ_0, radius a, thickness h), that has the same vertical axis Ox_3 than the container, and is clamped at the wall of the vessel. O is the center of the plate (Figure 1). The container contains a heavy, incompressible, inviscid liquid (density ρ) in contact with the plate; we call S the part of the wall of the vessel that is wetted by the liquid. The system container-liquid rotates with the constant angular velocity ω_0 about Ox_3 (Figure 1). We study, in linear theory, the small oscillations of the system liquid-plate about its equilibrium position, assuming that the liquid remains in contact with the container and the plate and the vibrations of the plate are transverse.

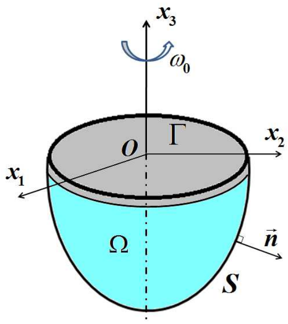

FIGURE 1. Model of the system

Let $\vec{u}(r,\theta,x_3,t)$ be the small displacement of a particle M of the liquid with respect to its relative equilibrium position: $(r,\theta,x_3$ are the cylindrical coordinates of M in a system of axes rotating about Ox_3 with the angular velocity ω_0).

Taking into account of Coriolis theorem, Euler's equation can be written, if $\mathcal{P}$ is the pressure of the liquid in M:

$$\ddot{\vec{u}} + 2\omega_0\vec{x}_3 \times \dot{\vec{u}} + \omega_0\vec{x}_3 \times \left(\omega_0\vec{x}_3 \times \overrightarrow{OM}\right) = -\frac{1}{\rho}\overrightarrow{\mathrm{grad}}\,\mathcal{P} - g\vec{x}_3,$$

where g is the acceletation due to the gravity, or

$$\ddot{\vec{u}} - 2\omega_0\dot{\vec{u}} \times \vec{x}_3 = -\frac{1}{\rho}\overrightarrow{\mathrm{grad}}\Big(\mathcal{P} + \rho g x_3 - \frac{\rho\omega_0^2}{2}r^2\Big). \tag{2.1}$$

The pressure in the relative equilibrium position is

$$P_{\mathrm{st}} = p_0 - \rho g x_3 + \tfrac{1}{2}\rho\omega_0^2 r^2,$$

p_0 being the liquid pressure in O. Introducing the dynamic pressure $p = \mathcal{P} - \mathcal{P}_{\mathrm{st}}$, we have, instead of (2.1), the equation

$$\ddot{\vec{u}} - 2\omega_0\dot{\vec{u}} \times \vec{x}_3 = -\frac{1}{\rho}\overrightarrow{\mathrm{grad}}\,p \quad \text{in} \quad \Omega. \tag{2.2}$$

We must add the incompressibility condition, i.e., after integration

$$\mathrm{div}\vec{u} = 0 \quad \text{in} \quad \Omega \tag{2.3}$$

and the boudary condition on S

$$u_{n|S} \stackrel{\mathrm{def}}{=} \vec{u}\cdot\vec{n}_{|S} = 0, \tag{2.4}$$

$\vec{n}$ being the external normal unit vector to S.

We call $w_0(r,\theta)$ the normal displacement of the plate in relative equilibrium position with respect to the planar position; we have, p_a being the constant atmospheric pressure:

$$0 = \frac{h}{r}\frac{\partial}{\partial r}\Big(Pr\frac{\partial w_0}{\partial r}\Big) + \frac{hQ}{r^2}\frac{\partial^2 w_0}{\partial r^2} - D\Delta\Delta w_0 - p_a + p_0 - \rho g w_0 + \frac{\rho\omega_0^2}{2}r^2$$

and the boundary conditions

$$w_0 = 0, \quad \frac{\partial w_0}{\partial r} = 0 \quad \text{for} \quad r = a.$$

We call now $w(r,\theta,t)$ the normal displacement of the plate with respect to its planar position; we have:

$$\rho_0 h \frac{\partial^2 w}{\partial t^2} = \frac{h}{r}\frac{\partial}{\partial r}\Big(Pr\frac{\partial w}{\partial r}\Big) + \frac{hQ}{r^2}\frac{\partial^2 w}{\partial \theta^2} - D\Delta\Delta w - p_a + \mathcal{P}_{\text{plate}}$$

and the boundary conditions

$$w = 0, \quad \frac{\partial w}{\partial r} = 0 \quad \text{for} \quad r = a.$$

Setting $\zeta(r,\theta,t) = w - w_0$, that is the displacement of a point of the plate with respect to its relative equilibrium position, we obtain by substracting and using the formula

$$\mathcal{P}_{\text{plate}} = p_{|\Gamma} + p_0 - \rho g w + \tfrac{1}{2}\rho\omega_0^2 r^2,$$

the equation

$$\rho_0 h\ddot{\zeta} = \frac{h}{r}\frac{\partial}{\partial r}\Big(Pr\frac{\partial \zeta}{\partial r}\Big) + \frac{hQ}{r^2}\frac{\partial^2 \zeta}{\partial \theta^2} - D\Delta\Delta\zeta + p_{|\Gamma} - \rho g\zeta. \tag{2.5}$$

We must add

$$u_{n|\Gamma} \overset{\text{def}}{=} \vec{u}\cdot\vec{n}_{|\Gamma} = \zeta, \tag{2.6}$$

$$\zeta = 0, \quad \frac{\partial \zeta}{\partial r} = 0 \quad \text{for} \quad r = a, \tag{2.7}$$

$$\zeta 2\pi \text{ periodic in } \theta, \tag{2.8}$$

$$\int_\Gamma \zeta \,\mathrm{d}\Gamma = 0,$$

the last equation expressing that the volume of the liquid remains constant.

We are going to deduce from the equation (2.5) an integral relation that will be useful in the following.

Multiplying the equation (2.5) by $\bar{\tilde{\zeta}}$, $\tilde{\zeta}$ being a sufficiently regular function, 2π-periodic in θ and verifying $\tilde{\zeta} = 0$, $\frac{\partial\tilde{\zeta}}{\partial r} = 0$ on $r = a$, and integrating on Γ, we obtain

$$\int_\Gamma (p_{|\Gamma} - \rho_0 h\ddot{\zeta})\bar{\tilde{\zeta}}\,\mathrm{d}\Gamma = \int_\Gamma \Big[-\frac{h}{r}\frac{\partial}{\partial r}\Big(Pr\frac{\partial\zeta}{\partial r}\Big) - \frac{hQ}{r^2}\frac{\partial^2\zeta}{\partial\theta^2} + D\Delta\Delta\zeta + \rho g\zeta\Big]\bar{\tilde{\zeta}}\,\mathrm{d}\Gamma.$$

In Gauss formula

$$\iint_\Gamma (u\Delta v - v\Delta u)\mathrm{d}\Gamma = \int_{\partial\Gamma}\Big(\frac{\partial u}{\partial n}v - \frac{\partial v}{\partial n}u\Big)\mathrm{d}(\partial\Gamma),$$

we take $u = \Delta\zeta$, $v = \bar{\tilde{\zeta}}$, and we obtain

$$\int_\Gamma \Delta\Delta\zeta\cdot\bar{\tilde{\zeta}}\,\mathrm{d}\Gamma = \int_\Gamma \Delta\zeta\cdot\Delta\bar{\tilde{\zeta}}\,\mathrm{d}\Gamma.$$

Integrating by parts the other integrals, we deduce from (2.5) the equation

$$(2.9)\qquad \int_\Gamma (p_{|\Gamma} - \rho_0 h\ddot{\zeta})\bar{\tilde{\zeta}}\,\mathrm{d}\Gamma = \int_\Gamma \Big[h\Big(P\zeta_r\bar{\tilde{\zeta}}_r + \frac{Q}{r^2}\zeta_\theta\bar{\tilde{\zeta}}_\theta\Big) + D\Delta\zeta\cdot\Delta\bar{\tilde{\zeta}} + \rho g\zeta\bar{\tilde{\zeta}}\Big]\mathrm{d}\Gamma,$$

where

$$\zeta_r = \frac{\partial\zeta}{\partial r},\quad \zeta_\theta = \frac{\partial\zeta}{\partial\theta}.$$

Now, we assume that

$$\zeta, \tilde{\zeta} \in \tilde{H}_0^2(\Gamma) = \Big\{\zeta \in H^2(\Gamma);\ \zeta = 0;\ \frac{\partial\zeta}{\partial r} = 0 \text{ on } \partial\Gamma,\ \int_\Gamma \zeta\,\mathrm{d}\Gamma = 0\Big\}$$

and we set

$$b(\zeta, \tilde{\zeta}) = \int_\Gamma \Big[h\Big(P\zeta_r\bar{\tilde{\zeta}}_r + \frac{Q}{r^2}\zeta_\theta\bar{\tilde{\zeta}}_\theta\Big) + D\Delta\zeta\cdot\Delta\bar{\tilde{\zeta}} + \rho g\zeta\bar{\tilde{\zeta}}\Big]\mathrm{d}\Gamma.$$

It is a hermitian sesquilinear form on $H_0^2(\Gamma)$.

It is known [9] that, on $\tilde{H}_0^2(\Gamma)$, $(\int_\Gamma |\Delta\zeta|^2\,\mathrm{d}\Gamma)^{1/2}$ defines a norm that is equivalent to the classical norm of $H^2(\Gamma)$.

On the other hand, since $P \geqslant 0$, $Q > 0$, we have

$$0 \leqslant \int_\Gamma \Big(P|\zeta_r|^2 + \frac{Q}{r^2}|\zeta_\theta|^2\Big)\mathrm{d}\Gamma \leqslant C_1\|\zeta\|^2_{H^1(\Gamma)} \quad (C_1 > 0).$$

Then, $b(\zeta, \tilde{\zeta})$ is continuous and coercive on $\tilde{H}_0^2(\Gamma)$.

On the other hand, the embedding $\tilde{H}_0^2(\Gamma) \subset \tilde{L}^2(\Gamma) = \{f \in L^2(\Gamma), \int_\Gamma f\,\mathrm{d}\Gamma = 0\}$ is obviously continuous and compact.

Therefore, there exists an unbounded operator $\mathcal{B}$ of $\tilde{L}^2(\Gamma)$, selfadjoint, strongly positive, with compact, positive definite, selfadjoint inverse, such that

$$b(\zeta, \tilde{\zeta}) = (\mathcal{B}\zeta, \tilde{\zeta})_{\tilde{L}^2(\Gamma)} \quad \forall\,\zeta \in D(\mathcal{B}),\ \forall\,\tilde{\zeta} \in \tilde{H}_0^2(\Gamma).$$

Since p appears in Euler's equation by its gradient, we assume that $p \in H^1(\Omega)$; besides, p being indeterminate to the extend of an additive function of the time, we can suppose $p_{|\Gamma} \in \tilde{L}^2(\Gamma)$.

Then, the variational equation (2.9), that can be written

$$b(\zeta, \tilde{\zeta}) = (p_{|\Gamma} - \rho_0 h\ddot{\zeta}, \tilde{\zeta})_{\tilde{L}^2(\Gamma)} \quad \forall\tilde{\zeta} \in \tilde{H}_0^2(\Gamma)$$

is equivalent [6] to the equation

$$(2.10)\qquad p_{|\Gamma} - \rho_0 h\ddot{\zeta} = \mathcal{B}\zeta.$$

Finally the equations of the problem are (2.2)–(2.4), (2.6)–(2.8), (2.10).

3. Operatorial equations of the problem

3.1. *We recall Weyl's orthogonal decomposition*

$$\mathscr{L}^2(\Omega) \overset{\text{def}}{=} [L^2(\Omega)]^3 = J_0(\Omega) \oplus \mathscr{G}(\Omega);$$
$$J_0(\Omega) = \{\vec{v} \in \mathscr{L}^2(\Omega);\ \operatorname{div}\vec{v} = 0;\ v_{n|\partial\Omega} = 0\};$$
$$\mathscr{G}(\Omega) = \{\overrightarrow{\operatorname{grad}}\, q;\ q \in H^1(\Omega)\}.$$

Since $\operatorname{div} \vec{u} = 0$ in Ω, $u_{n|S} = 0$, we seek $\vec{u}$ in the space

$$J_{0,S}(\Omega) \stackrel{\text{def}}{=} \left\{ \vec{u} \in \mathscr{L}^2(\Omega);\ \operatorname{div} \vec{u} = 0;\ u_{n|S} = 0 \right\}.$$

By virtue of the orthogonal decomposition [3] $J_{0,S}(\Omega) = J_0(\Omega) \oplus \mathscr{G}_{h,S}(\Omega)$, we write $\vec{u}$ in the form

$$\vec{u} = \vec{v} + \vec{U} : \begin{cases} \vec{v} \in J_0(\Omega) \\ \vec{U} \in \mathscr{G}_{h,S}(\Omega) = \left\{ \vec{U} = \overrightarrow{\operatorname{grad}}\, \Phi;\ \Phi \in \tilde{H}^1(\Omega);\ \Delta\Phi = 0 \text{ in } \Omega;\ \frac{\partial \Phi}{\partial n}|_S = 0 \right\}; \end{cases}$$

with

$$\tilde{H}^1(\Omega) = \left\{ \Phi \in H^1(\Omega),\ \int_\Omega \Phi \, \mathrm{d}\Omega = 0 \right\}.$$

The Euler's equation (2.2) can be written

$$\ddot{\vec{v}} + \ddot{\vec{U}} - 2\omega_0 \big(\dot{\vec{v}} + \dot{\vec{U}}\big) \times \vec{x}_3 + \frac{1}{\rho} \overrightarrow{\operatorname{grad}}\, p = 0. \tag{3.1}$$

We introduce a function $\tilde{\vec{u}}(x_1, x_2, x_3) \in J_{0,S}(\Omega)$ and its analogous decomposition $\tilde{\vec{u}} = \tilde{\vec{v}} + \tilde{\vec{U}}$. We have

$$\int_\Omega \rho \big(\ddot{\vec{v}} + \ddot{\vec{U}}\big) \cdot \big(\tilde{\vec{v}} + \tilde{\vec{U}}\big) \mathrm{d}\Omega - 2\rho\omega_0 \int_\Omega \left[\big(\dot{\vec{v}} + \dot{\vec{U}}\big) \times \vec{x}_3 \right] \cdot \big(\tilde{\vec{v}} + \tilde{\vec{U}}\big) \mathrm{d}\Omega$$
$$+ \int_\Omega \overrightarrow{\operatorname{grad}}\, p \cdot \tilde{\vec{u}} \, \mathrm{d}\Omega = 0.$$

The Green formula gives

$$\int_\Omega \overrightarrow{\operatorname{grad}}\, p \cdot \tilde{\vec{u}} \, \mathrm{d}\Omega = \int_\Gamma p_{|\Gamma} \tilde{\vec{u}}_{n|\Gamma} \mathrm{d}\Gamma = \int_\Gamma (\mathcal{B}\zeta + \rho_0 h \ddot{\zeta}) \tilde{\zeta} \, \mathrm{d}\Gamma = b(\zeta, \tilde{\zeta}) + \rho_0 h \int_\Gamma \ddot{\zeta} \tilde{\zeta} \, \mathrm{d}\Gamma,$$

so that the precedent equation becomes

$$\int_\Omega \rho \big(\ddot{\vec{v}} \cdot \tilde{\vec{v}} + \ddot{\vec{U}} \cdot \tilde{\vec{U}}\big) \mathrm{d}\Omega - 2\rho\omega_0 \int_\Omega \left[\big(\dot{\vec{v}} + \dot{\vec{U}}\big) \times \vec{x}_3 \right] \cdot \tilde{\vec{v}} \, \mathrm{d}\Omega$$
$$- 2\rho\omega_0 \int_\Omega \left[\big(\dot{\vec{v}} + \dot{\vec{U}}\big) \times \vec{x}_3 \right] \cdot \tilde{\vec{U}} \, \mathrm{d}\Omega + b(\zeta, \tilde{\zeta}) + \rho_0 h \int_\Gamma \ddot{\zeta} \tilde{\zeta} \, \mathrm{d}\Gamma = 0$$

$\vec{U}$ and $\overrightarrow{\operatorname{grad}}\, p$ being orthogonal to $J_0(\Omega)$, (3.1) gives

$$\int_\Omega \rho \ddot{\vec{v}} \cdot \tilde{\vec{v}} \, \mathrm{d}\Omega - 2\rho\omega_0 \int_\Omega \left[\big(\dot{\vec{v}} + \dot{\vec{U}}\big) \times \vec{x}_3 \right] \cdot \tilde{\vec{v}} \, \mathrm{d}\Omega = 0,$$

so that, since $\zeta = u_{n|\Gamma} = v_{n|\Gamma} + U_{n|\Gamma} = U_{n|\Gamma}$, the precedent variational equation becomes

$$\int_\Omega \rho \ddot{\vec{U}} \cdot \tilde{\vec{U}} \, \mathrm{d}\Omega - 2\rho\omega_0 \int_\Omega \big(\dot{\vec{v}} \times \vec{x}_3\big) \cdot \tilde{\vec{U}}$$
$$- 2\rho\omega_0 \int_\Omega \big(\dot{\vec{U}} \times \vec{x}_3\big) \cdot \tilde{\vec{U}} \, \mathrm{d}\Omega$$
$$+ b(U_{n|\Gamma}, \tilde{U}_{n|\Gamma}) + \rho_0 h \int_\Gamma \ddot{U}_{n|\Gamma} \tilde{U}_{n|\Gamma} \, \mathrm{d}\Gamma = 0, \tag{3.2}$$

that has only one term containing $\vec{v}$.

3.2.1. *Let us introduce the space*

$$\mathcal{V} = \left\{\vec{U} = \overrightarrow{\text{grad}}\,\Phi;\ \Phi \in \tilde{H}^1(\Omega);\ U_{n|S} = 0;\ U_{n|\Gamma} \in \tilde{H}^2_0(|\Gamma)\right\};$$

equipped with the norm defined by

$$\|\vec{U}\|^2_{\mathcal{V}} = \int_\Omega |\vec{U}|^2 \mathrm{d}\Omega + \|U_{n|\Gamma}\|^2_2,$$

where $\|\cdot\|_2$ is the classical norm of $\tilde{H}^2_0(\Gamma)$, and χ completion of $\mathscr{V}$ for the norm associated to the scalar product

$$\left(\vec{U}, \tilde{\vec{U}}\right)_\chi = \int_\Omega \vec{U}\cdot\bar{\tilde{\vec{U}}}\,\mathrm{d}\Omega + \frac{\rho_0 h}{\rho}\int_\Gamma U_{n|\Gamma}\bar{\tilde{U}}_{n|\Gamma}\,\mathrm{d}\Gamma$$

3.2.2. *Now, we introduce a few operators.* Let P_0 the orthogonal projector from $\mathscr{L}^2(\Omega)$ into $J_0(\Omega)$; we set:

$$P_0(\vec{v}\times\vec{x}_3) = iA_{11}\vec{v};\quad A_{11} \text{ bounded from } J_0(\Omega) \text{ into } J_0(\Omega);$$

$$P_0(\vec{U}\times\vec{x}_3) = iA_{12}\vec{U};\quad A_{12} \text{ bounded from } \chi \text{ into } J_0(\Omega).$$

Applying P_0 to the equation (3.1), we obtain

$$\ddot{\vec{v}} - 2\omega_0 P_0\left[\left(\dot{\vec{v}} + \dot{\vec{U}}\right)\times\vec{x}_3\right] = 0,$$

i.e., the operatorial equation

$$\ddot{\vec{v}} - 2i\omega_0 A_{11}\vec{v} - 2i\omega_0 A_{12}\vec{U} = 0. \tag{3.3}$$

Since we have

$$\left|\int_\Omega (\vec{v}\times\vec{x}_3)\cdot\bar{\tilde{\vec{U}}}\,\mathrm{d}\Omega\right| \leqslant c\|\vec{v}\|_{J_0(\Omega)}\|\tilde{\vec{U}}\|_\chi \quad (c>0),$$

we can set

$$\int_\Omega (\vec{v}\times\vec{x}_3)\cdot\bar{\tilde{\vec{U}}}\,\mathrm{d}\Omega = \left(iA_{21}\vec{v}, \tilde{\vec{U}}\right)_\chi,\quad A_{21} \text{ bounded from } J_0(\Omega) \text{ into } \chi,$$

and, in the same manner

$$\int_\Omega (\vec{U}\times\vec{x}_3)\cdot\bar{\tilde{\vec{U}}}\,\mathrm{d}\Omega = \left(iA_{22}\vec{U}, \tilde{\vec{U}}\right)_\chi,\quad A_{22} \text{ bounded from } \chi \text{ into } \chi.$$

3.2.3. *We are going to give a few properties of these operators.*

- A_{11} is the well-known Coriolis operator [3]. It is selfadjoint; $\|A_{11}\| = 1$; its spectrum $\sigma(A_{11})$ coincides with its essential spectrum $\sigma_{ess}(A_{11})$ and fills the closed interval $[-1,1]$.
- A_{21} and A_{12} are mutually adjoint. Indeed, we have

$$\begin{aligned}(A_{12}\vec{U},\vec{v})_{J_0(\Omega)} &= -i\int_\Omega (\vec{U}\times\vec{x}_3)\cdot\bar{\vec{v}}\,\mathrm{d}\Omega = i\int_\Omega (\bar{\vec{v}}\times\vec{x}_3)\cdot\vec{U}\,\mathrm{d}\Omega \\ &= \overline{-i\int_\Omega (\vec{v}\times\vec{x}_3)\cdot\bar{\vec{U}}\,\mathrm{d}\Omega} = \overline{(A_{21}\vec{v},\vec{U})_\chi} = (\vec{U}, A_{21}\vec{v})_\chi.\end{aligned}$$

- A_{22} is selfadjoint.
- We have $\|A_{12}\| = \|A_{21}\| \leqslant 1$, $\|A_{22}\| \leqslant 1$.

3.3. *Then, the variational equation* (3.2) *can be written*

$$(3.4)\qquad \big(\ddot{\vec{U}},\tilde{\vec{U}}\big)_\chi - 2i\omega_0\big(\big[A_{21}\dot{\vec{v}} + A_{22}\dot{\vec{U}}\big],\tilde{\vec{U}}\big)_\chi + \frac{1}{\rho}b\big(U_{n|\Gamma},\tilde{U}_{n|\Gamma}\big) = 0, \quad \forall \tilde{\vec{U}} \in \mathcal{V}.$$

We are going to deduce from (3.4) the second operatorial equation of the problem.

a). The embedding $\mathcal{V} \subset \chi$ is obviously dense and continuous, let us prove that it is compact.

Since $\vec{U} \in \mathcal{V}$, we can write

$$\vec{U} = \overrightarrow{\mathrm{grad}}\,\Phi; \quad \Phi \in \tilde{H}^1(\Omega); \quad \Delta\Phi = 0 \ \text{ in } \ \Omega; \quad \frac{\partial\Phi}{\partial n}|_S = 0; \quad \frac{\partial\Phi}{\partial n}|_\Gamma \in \tilde{H}^2_0(\Gamma).$$

Let us consider [8] the Newmann problem

$$-\Delta\Phi = 0 \ \text{ in } \ \Omega; \quad \frac{\partial\Phi}{\partial n}|_S = 0; \quad \frac{\partial\Phi}{\partial n}|_\Gamma = \eta \in \tilde{H}^2_0(\Gamma).$$

For each $\Psi \in H^1(\Omega)$, we have, by virtue of Green's formula:

$$0 = -\int_\Omega \Delta\Phi \cdot \bar{\Psi}\,\mathrm{d}\Omega = \int_\Omega \overrightarrow{\mathrm{grad}}\,\Phi \cdot \overrightarrow{\mathrm{grad}}\,\bar{\Psi}\,\mathrm{d}\Omega - \int_\Gamma \eta\bar{\Psi}|_\Gamma\,\mathrm{d}\Gamma.$$

The compatibility condition for η being $\int_\Gamma \eta\,\mathrm{d}\Gamma = 0$, the precedent equation is unchanged if we add a constant to Ψ and we have

$$\int_\Omega \overrightarrow{\mathrm{grad}}\,\Phi \cdot \overrightarrow{\mathrm{grad}}\,\bar{\Psi}\,\mathrm{d}\Omega = \int_\Gamma \eta\bar{\Psi}|_\Gamma\,\mathrm{d}\Gamma \quad \forall\,\Psi \in \tilde{H}^1(\Omega).$$

Then, setting $\Psi = \Phi$, we have

$$\int_\Omega \big|\overrightarrow{\mathrm{grad}}\,\Phi\big|^2\mathrm{d}\Omega = \int_\Gamma \eta\bar{\Phi}|_\Gamma\,\mathrm{d}\Gamma,$$

and, by a trace theorem

$$\int_\Omega \big|\overrightarrow{\mathrm{grad}}\,\Phi\big|^2\mathrm{d}\Omega \leqslant C\|\eta\|_{\tilde{L}^2(\Gamma)}\|\Phi\|_{\tilde{H}^1(\Omega)} \qquad (C > 0).$$

By virtue of the Poincaré inequality, if $\Phi \in \tilde{H}^1(\Omega)$, $\|\Phi\|_{\tilde{H}^1(\Omega)}$ and $\|\overrightarrow{\mathrm{grad}}\,\Phi\|_{\mathscr{L}^2(\Omega)}$ are equivalent norms, so that we have

$$\Big(\int_\Omega |\overrightarrow{\mathrm{grad}}\,\Phi|^2\mathrm{d}\Omega\Big)^{1/2} \leqslant C'\|\eta\|_{\tilde{L}^2(\Gamma)} \qquad (C' > 0)$$

and then

$$\int_\Omega |\vec{U}|^2\mathrm{d}\Omega \leqslant \big(C'\|U_{n|\Gamma}\|_{\tilde{L}^2(\Gamma)}\big)^2 \qquad \forall\,\vec{U} \in \mathcal{V}$$

and, finally

$$\|\vec{U}\|^2_\chi \leqslant C''\|U_{n|\Gamma}\|^2_{\tilde{L}^2(\Gamma)} \qquad \forall\,\vec{U} \in \mathcal{V} \quad (C'' > 0).$$

Then, let a sequence $\{\vec{U}^p\}$ weakly convergent in $\mathcal{V}$ to $\vec{U}^* \in \mathcal{V} \subset \chi$.

Since $\vec{U}^p - \vec{U}^* \in \mathcal{V}$, we have

$$\|\vec{U}^p - \vec{U}^*\|^2_\chi \leqslant C''\|U^p_{n|\Gamma} - |U^*_{n|\Gamma}\|^2_{\tilde{L}^2(\Gamma)}.$$

The sequence $\{U^p_{n|_\Gamma}\}$ converges weakly in $\tilde{H}^2_0(\Gamma)$, thus strongly in $\tilde{L}^2(\Gamma)$, so that $\|\vec{U}^p - \vec{U}^*\|_\chi \to 0$ when $p \to \infty$ and the embedding $\mathcal{V} \subset \chi$ is compact.

b). *We remark that, by virtue of the definition of* $\|\cdot\|_\mathcal{V}$, we have

$$\|\vec{U}\|^2_\mathcal{V} \geqslant \|U_{n|_\Gamma}\|^2_2.$$

On the other hand, we have

$$\int_\Omega |\vec{U}|^2 \mathrm{d}\Omega \leqslant C' \|U_{n|_\Gamma}\|^2_{\tilde{L}^2(\Gamma)} \leqslant C' \|U_{n|_\Gamma}\|^2_2,$$

so that

$$\|\vec{U}\|^2_\mathcal{V} = \int_\Omega |\vec{U}|^2 \mathrm{d}\Omega + \|U_{n|\Gamma}\|^2_2 \leqslant (C'+1)\|U_{n|_\Gamma}\|^2_2.$$

Consequently, $\|U_{n|_\Gamma}\|_2$ defines a norm that is equivalent to $\|\vec{U}\|_\mathcal{V}$. Then, $b(U_{n|\Gamma}, \tilde{U}_{n|\Gamma})$ is a coercive, continuous, hermitian sesquilinear form on $\mathcal{V}$.

Therefore, to $\frac{1}{\rho} b(U_{n|\Gamma}, \tilde{U}_{n|\Gamma})$ and the pair $(\mathcal{V}, \chi)$, we can associate an unbounded operator Q_0 of χ, that is selfadjoint and strongly positive and that has a compact, positive definite, selfadjoint inverse.

So, the variational equation (3.4) is equivalent [6] to the operatorial equation

$$\ddot{\vec{U}} - 2i\omega_0 \big(A_{21}\dot{\vec{v}} + A_{22}\dot{\vec{U}}\big) + Q_0 \vec{U} = 0. \tag{3.5}$$

The equations (3.3) and (3.5), where $\vec{v} \in J_0(\Omega)$ and $\vec{U} \in \mathcal{V}$ are the operatorial equations of the problem. By means of these equations, we are going study the spectrum of the problem.

4. The spectrum of the problem is real

Let us seeking the solutions depending on time according to the law $e^{i\omega t}$; we have

$$\begin{cases} -\omega^2 \vec{v} + 2\omega_0\omega A_{11}\vec{v} + 2\omega_0\omega A_{12}\vec{U} = 0 \\ -\omega^2 \vec{U} + 2\omega_0\omega (A_{21}\vec{v} + A_{22}\vec{U}) + Q_0\vec{U} = 0 \end{cases} \tag{4.1}$$

$\omega = 0$ is an eigenvalue. Indeed, for $\omega = 0$, the first equation (4.1) is verified and the second gives $Q_0\vec{U} = 0$, so $\vec{U} = 0$. The corresponding eigenspace is the space of the $(\vec{v}, 0)^t$, with $\vec{v}$ arbitrary in $J_0(\Omega)$. Discarding $\omega = 0$ and dividing the equations (4.1) by ω, we obtain

$$\begin{cases} -\omega \vec{v} + 2\omega_0 A_{11}\vec{v} + 2\omega_0 A_{12}\vec{U} = 0 \\ -\omega \vec{U} + 2\omega_0 A_{21}\vec{v} + 2\omega_0 A_{22}\vec{U} + \omega^{-1} Q_0 \vec{U} = 0 \end{cases} \tag{4.2}$$

In order to eliminate ω^{-1}, we set $Q_0^{1/2}\vec{U} = \omega \vec{U}_0$, and we obtain, instead of (4.2), the system for the unknowns $\vec{v}, \vec{U}, \vec{U}_0$, that we can write

$$\begin{pmatrix} 2\omega_0 A_{11} & 2\omega_0 A_{12} & 0 \\ 2\omega_0 A_{21} & 2\omega_0 A_{22} & Q_0^{1/2} \\ 0 & Q_0^{1/2} & 0 \end{pmatrix} \begin{pmatrix} \vec{v} \\ \vec{U} \\ \vec{U}_0 \end{pmatrix} = \omega \begin{pmatrix} \vec{v} \\ \vec{U} \\ \vec{U}_0 \end{pmatrix}.$$

The operator of the lefthand-side is a selfadjoint unbounded operator of $J_0(\Omega) \oplus \chi \oplus \chi$. Consequently, the spectrum of the problem is real.

5. The spectrum outside the interval $[-2\omega_0, 2\omega_0]$

We set $\lambda = \frac{2\omega_0}{\omega}$, so that $|\lambda| < 1$.

The equations (4.2) become

$$\begin{cases} \lambda A_{11}\vec{v} + \lambda A_{12}\vec{U} = \vec{v} \\ \lambda A_{21}\vec{v} + \lambda A_{22}\vec{U} + \frac{\lambda^2}{4\omega_0^2} Q_0\vec{U} = \vec{U} \end{cases} \tag{5.1}$$

The first equation can be written

$$(I_{J_0(\Omega)} - \lambda A_{11})\vec{v} = \lambda A_{12}\vec{U}.$$

Since $\|A_{11}\| = 1$, $I_{J_0(\Omega)} - \lambda A_{11}$ has an inverse $R(\lambda)$ holomorphic in $|\lambda| < 1$, and we have $\vec{v} = \lambda R(\lambda) A_{12}\vec{U}$. Carrying out in the second equation (5.1), we obtain

$$\vec{U} = \lambda^2 A_{21} R(\lambda) A_{12}\vec{U} + \lambda A_{22}\vec{U} + \frac{\lambda^2}{4\omega_0^2} Q_0\vec{U}. \tag{5.2}$$

In order to eliminate the unbounded operator Q_0, we set, like above

$$Q_0^{1/2}\vec{U} = \omega\vec{U}_0 = \frac{2\omega_0}{\lambda}\vec{U}_0,$$

i.e.,

$$Q_0^{-1/2}\vec{U}_0 = \frac{\lambda}{2\omega_0}\vec{U}.$$

Applying to the equation (5.2) the operator $Q_0^{-1/2}$, dividing by $\frac{\lambda}{2\omega_0}$ and choosing as unknown $\vec{U}_0$, we obtain

$$\Big(I_\chi + 4\omega_0^2 Q_0^{-1/2} A_{21} R(\lambda) A_{12} Q_0^{-1/2} + \frac{4\omega_0^2}{\lambda} Q_0^{-1/2} A_{22} Q_0^{-1/2} - \frac{4\omega_0^2}{\lambda^2} Q_0^{-1}\Big)\vec{U}_0 = 0$$

The operators are selfadjoint and, except I_χ, compact. Then, we have to deal with a Fredholm's pencil [3] in the domain $\lambda \in |\lambda| < 1 - \{0\}$, regular since there are not complex eigenvalues.

Therefore, we have a discret spectrum in the domain $-1 < \lambda\,\text{real} < 1 - \{0\}$, the possible points of accumulation of the eigenvalues being $\lambda = 0$ and $\lambda = \pm 1$.

The eigenvalues of our problem $\omega = \frac{2\omega_0}{\lambda}$ have finite multiplicities, are outside the closed interval $[-2\omega_0, 2\omega_0]$ and can have as points of accumulation $\omega = \pm 2\omega_0$, $\omega = \pm\infty$.

6. The essential spectrum

We are going to seek the part of the spectrum inside $[-2\omega_0, 2\omega_0]$.

We can write the equations (4.1)

$$\begin{cases} A_{11}\vec{v} + A_{12}\vec{U} = \frac{\omega}{2\omega_0}\vec{v} \\ \mathscr{C}(\omega)\vec{U} = -2\omega_0\omega A_{21}\vec{v} \end{cases}$$

with

$$\mathscr{C}(\omega) = Q_0 + 2\omega_0\omega A_{22} - \omega^2 I$$

Let us prove that $\mathscr{C}(\omega)$ has a denumerable set of real eigenvalues $\{\omega_n\}$ that has the infinity as point of accumulation. We are going to show that the equation $\mathscr{C}(\omega)\vec{U} = 0$ can be replaced by a suitable system of two equations. Introducing still the unknown function $\vec{U}_0$ by

$$Q_0^{-1/2}\vec{U}_0 = \omega^{-1}\vec{U}, \tag{6.1}$$

we see easily that $\mathscr{C}(\omega)\vec{U} = 0$ can be written

$$Q_0^{-1/2}\vec{U} - 2\omega_0\omega Q_0^{-1/2}A_{22}Q_0^{-1/2}\vec{U}_0 = \omega^{-1}\vec{U}_0,$$

and we obtain the equivalent system

$$\begin{pmatrix} 0 & Q_0^{-1/2} \\ Q_0^{-1/2} & -2\omega_0 Q_0^{-1/2}A_{22}Q_0^{-1/2} \end{pmatrix} \begin{pmatrix} \vec{U} \\ \vec{U}_0 \end{pmatrix} = \omega^{-1} \begin{pmatrix} \vec{U} \\ \vec{U}_0 \end{pmatrix}.$$

Since the operator of the left-hand side is self-adjoint and compact, it has a denumerable set of real eigenvalues $\{\omega_n^{-1}\}$ having zero as point of accumulation. Consequently, the property of $\mathscr{C}(\omega)$ is proved.

We have

$$(\mathscr{C}(\omega)\vec{U}, \vec{U})_\chi = (Q_0\vec{U}, \vec{U})_\chi + 2\omega_0\omega(A_{22}\vec{U}, \vec{U})_\chi - \omega^2\|\vec{U}\|_\chi^2,$$

so that, if $\lambda_1(Q_0) > 0$ is the smallest eigenvalue of Q_0, we can write

$$(\mathscr{C}(\omega)\vec{U}, \vec{U})_\chi \geqslant (\lambda_1(Q_0) - 8\omega_0^2)\|\vec{U}\|_\chi^2.$$

We must distinguish two cases

1). $\lambda_1(Q_0) > 8\omega_0^2$

In this case, from $\mathscr{C}(\omega)\vec{U} = 0$, we deduce $\vec{U} = 0$ and $\mathscr{C}(\omega)$ has an inverse. Since

$$\mathscr{C}(\omega) = Q_0^{1/2}(I + 2\omega_0\omega Q_0^{-1/2}A_{22}Q_0^{-1/2} - \omega^2 Q_0^{-1})Q_0^{1/2},$$

We have

$$[\mathscr{C}(\omega)]^{-1} = Q_0^{-1/2}(I + 2\omega_0\omega Q_0^{-1/2}A_{22}Q_0^{-1/2} - \omega^2 Q_0^{-1})^{-1}Q_0^{-1/2},$$

so that $[\mathscr{C}(\omega)]^{-1}$ is compact like $Q_0^{-1/2}$. Then, we have

$$\vec{U} = -2\omega_0\omega[\mathscr{C}(\omega)]^{-1}A_{21}\vec{v}.$$

Carrying in the first equation (4.1), we obtain

$$\mathcal{M}(\omega)\vec{v} = \omega\vec{v}, \qquad \vec{v} \in J_0(\Omega) \tag{6.2}$$

with

$$\mathcal{M}(\omega) = 2\omega_0(A_{11} - 2\omega_0\omega A_{12}[\mathscr{C}(\omega)]^{-1}A_{21}).$$

We can thus apply a well-known method [3]. $2\omega_0 A_{11}$ is bounded, selfadjoint and its spectrum, that coincides with its essential spectrum, is the closed interval $[-2\omega_0, 2\omega_0]$. The operatorial function $V_0(\omega) = 4\omega_0^2\omega A_{12}[\mathscr{C}(\omega)]^{-1}A_{21}$ is analytical in ω ; for each ω; it is a compact selfadjoint operator.

Let $\omega_1 \in [-2\omega_0, 2\omega_0]$. Then, the operator

$$\mathcal{M}(\omega_1) = 2\omega_0 A_{11} - V_0(\omega_1)$$

has, by virtue of a Weyl's theorem [3,8], an essential spectrum that coincides with the spectrum of $2\omega_0 A_{11}$, i.e., $[-2\omega_0, 2\omega_0]$. Taking $\omega_2 \in \sigma_{ess}[\mathcal{M}(\omega_1)]$, it is possible to associate to ω_2, a Weyl's sequence $\{\vec{v}_n\} \in J_0(\Omega)$, depending on ω_1 and ω_2, i.e., [3,8]:

$$\vec{v}_n \to 0 \text{ weakly; } \inf \|\vec{v}_n\| > 0; \qquad (\mathcal{M}(\omega_1) - \omega_2 I)\vec{v}_n \to 0 \ \text{ strongly.}$$

Let us choose $\omega_2 = \omega_1$; then, there exists a Weyl's sequence $\{\vec{v}\,'_n\} \in J_0(\Omega)$, depending on ω_1 only, such that

$$\vec{v}\,'_n \to 0 \ \text{ weakly; } \inf \|\vec{v}\,'_n\| > 0; \qquad (\mathcal{M}(\omega_1) - \omega_2 I)\vec{v}\,'_n \to 0 \ \text{ strongly.}$$

Consequently, ω_1 belongs to the essential spectrum of the problen (6.1). So, if $\lambda_1(Q_0) > 8\omega_0^2$, each point of the closed interval $[-2\omega_0, 2\omega_0]$ belongs to the essential spectrum of the problem.

2). $\lambda_1(Q_0) \leqslant 8\omega_0^2$
Since $\mathscr{C}(\omega)$ has a denumerable set of real eigenvalues ω_n that have the infinity as point of accumulation, the interval $[-2\omega_0, 2\omega_0]$ includes no more than a finite number of ω_n. At the points of $[-2\omega_0, 2\omega_0]$ different from these ω_n, $\mathscr{C}(\omega)$ has a compact inverse and the above proof can be carried once again. Such points belong to the essential spectrum. Since the essential spectrum is closed, it is the closed interval $[-2\omega_0, 2\omega_0]$.

7. Conclusion

The spectrum of the problem is real and formed by:

1) an essential spectrum that fills the closed interval $[-2\omega_0, 2\omega_0]$,
2) a discrete spectrum that is located outside this interval with possible points of accumulation $\omega = \pm\infty$, $\omega = \pm 2\omega_0$.

8. Existence and unicity of the solution of the associated evolution problem

1). Like in the spectral problem, we introduce a new unknown function $\vec{W}$ defined by

$$\dot{\vec{W}} = iQ_0^{1/2}\vec{U} \qquad \vec{W}(0) = 0$$

Setting $A'_{ij} = 2\omega_0 A_{ij}$ $(i, j = 1, 2)$, we replace the equations (3.3), (3.5) by the system

$$\begin{cases} \ddot{\vec{v}} - iA'_{11}\dot{\vec{v}} - iA'_{12}\dot{\vec{U}} = 0 \\ \ddot{\vec{U}} - iA'_{21}\dot{\vec{v}} - iA'_{22}\dot{\vec{U}} - iQ_0^{1/2}\dot{\vec{W}} = 0 \\ \ddot{\vec{W}} \qquad\quad - iQ_0^{1/2}\dot{\vec{U}} = 0 \end{cases}$$

Setting still

$$\mathcal{A} = \begin{pmatrix} A'_{11} & A'_{12} & 0 \\ A'_{21} & A'_{22} & Q_0^{1/2} \\ 0 & Q_0^{1/2} & 0 \end{pmatrix}, \qquad \vec{w} = \begin{pmatrix} \vec{v} \\ \vec{U} \\ \vec{W} \end{pmatrix},$$

we obtain the equation

$$\ddot{\vec{w}} = i\mathcal{A}\dot{\vec{w}},$$

$-i\mathcal{A}$ is considered as an unbounded operator of $H_0 = J_0(\Omega) \oplus \chi \oplus \chi$ and

$$D(-i\mathcal{A}) = J_0(\Omega) \oplus \mathcal{V} \oplus \mathcal{V}.$$

2a). $-i\mathcal{A}$ is accretive and closed.

Indeed, we have

$$\begin{aligned}(-i\mathcal{A}\vec{w}, \vec{w})_{H_0} = &- i\big[(A'_{11}\vec{v} + A'_{12}\vec{U}, \vec{v})_{J_0(\Omega)} + (A'_{21}\vec{v} + A'_{22}\vec{U}, \vec{U})_\chi\big] \\ &- i\big[(Q_0^{1/2}\vec{W}, \vec{U})_\chi + (Q_0^{1/2}\vec{U}, \vec{W})_\chi\big]\end{aligned}$$

A simple calculation proves that the first bracket is real; the second is obviously real.

Then, we have

$$\Re(-i\mathcal{A}\vec{w}, \vec{w})_{H_0} = 0.$$

On the other hand, it is obvious that $-i\mathcal{A}$ is closed.

2b). The adjoint $(-i\mathcal{A})^*$ of $-i\mathcal{A}$ is accretive.

Indeed, we have

$$(-i\mathcal{A})^* = i\mathcal{A}^* = i\mathcal{A},$$

so that

$$\Re((-i\mathcal{A})^*\vec{w}, \vec{w})_{H_0} = 0.$$

Consequently, $-i\mathcal{A}$ is the generator of a contraction semigroup.

3). $i(-i\mathcal{A}) = \mathcal{A}$ is selfadjoint, so that, by virtue a Stone's theorem [7, 8], $-i\mathcal{A}$ is the generator of a group of unitary operators, denoted by

$$G(t) = e^{it\mathcal{A}}, \qquad -\infty < t < +\infty$$

4). Let us take as initial conditions

$$\vec{w}(0) = \vec{w}^{\,0} = \begin{pmatrix}\vec{v}^{\,0}\\ \vec{U}^0\\ 0\end{pmatrix}; \qquad \dot{\vec{w}}(0) = \dot{\vec{w}}^{\,0} = \begin{pmatrix}\vec{v}^{\,1}\\ \vec{U}^1\\ \vec{W}^1 = iQ_0^{1/2}\vec{U}^0\end{pmatrix}.$$

We have, from (6.2):

$$\dot{\vec{w}} = G(t)\dot{\vec{w}}^{\,0}$$

and then

$$\vec{w} = \vec{w}^{\,0} + \int_0^t G(s)\mathrm{d}s \cdot \dot{\vec{w}}^{\,0}.$$

Therefore [3], under the conditions

$$\begin{pmatrix}\vec{v}^{\,0}\\ \vec{U}^0\end{pmatrix} \text{ and } \begin{pmatrix}\vec{v}^{\,1}\\ \vec{U}^1\end{pmatrix} \in J_0(\Omega) \oplus \mathcal{V},$$

the evolution problem has a generalized solution continuously differentiable and such that $\vec{v}(t) \in J_0(\Omega)$ and $\vec{U}(t) \in \mathcal{V}$ for each t.

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References

1. H. F. Bauer, W. Eidel, *Transverse vibration and stability of spinning circular plates of constant thickness and different boundary conditions*, J. Sound Vib. **300** (2007), 877–895.
2. R. Dautray, J. L. Lions, *Analyse Mathématique et calcul numérique*, Vol. 4, Masson, Paris, 1988.
3. N. D. Kopachevskii, S. G. Krein, *Operator approach to linear problems of hydrodynamics*, Vol. 1, Birkhauser Verlag, Basel, 2001.
4. G. Krall, *Meccanica Tecnica delle vibrazioni*, Parte 2^a, Eredi Virgilio Veschi, Roma, 1961.
5. H. Lamb, H. V. Southwell, *The vibration of a spining disk*, Proc. R. Soc. Lond., Ser. A **99** (1921), 272–280.
6. J. L. Lions, *Equations différentielles opérationnelles et problèmes aux limites*, Vol. 8, Springer Verlag, Berlin, 1961.
7. F. Riesz, B. Sz. Nagy, *Leçons d'analyse fonctionnelle* , Gauthier villars, Paris, 1968.
8. J. Sanchez Hubert, E. Sanchez Palencia, *Vibration and Coupling of Continuous Systems: Asymptotic Methods*, Springer Verlag, Berlin, 1989.
9. W. Velte, *Direkte Methoden der Variationsrechnung*, Vieweg+Teubner Verlag, Wiesbaden, 1976.

МАЛЕ ОСЦИЛАЦИЈЕ ИДЕАЛНЕ ТЕЧНОСТИ У ПОСУДИ СА ЕЛАСТИЧНОМ КРУЖНОМ ОБЛОГОМ, ПРИ РАВНОМЕРНОЈ РОТАЦИЈИ

Резиме. Проблем малих осцилација идеалне течности унутар потпуно чврсте посуде при равномерној ротацији проучавали су Копачевски и Крајн [3]. У овом раду је предложено уопштење тог модела за случај посуде са еластичном кружном облогом.

У том контексту, изведене су линеаризоване једначине кретања система облога-течност. Добијене су варијационе једначине малих осцилација око равнотежног положаја система, а потом су изведене и анализиране и две операторске једначине у одговарајућем Хилбертовом простору. Показано је да је спектар система реалан и да се састоји од пребројивог скупа сопствених вредности као и есенцијално непрекидног спектра који испуњава интервал. Коначно, помоћу теорије полугрупа, доказана је теорема егзистенције и јединствености за решење придруженог еволутивног проблема.

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