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ON INTEGRALS, HAMILTONIAN AND METRIPLECTIC FORMULATIONS OF POLYNOMIAL SYSTEMS IN 3D

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Abstract. The first integrals of the reduced three-wave interaction problem, the Rabinovich system, the Hindmarsh–Rose model, and the Oregonator model are derived using the method of Darboux polynomials. It is shown that, the reduced three-wave interaction problem, the Rabinovich system, the Hindmarsh–Rose model can be written in a bi-Hamiltonian/Nambu metriplectic form.

1. Introduction

In this present paper, we are interested in the integrability and the Hamiltonian analysis of three dimensional polynomial systems [42]. Even though various methods for the determinations of the first integrals of non-planar systems can be found in the literature, only a few of them are really satisfactory [32]. For example, in [19], an ansatz for finding a polynomial invariant was proposed which is applicable only for some special cases. In [21], another method, based on the Frobenius integrability theorem, is proposed. In a similar programme, [11] investigated a three-dimensional Hamiltonian system with quartic potentials.

The goal of this paper is to obtain the first integrals of the reduced three-wave interaction model, the Rabinovich system, the Hindmarsh–Rose model and the Oregonator model, using the method of Darboux polynomials. After deriving the first integrals, we shall further investigate the possible Hamiltonian formulations, bi-Hamiltonian representations and metriplectic realizations of these systems.

In order to achieve these goals, the paper is organized as follows. For the sake of completeness, and in order to widen the spectrum of the potential readers, we shall reserve the following section for some necessary theoretical issues. Accordingly, in the theorem 2.1, we shall exhibit a way to write a time-dependent first integral of a system if the complete set of Darboux polynomials are determined. Then, we shall recall Poisson, Nambu–Poisson and the metriplectic characterizations of 3D systems.

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The rest of the paper is reserved for applications of the theorems to particular problems. In the proposition 3.1, the first integrals of the reduced three-wave interaction problem will be exhibited for various different choices of its parameters. It will be shown in the proposition 3.2 that the reduced three-wave interaction problem is in the bi-Hamiltonian/Nambu metriplectic form. We shall present the first integrals of the Rabinovich system in the proposition 4.1. It will be stated in the proposition 4.2 that the Rabinovich system is in the bi-Hamiltonian/Nambu metriplectic form as well. In the proposition 5.1, the first integrals will be derived for several subsystems of the Hindmarsh–Rose model. The bi-Hamiltonian/Nambu metriplectic realization of the Hindmarsh–Rose model will be given in the proposition 5.2. In the last section, we shall obtain a first integral of the Oregonator model and present its Hamiltonian character.

2. Polynomial Systems in $3D$

2.1. Darboux' Polynomials. Consider a polynomial three dimensional ODE system

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$$

generated by a vector field $\mathbf{X}$. A function $I = I(t, \mathbf{x})$ is called the first integral of the system if it remains constant on any integral curve [14]. A function g is called the second integral of the system if it satisfies

$$\mathbf{X}(g) = \lambda g \tag{2.1}$$

for some cofactor λ.

Polynomial second integrals for the polynomial vector fields are called Darboux polynomials [10]. Darboux polynomials simplify the determination of possible first integrals. For example, if there exist two relatively prime Darboux polynomials having a common cofactor then their fraction is a first integral. The inverse of this statement is also true. That is, if we have a rational first integral P_1/P_2 of a vector field X, then both P_1 and P_2 are Darboux polynomials of X.

For the case of planar polynomial vector fields, there are stronger tools for determination of the first integrals. First of all, there exists a semi-algorithm, called Prelle–Singer method [45, 56, 57]. Further, if one has a certain number of relatively prime irreducible Darboux polynomials, not necessarily having a common cofactor, it is possible to write the first integrals using the polynomials [8, 12, 38]. Unfortunately, these methods can not directly be applied for non-planar systems.

We state the following observation which enables one to arrive at a time-dependent first integral of a system when it possesses autonomous Darboux polynomials. Note that, the theorem is valid in any dimensions.

Theorem 2.1. *If g_α's are Darboux Polynomials for an autonomous system $\dot{\mathbf{x}} = \mathbf{X}$, and if there exist constants n_α's, not all zero, satisfying the equality*

$$\sum_{\alpha=1}^{k} n_\alpha \lambda_\alpha = r, \tag{2.2}$$

for some real number $r \in \mathbb{R}$, then the function

$$I = e^{-rt} \prod_{\alpha=1}^{k} g_\alpha^{n_\alpha} \tag{2.3}$$

is a time-dependent first integral of the system.

PROOF. To prove the assertion, in the following calculation, we shall show that the total derivative of the function I in (2.3) with respect to time variable is zero.

$$\begin{aligned}
\Big(\frac{\partial}{\partial t} + X\Big)(I) &= \frac{\partial}{\partial t}\left(e^{-rt} \prod_{\alpha=1}^{k} g_\alpha^{n_\alpha}\right) + e^{-rt} X\left(\prod_{\alpha=1}^{k} g_\alpha^{n_\alpha}\right) \\
&= -re^{-rt} \prod_{\alpha=1}^{k} g_\alpha^{n_\alpha} + e^{-rt}\left(\prod_{\alpha=1}^{k} g_\alpha^{n_\alpha - 1}\right)\left(\sum_{\beta=1}^{k} (n_\beta g_1 \dots X(g_\beta) \dots g_k)\right) \\
&= -re^{-rt} \prod_{\alpha} g_\alpha^{n_\alpha} + e^{-rt}\left(\prod_{\alpha=1}^{k} g_\alpha^{n_\alpha}\right)\left(\sum_{\beta=1}^{k} n_\beta \lambda_\beta\right) \\
&= -re^{-rt} \prod_{\alpha} g_\alpha^{n_\alpha} + re^{-rt} \prod_{\alpha=1}^{k} g_\alpha^{n_\alpha} = 0
\end{aligned}$$

where in the first line we have assumed that the Darboux' polynomials g_α's do not depend on time variable explicitly, and, in the last line, we have employed the equality (2.2). □

To the best of our knowledge, in the literature, the case where $\sum_\alpha n_\alpha \lambda_\alpha \neq r$ is still open. We refer [44] for a variant of the Prelle–Singer/Darboux method to derive called quasi-rational first integrals of non-planar systems.

2.2. Poisson Systems. A Poisson bracket is a skew-symmetric binary operation $\{\bullet, \bullet\}$ on the space of smooth functions satisfying the Leibnitz and the Jacobi identities [40, 41, 51, 61]. On an n-dimensional Euclidean space, we define the Poisson bracket of F and H by

$$\{F, H\} = \nabla F \cdot N \nabla H,$$

where N is an $n \times n$ Poisson matrix. Here, ∇F and ∇H are the gradients of F and H, respectively. A function C is called a Casimir function if $\{C, F\} = 0$ for all F. It is evident that, in order to have a non-constant Casimir function, the Poisson matrix N must be degenerate.

A system of ODEs is called a Hamiltonian system if it can be written in the form of Hamilton's equations

$$\dot{\mathbf{x}} = \{\mathbf{x}, H\} = N \nabla H \tag{2.4}$$

for a Hamiltonian function H [26, 28]. A system is called a bi-Hamiltonian system if it admits two different Hamiltonian formulations

$$\dot{\mathbf{x}} = N_1 \nabla H_2 = N_2 \nabla H_1, \tag{2.5}$$

with the requirement that the Poisson matrices N_1 and N_2 be compatible [7,43,51]. For some discussions on the integrability of the bi-Hamiltonian systems, we cite [15,50].

The space of three dimensional vectors and the space of three by three skew-symmetric matrices are isomorphic. Existence of this isomorphism enables us to identify a three by three Poisson matrix N with a three dimensional Poisson vector field $\mathbf{J}$ [13,27]. Under this isomorphism, the Hamilton's equations (2.4) take the particular form

$$\dot{\mathbf{x}} = \mathbf{J} \times \nabla H, \tag{2.6}$$

whereas a bi-Hamiltonian system (2.5) turns out to be

$$\dot{\mathbf{x}} = \mathbf{J}_1 \times \nabla H_2 = \mathbf{J}_2 \times \nabla H_1.$$

Here, $\mathbf{J}_1$ and $\mathbf{J}_2$ are the Poisson vectors associated with the Poisson matrices N_1 and N_2, respectively. In terms of the Poisson vectors, the Jacobi identity becomes

$$\mathbf{J} \cdot (\nabla \times \mathbf{J}) = 0. \tag{2.7}$$

For the proof of the following theorem, we refer [2,29–31].

Theorem 2.2. *The general solution of the Jacobi identity* (2.7) *is*

$$\mathbf{J} = \frac{1}{M} \nabla H_1, \tag{2.8}$$

where M and H_1 are arbitrary functions.

Existence of the scalar multiple $1/M$ in (2.8) is a manifestation of the conformal invariance of the Jacobi identity. In the literature, M is called Jacobi's last multiplier [20,34,35,62]. Potential function H_1 in (2.8) is a Casimir function of the Poisson vector field $\mathbf{J}$. The kernel of $\mathbf{J}$ is one dimensional hence any other Casimir of $\mathbf{J}$ has to be dependent to the potential function H_1. Substitution of the general solution (2.8) of $\mathbf{J}$ into the Hamilton's equations (2.6) results with the general form

$$\dot{\mathbf{x}} = \frac{1}{M} \nabla H_1 \times \nabla H_2 \tag{2.9}$$

of the Hamiltonian systems in three dimensions.

While writing a non-autonomous system in form of the Hamilton's equations (2.6), it is evident that the Poisson vector $\mathbf{J}$ or the Hamiltonian function H must explicitly depend on the time variable t. If the Hamiltonian function depends on t then it fails to be an integral invariant of the system. We present the following calculation

$$\begin{aligned} \frac{d}{dt} H(\mathbf{x}, t) &= \nabla H(\mathbf{x}, t) \cdot \dot{\mathbf{x}} + \frac{\partial}{\partial t} H(\mathbf{x}, t) \\ &= \nabla H \cdot (\mathbf{J} \times \nabla H) + \frac{\partial}{\partial t} H(\mathbf{x}, t) = \frac{\partial}{\partial t} H(\mathbf{x}, t), \end{aligned}$$

showing that the Hamiltonian function is a constant of the motion if and only if it does not depend on t.

2.3. Nambu–Poisson Systems. On a Euclidean space, a skew-symmetric ternary operation $\{\bullet,\bullet,\bullet\}$ is called the Nambu–Poisson bracket if it satisfies the generalized Leibnitz identity

$$\{F_1, F_2, FH\} = \{F_1, F_2, F\}H + F\{F_1, F_2, H\}$$

and the fundamental (or Takhtajan) identity

$$\{F_1, F_2, \{H_1, H_2, H_3\}\} = \sum_{k=1}^{3} \{H_1, \dots, H_{k-1}, \{F_1, F_2, H_k\}, H_{k+1}, \dots, H_3\},$$

for arbitrary functions F, F_1, F_2, H, H_1, H_2 [49, 58]. An ODE system is called a Nambu–Hamiltonian system if it can be written as

$$\dot{\mathbf{x}} = \{\mathbf{x}, H_1, H_2\}. \tag{2.10}$$

Note that, in this case, the dynamics is generated by a pair (H_1, H_2) of Hamiltonian functions.

One can write a Nambu-Hamiltonian system (2.10) in the bi-Hamiltonian form as follows

$$\dot{\mathbf{x}} = \{\mathbf{x}, H_1\}^{H_2} = \{\mathbf{x}, H_2\}^{H_1},$$

where the Poisson brackets are defined by

$$\{F, H\}^{H_2} = \{F, H, H_2\}, \qquad \{F, H\}^{H_1} = \{F, H_1, H\},$$

respectively, [25].

In three dimensions, we define a Nambu–Poisson bracket of F, H_1 and H_2 by taking the triple product of their gradients, that is

$$\{F, H_1, H_2\} = \frac{1}{M} \nabla F \cdot \nabla H_1 \times \nabla H_2, \tag{2.11}$$

where M being an arbitrary function. It is immediate to observe that the Hamilton's equations (2.9) are in the Nambu–Hamiltonian form (2.10) generated by the Hamiltonian pair (H_1, H_2), see [25, 59]. Hence, the system (2.9) is in bi-Hamiltonian form as well. For the proof of the following theorem, we cite [13, 17].

THEOREM 2.3. *Let $\dot{\mathbf{x}} = \mathbf{X}$ be a three dimensional dynamical system and assume that it has a time independent first integral. Then $\dot{\mathbf{x}} = \mathbf{X}$ is a bi-Hamiltonian and a Nambu-Hamiltonian system if and only if there exists a real valued function M satisfying $\nabla \cdot (M\mathbf{X}) = 0$.*

In the statement of the theorem 2.3, the function M is the Jacobi's last multiplier. For the use of the method Jacobi's last multiplier in the Lagrangian framework, we refer [18]. If, particularly, $F = \frac{1}{2}\|\mathbf{x}\|^2$, then the Nambu-Poison bracket (2.11) reduces to the Lie–Poisson bracket

$$\{H_1, H_2\}_{LP} = \frac{1}{M} \mathbf{x} \cdot \nabla H_1 \times \nabla H_2,$$

see, for example, [5].

2.4. Metriplectic Systems. If a system is dissipative in nature then one can not write it in the Hamiltonian form. Some dissipative systems can be written as a gradient system, whereas some others can be written as a metriplectic system which is a sum of a Hamiltonian system and a gradient system. Let us now discuss this in more technical terms.

Let G be a positive semi-definite symmetric matrix on an Euclidean space. Consider the symmetric bracket of two functions

$$(F, S) = \nabla F \cdot G \nabla S.$$

An ODE system is called a metric or a gradient system if it can be written in the form

$$\dot{\mathbf{x}} = (\mathbf{x}, S) = G \nabla S,$$

see [53]. The generating function S is not necessarily a conserved quantity of the system instead we have $\dot{S} = (S, S) \geqslant 0$. In the literature, the function S is commonly called as entropy. We cite two recent studies on the theory of non-equilibrium thermodynamics in the framework of the gradient system [36, 37].

A metriplectic bracket is a sum of a Poisson bracket and a metric bracket. Explicitly, we define it by

$$\{\{F, E\}\} = \{F, E\} + a(F, E) = \nabla F \cdot N \nabla E + a \nabla F \cdot G \nabla E, \tag{2.12}$$

for any scalar a, [4, 5, 22, 39, 46, 47, 54]. The metriplectic structures are examples of the Leibnitz brackets [24, 52]. We refer an interesting study [23] for the relation between the metriplectic structure and the contact structure.

In the literature, there are two main approaches while generating a dynamics with the metriplectic bracket (2.12). In the first one, generating function F, called the generalized free energy, is chosen to be the difference of a Hamiltonian function H and a entropy function S, that is $F = H - S$. In this case, we require that ∇S lives in the kernel of N and ∇H lives in the kernel of G, that is

$$N \nabla S = 0, \qquad G \nabla H = 0.$$

In this case, the equation of motion is given by

$$\dot{\mathbf{x}} = \{\{\mathbf{x}, F\}\} = \{\mathbf{x}, F\} + a(\mathbf{x}, F) = \{\mathbf{x}, H\} - a(\mathbf{x}, S).$$

In this first type, we have a conservation law $\dot{H} = \{\{H, F\}\} = 0$ and a dissipation $\dot{S} = \{\{S, F\}\} \leqslant 0$. A second type of metriplectic dynamics is generated by a single function, say H, and written as

$$\dot{\mathbf{x}} = \{\{\mathbf{x}, H\}\} = \{\mathbf{x}, H\} + a(\mathbf{x}, H)$$

without any restriction on H.

In three dimensions, the Hamiltonian (reversible) part of a metriplectic system can be written in the terms of bi-Hamiltonian/Nambu bracket, hence we write the total system in a bi-Hamiltonian/Nambu metriplectic form given by

$$\dot{\mathbf{x}} = \{\mathbf{x}, H_1, H_2\} - a(\mathbf{x}, S) = \frac{1}{M} \nabla H_1 \times \nabla H_2 - aG\nabla S,$$

see, for example, [3].

3. Reduced three-wave interaction model

The reduced three-wave interaction model [20, 55] is given by the following system of ODEs

$$
(3.1)\qquad \begin{cases} \dot{x} = -2y^2 + \gamma x + z + \delta y \\ \dot{y} = 2xy + \gamma y - \delta x \\ \dot{z} = -2xz - 2z. \end{cases}
$$

where three quasisynchronous waves interact in a plasma with quadratic nonlinearities. In the literature, there exist some studies on the integrability of (3.1), see, for example, [6, 19].

In this section, we have two main goals. The first one is to show how the previous results on the integrals of (3.1) can be obtained in a more simplified manner using Darboux polynomials. The second goal is to exhibit the bi-Hamiltonian/Nambu metriplectic realization of the system (3.1). Accordingly, we start with the following proposition characterizing the integrals of the system for some certain parameters. We shall prove the proposition for each subcases. After the proof is accomplished, in the last part of this section, we shall present the metriplectic realization of the most general form of reduced three-wave interaction model given in Eqs. (3.1).

PROPOSITION 3.1. *The reduced three-wave interaction problem presented in* (3.1) *has the following first integrals.*

1. *If δ is arbitrary and $\gamma = 0$, then $I = e^{2t} z(y - \delta/2)$.*
2. *If δ is arbitrary and $\gamma = -1$, then $I = e^{2t}(x^2 + y^2 + z)$.*
3. *If $\delta \neq 0$ and $\gamma = -2$, then $I = e^{4t}(x^2 + y^2 + 2/\delta\, yz)$.*
4. *If $\delta = 0$ and $\gamma \neq 2$, then $I = e^{2-\gamma} yz$.*
5. *If $\delta = 0$ and $\gamma = -1$, then $I_1 = e^{2t}(x^2 + y^2 + z)$, $I_2 = e^{3t} yz$.*

We start with a second degree Darboux' polynomial g in the form

$$
g = Ax^2 + By^2 + Cz^2 + Exy + Fxz + Gyz + Jx + Ky + Lz.
$$

By studying the eigenvalue problem (2.1) for the case of reduced three-wave interaction model (3.1), we arrive at the following equations

$$
(3.2)\qquad A = B, \qquad E = F = C = 0
$$

$$
(3.3)\qquad \begin{cases} 2A\gamma - E\delta = \lambda A, & \gamma + E\delta - 2J = \lambda B, \\ F - 4C = \lambda C, & \delta + 2E\gamma - 2B\delta + 2K = \lambda E, \\ 2A + (\gamma - 2)F - G\delta - 2L = \lambda F, & E + F\delta + (\gamma - 2)G = \lambda G \end{cases}
$$

$$
(3.4)\qquad J\gamma - K\delta = \lambda J, \qquad J\delta + K\gamma = \lambda K, \qquad J - 2L = \lambda L.
$$

determining the coefficients of g. Here, λ is the cofactor associated with g. In each of the cases presented in the proposition 3.1, the system (3.2)–(3.4) of equations takes particular forms. Now, we shall derive the integrals proposed in the proposition 3.1 case by case. Further, we shall study the Hamiltonian character of each of the cases, and present their bi-Hamiltonian formulations by employing the theorem 2.3.

3.1. Case 1: δ is arbitrary and $\gamma = 0$. These choices reduce the three-wave interaction model (3.1) to a particular form

$$\begin{cases} \dot{x} = -2y^2 + +z + \delta y \\ \dot{y} = 2xy - \delta x \\ \dot{z} = -2xz - 2z. \end{cases} \tag{3.5}$$

whereas the equations (3.2)–3.4 defining the Darboux' polynomial g turn out to be

$$A = B = C = E = F = K = J = 0, \qquad L = -\frac{\delta}{2}G, \qquad \lambda = -2.$$

Here, G is an arbitrary function. Additionally, by choosing $G = 1$, we obtain the Darboux' polynomial

$$g = zy - \frac{\delta}{2}z.$$

The condition (2.2) implies the requirement $-r + n\lambda = 0$ with $\lambda = -2$. For $r = -2$, we have $n = 1$, so that an integral of the motion equals to

$$I = e^{2t}\Big(zy - \frac{\delta}{2}z\Big). \tag{3.6}$$

Now we introduce the change of dependent variable z by w according to the rule $w = e^{2t}z$ in order to exhibit the bi-Hamiltonian character of the first case (3.5). By choosing $\gamma = 0$ and performing the substitution $w = e^{2t}z$ in Eqs.(3.5), we arrive at the following reduced form

$$\begin{cases} \dot{x} = -2y^2 + we^{-2t} + \delta y \\ \dot{y} = 2xy - \delta x \\ \dot{w} = -2xw \end{cases} . \tag{3.7}$$

This reduced system is divergence free hence Jacobi's last multiplier for the system is a constant function, say $M = 1$. According to theorem 2.3, the system (3.7) can be written in the bi-Hamiltonian form. Using the pair of Hamiltonian functions

$$H_1 = wy - \frac{\delta}{2}w, \qquad H_2 = x^2 + y^2 + e^{-2t}w$$

we write the system (3.7) as in the form of cross product of two gradients

$$(\dot{x}, \dot{y}, \dot{w})^T = \nabla H_1 \times \nabla H_2 = \mathbf{J}_1 \times \nabla H_2 = \mathbf{J}_2 \times \nabla H_1$$

with Poisson vector fields $\mathbf{J}_1 = \nabla H_1$ and $\mathbf{J}_2 = -\nabla H_2$, respectively. In this realization, H_1 has been obtained simply by substituting $w = e^{2t}z$ into the integral I presented in (3.6).

3.2. Case 2: δ is arbitrary and $\gamma = -1$. These choices reduce the three-wave interaction model (3.1) to a particular form

$$\begin{cases} \dot{x} = -2y^2 - x + z + \delta y \\ \dot{y} = 2xy - y - \delta x \\ \dot{z} = -2xz - 2z. \end{cases} \tag{3.8}$$

In the case, the system of equations (3.2)–(3.4) becomes

$$C = E = F = G = K = J = 0, \qquad A = B = L, \qquad \lambda = -2,$$

so that the Darboux' polynomial turns out to be $g = A(x^2 + y^2 + z)$ with cofactor $\lambda = -2$. The condition for I being a first integral, namely $-r + n\lambda = 0$, has a solution $n = 1$ and $r = -2$, hence we have that

$$I = e^{2t}(x^2 + y^2 + z).$$

Now, let us present the bi-Hamiltonian realization of this second case (3.8). To this end, we introduce the following change of variables

$$u = xe^t, \qquad v = ye^t, \qquad w = ze^{2t},$$

and we rescale the time variable by $\bar{t} = e^t$. Substituting these into the system (3.8), we arrive at the following non-autonomous system

$$\begin{cases} \acute{u} = -2v^2 + w + \delta v\bar{t} \\ \acute{v} = 2uv - \delta u\bar{t} \\ \acute{w} = -2uw \end{cases} \tag{3.9}$$

where prime denotes the derivative with respect to the new time variable $\bar{t} = e^t$. The system (3.9) is divergence free, hence we can take the Jacobi's last multiplier M as the unity. We can write the system (3.9) as in the bi-Hamiltonian form

$$(\acute{u}, \acute{v}, \acute{w})^T = \nabla H_1 \times \nabla H_2 = \mathbf{J}_1 \times \nabla H_2 = \mathbf{J}_2 \times \nabla H_1,$$

generated by the pair of Hamiltonian functions

$$H_1 = u^2 + v^2 + w, \qquad H_2 = vw + \delta\frac{v^2}{2}\bar{t} - \delta\frac{u^2}{2}\bar{t}.$$

3.3. Case 3: $\delta \neq 0$ and $\gamma = -2$. For the above choice of parameters $\delta \neq 0$ and $\gamma = -2$, it may be verified that, we have

$$\begin{cases} \dot{x} = -2y^2 - 2x + z + \delta y \\ \dot{y} = 2xy - 2y - \delta x \\ \dot{z} = -2xz - 2z. \end{cases} \tag{3.10}$$

whereas the system of equations (3.2)–(3.4) defining the Darboux' polynomial turns out to be

$$C = E = F = J = K = L = 0, \qquad A = B, \qquad G = \frac{2}{\delta}A, \qquad \lambda = -4.$$

This leads to the eigenfunction $g = A(x^2 + y^2 + \frac{2}{\delta}yz)$ with $\lambda = -4$. According to the theorem 2.1 and by choosing $A = 1$ we obtain the following first integral

$$I = e^{4t}(x^2 + y^2 + \frac{2}{\delta}yz)$$

of the system (3.10).

To arrive the Hamiltonian formulation of the third case (3.10), we first make the substitutions

$$u = xe^{2t}, \qquad v = ye^{2t}, \qquad w = ze^{2t}$$

into (3.10) which result with the following non-autonomous divergence free system

$$\begin{cases} \dot{u} = -2v^2e^{-2t} + w + \delta v \\ \dot{v} = 2uve^{-2t} - \delta u \\ \dot{z} = -2uwe^{-2t} \end{cases} \tag{3.11}$$

A direct calculation proves that the system (3.11) is bi-Hamiltonian

$$(\dot{u}, \dot{v}, \dot{w})^T = \nabla H_1 \times \nabla H_2 = \mathbf{J}_1 \times \nabla H_2 = \mathbf{J}_2 \times \nabla H_1$$

generated by the pair of Hamiltonian functions

$$H_1 = \frac{\delta}{2}(u^2e^{-2t} + v^2e^{-2t} + w), \quad H_2 = u^2 + v^2 + \frac{2}{\delta}vw.$$

3.4. Case 4: $\delta = 0$ and $\gamma \neq 2$. In this case, we have that

$$\begin{cases} \dot{x} = -2y^2 + \gamma x + z \\ \dot{y} = 2xy + \gamma y \\ \dot{z} = -2xz - 2z. \end{cases} \tag{3.12}$$

It is a straightforward matter to verify that $g_1 = y$ and $g_2 = z$ are Darboux polynomials with cofactors $\lambda_1 = 2x - 1$ and $\lambda_2 = -2x - 2$, respectively. The condition (2.2) now leads to

$$-r + \sum_{\alpha} n_\alpha g_\alpha = 0 \Rightarrow -r + n_1(2x + \gamma) + n_2(-2x - 2) = 0.$$

Setting $r = -1$ we obtain the following equations

$$n_1 - n_2 = 0, \quad \gamma n_1 - 2n_2 + 1 = 0.$$

So that we have $n_1 = n_2 = \frac{1}{2-\gamma}$, and the first integral

$$I = e^{(2-\gamma)t}yz.$$

In order to exhibit the Hamiltonian formulation of the system (3.12), we introduce the following change of coordinates

$$u = xe^{-\gamma t}, \qquad v = ye^{-\gamma t}, \qquad w = ze^{2t}.$$

Substitutions of these into (3.12) lead to a non-autonomous divergence free system

$$\begin{cases} \dot{u} = -2v^2e^{\gamma t} + we^{-(2+\gamma)t} \\ \dot{v} = 2uve^{\gamma t} \\ \dot{w} = -2uwe^{\gamma t} \end{cases} .$$

which is in the bi-Hamiltonian form

$$(\dot{u}, \dot{v}, \dot{w})^T = \nabla H_1 \times \nabla H_2 = \mathbf{J}_1 \times \nabla H_2 = \mathbf{J}_2 \times \nabla H_1$$

generated by the pair

$$H_1 = u^2e^{\gamma t} + v^2e^{\gamma t} + e^{-(2+\gamma)t}w, \qquad H_2 = vw.$$

3.5. Case 5: $\delta = 0$ and $\gamma = -1$. In this case, we have

$$\begin{cases} \dot{x} = -2y^2 - x + z \\ \dot{y} = 2xy - y \\ \dot{z} = -2xz - 2z. \end{cases} \tag{3.13}$$

For this case, in addition to the Darboux polynomials $g_1 = y$ and $g_2 = z$ introduced in the previous case, we have $g_3 = x^2 + y^2 + z$ with cofactor $\lambda_3 = -2$. The condition (2.2) becomes

$$2(n_1 - n_2) - (n_1 + 2n_2 + 2n_3) = r.$$

We make the standardization $r = -1$ and obtain the following set of equations

$$n_1 = n_2 \text{ and } n_1 + 2n_2 + 2n_3 = 1.$$

This leads to the following subcases: (a) $n_3 = 0$ and $n_1 = n_2 = \frac{1}{3}$, and (b) $n_1 = n_2 = 0$ and $n_3 = \frac{1}{2}$. So that, we have two time-dependent integrals of the motion

$$I_1(x, y, z) = e^t (yz)^{\frac{1}{3}} \text{ and } I_2(x, y, z) = e^t (x^2 + y^2 + z)^{\frac{1}{2}}, \tag{3.14}$$

respectively.

Now, we shall exhibit the bi-Hamiltonian character of the system (3.13). We make the change of variables $u = xe^t$, $v = ye^t$, and $w = ze^{2t}$ and rescale the time variable by $\bar{t} = e^t$, then arrive the autonomous system

$$\begin{cases} \acute{u} = -2v^2 + w \\ \acute{v} = 2uv \\ \acute{w} = -2uw \end{cases} \tag{3.15}$$

where prime denotes the derivative with respect to the new time variable $\bar{t} = e^t$. Note that, this system is divergence free, hence we can take the Jacobi's last multiplier as the unity. In the new coordinate system, the integrals of the system presented in (3.14) become the Hamiltonian functions

$$H_1 = vw, \qquad H_2 = u^2 + v^2 + w$$

of the system (3.15). This enables us to write the system (3.15) as

$$(\acute{u}, \acute{v}, \acute{w})^T = \nabla H_1 \times \nabla H_2.$$

3.6. Metriplectic realization. In this part, we shall present the metriplectic realization of the most general form of the reduced three-wave interaction model given in (3.1). The proof of the proposition is the matter of a direct calculation.

PROPOSITION 3.2. *The most general form of the reduced three-wave interaction problem* (3.1) *given by*

$$\begin{cases} \dot{x} = -2y^2 + \gamma x + z + \delta y \\ \dot{y} = 2xy + \gamma y - \delta x \\ \dot{z} = -2xz - 2z. \end{cases}$$

is in the bi-Hamiltonian/Nambu metriplectic form given by

$$(\dot{x}, \dot{y}, \dot{z})^T = \nabla H_1 \times \nabla H_2 - G\nabla H_2. \tag{3.16}$$

where the Hamiltonian functions are $H_1 = zy - \frac{\delta}{2}z$, *and* $H_2 = x^2 + y^2 + e^{-2t}z$, *and the metric tensor is*

$$G = \begin{pmatrix} -\gamma/2 & 0 & 0 \\ 0 & -\gamma/2 & 0 \\ 0 & 0 & 2ze^{2t} \end{pmatrix}.$$

See that, by replacing the roles of H_1 and H_2 in (3.16), and by modifying the metric, we may also generate the system (3.1) by the Hamiltonian H_1 as well. Let us study this symmetry on an autonomous particular case. For the particular choices of $\delta = 0$ and $\gamma = -1$ (the case 5) the general model (3.1) reduces to a more simple form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -2y^2 + z \\ 2xy \\ -2xz \end{pmatrix} - \begin{pmatrix} x \\ y \\ 2z \end{pmatrix} \tag{3.17}$$

where the first term at the right hand side is the conservative part of the system with two Hamiltonian functions $H_1 = yz$ and $H_2 = x^2 + y^2 + z$. The second term on the right hand side is the dissipative term and can be obtained by taking the metric two-form as

$$G = \begin{pmatrix} 0 & \frac{x}{z} & 0 \\ \frac{x}{z} & 0 & 1 \\ 0 & 1 & \frac{z}{y} \end{pmatrix}.$$

As a result, the model given in (3.17) can be written as

$$\dot{\mathbf{x}} = -\nabla H_2 \times \nabla H_1 - G\nabla H_1.$$

4. The Rabinovich system

This is described by the following system of equations:

$$\begin{cases} \dot{x} = hy - \nu_1 x + yz \\ \dot{y} = hx - \nu_2 y - xz \\ \dot{z} = -\nu_3 z + xy, \end{cases} \tag{4.1}$$

where h and ν_i are real constants.

In this subsection, we shall very briefly illustrate how the results of [19] for this system may be derived by the method of Darboux polynmials. In addition, we shall show that the Rabinovich system (4.1) can be written as a bi-Hamiltonian/Nambu metriplectic form.

Proposition 4.1. *The Rabinovich system* (4.1) *has the following first integrals.*

1. *If* $h = 0$, $\nu_2 = \nu_3$ *then* $I_1 = e^{2\nu_3 t}(y^2 + z^2)$
2. *If* $h = 0, \nu_1 = \nu_2$ *then* $I_2 = e^{2\nu_1 t}(x^2 + y^2)$.

Proof. First consider the vector field X generating the Rabinovich system (4.1). The action of X on the second order polynomial $g_1 = y^2 + z^2$ is computed to be

$$X(g_1) = 2hxy - 2(\nu_2 y^2 + \nu_3 z^2).$$

From this calculation, it is immediate to deduce that g_1 is a Darboux polynomial if $h = 0$ and $\nu_2 = \nu_3$. In this case, the cofactor $\lambda = -2\nu_3$ is a constant. So,

$$I_1 = e^{2\nu_3 t}(y^2 + z^2)$$

is a first integral of the system (4.1) when $h = 0$, $\nu_2 = \nu_3$ with ν_1 and ν_3 being arbitrary.

The action of the vector field X generating the Rabinovich system (4.1) on the polynomial $g_2 = x^2 + y^2$ results with

$$X(g_2) = 4hxy - 2(\nu_1 x^2 + \nu_2 y^2).$$

Consequently, g_2 becomes a Darboux polynomial when $h = 0, \nu_1 = \nu_2$. In this case, the eigenpolynomial being of degree zero *viz* $\lambda = -2\nu_1$. We are lead to the first integral

$$I_2 = e^{2\nu_1 t}(x^2 + y^2)$$

of the system (4.1) when $h = 0$, $\nu_1 = \nu_2$ with ν_1, ν_3 being arbitrary. □

Let us transform the Rabinovich system (4.1) in a form where we can write it as a bi-Hamiltonian/Nambu system. For the case of $\nu_1 = \nu_2 = \nu_3 = \nu$, we have two integrals I_1 and I_2 of the system (4.1). In this case, we apply a coordinate change

$$u = xe^{vt}, \quad v = ye^{vt}, \quad w = ze^{vt}$$

with the time rescaling $\bar{t} = \frac{1}{v}e^{vt}$ with $v \neq 0$, see also [9]. So, finally, we arrive at the following divergence free system

$$\acute{u} = vw, \quad \acute{v} = -uw, \quad \acute{w} = uv,$$

which is in the form of the bi-Hamiltonian and the Nambu–Poisson form

$$(\acute{u}, \acute{v}, \acute{w})^T = \nabla H_1 \times \nabla H_2$$

generated by the Hamiltonian functions pair

$$H_1 = \frac{1}{2}(v^2 + w^2), \qquad H_2 = \frac{1}{2}(u^2 + v^2).$$

For a similar for the case where h is nonzero and $\nu_1 = \nu_2 = v_3 = 0$, we refer [60].

In the following proposition, we shall exhibit a metriplectic realization of the Rabinovich system in its most general form given in (4.1).

Proposition 4.2. *The Rabinovich system* (4.1) *given by*

$$\begin{cases} \dot{x} = hy - \nu_1 x + yz \\ \dot{y} = hx - \nu_2 y - xz \\ \dot{z} = -\nu_3 z + xy, \end{cases}$$

is in the bi-Hamiltonian/Nambu metriplectic formulation given by

$$(\dot{x}, \dot{y}, \dot{z})^T = \nabla H_1 \times \nabla H_2 - G\nabla H_1.$$

where the Hamiltonian functions are $H_1 = \frac{1}{2}(x^2 + y^2)$*, and* $H_2 = \frac{1}{2}(y^2 + z^2)$*, and the metric tensor is*

$$G = \begin{pmatrix} \nu_1 & -h & 0 \\ -h & \nu_2 & \frac{z\nu_3}{y} \\ 0 & \frac{z\nu_3}{y} & 0 \end{pmatrix}.$$

5. Hindmarsh–Rose model

The Hindmarsh–Rose model of the action potential which is a modification of Fitzhugh model was proposed as a mathematical representation of the bursting behaviour of neurones, and was expected to simulate the repetitive, patterned and irregular activities seen in molluscan neurones [33]. The Hindmarsh–Rose model consists of a system of three autonomous differential equations, with mild nonlinearities for modelling neurons that exhibit triggered firing. The usual form of the equations are

$$\begin{cases} \dot{x} = y + \phi(x) - z - C \\ \dot{y} = \psi(x) - y \\ \dot{z} = r(s(x - x_R) - z) \end{cases}$$

where $\phi(x) = ax^2 - x^3$ and $\psi(x) = 1 - bx^2$. Here C is a control parameter while the remaining parameters are usually fixed. We rewrite the system in the following form appending two extra parameters

$$\begin{cases} \dot{x} = y - z - ax^3 + bx^2 + \alpha \\ \dot{y} = \beta - dx^2 - y \\ \dot{z} = px - rz - \gamma, \end{cases} \tag{5.1}$$

where $\alpha, \beta, \gamma, a, b, d, p, r$ are the constant parameters. Unfortunately we have not found a first integral with $a \neq 0$, which is the dominant nonlinear term here.

PROPOSITION 5.1. *The reduced Hindmarsh–Rose system*

$$\begin{cases} \dot{x} = y - z + bx^2 + \alpha \\ \dot{y} = \beta - dx^2 - y \\ \dot{z} = px - rz - \gamma \end{cases}$$

has the following first integrals.

(1) *If* $p = 0$ *then* $I = e^{rt}(rz + \gamma)$.
(2) *If* $d = 0$ *then* $I = e^{t}(y - \beta)$.
(3) *If* d, β, γ *are arbitrary,* $b = -d$, $p = -2$, $\alpha = \beta + \gamma$, *and* $r = 1$, *then* $I = e^{2t}(x - y + z)$.
(4) *If* α, γ *and* b *are arbitrary,* $p \neq 0$, $d = 2b$, $r = -(p+1)$, *and* $\beta = 2(\frac{\gamma}{p} - \alpha)$, *then* $I = e^{-t}(2x + y + \frac{2z}{p})$.
(5) *If* β, γ, r *are arbitrary,* $b \neq 0$, $d \neq 0$,

$$\alpha = -\frac{b(\gamma d + \beta d - b\beta + r\beta b)}{d(d - b + br)} \qquad \textit{and} \qquad p = \frac{(b-d)(d - b + br)}{b^2},$$

then the first integral becomes

$$I = e^{\frac{2(b-d)}{b}}(Ax^2 + By^2 + Cz^2 + Exy + Fxz + Gyz)$$

where the coefficients of the polynomial are given by

$$A = -\frac{(b-d)(d-b+br)}{b(-d+2b+br)}, \qquad B = -\frac{b(b-d)(d-b+br)}{d^2(-d+2b+br)},$$
$$C = -\frac{b(b-d)}{(d-b+br)(-d+2b+br)}, \qquad E = -2\frac{(b-d)(d-b+br)}{d(-d+2b+br)},$$
$$F = 2\frac{b-d}{-d+2b+br}, \qquad G = 2\frac{b(b-d)}{d(-d+2b+br)}.$$

(6) *If* $p = 0$, $b = d$, $r \neq 0$, β, γ *are arbitrary, and* $\alpha = -\frac{\beta r+\gamma}{r}$ *then* $I = rx + ry - z$. *When, additionally,* $r = -1$, *then* $I = x + y + z$.

The proof of these cases is similar to the ones done in the case of the reduced three-wave interaction model. In order to arrive the proposition, we have started with a generic second order Darboux' polynomial and runs the equation (2.1). This gives a set of equations determining the coefficients of the generic Darboux' polynomial. Then, a careful case analysis results the first integrals. To make a cross check of the validity of the proposition, one may, alternatively, take the total time derivatives of the integrals and show that they are zero.

In the following proposition, we are achieving to write the Hindmarsh–Rose model in a metriplectic form of the second kind.

Proposition 5.2. *The Hindmarsh–Rose model*

$$\begin{cases} \dot{x} = y - z - ax^3 + bx^2 + \beta - \gamma \\ \dot{y} = \beta - dx^2 - y \\ \dot{z} = px + z - \gamma, \end{cases}$$

the one presented in (5.1) *with* $r = -1$ *and* $\alpha = \beta - \gamma$, *is in the bi-Hamiltonian Nambu metriplectic form*

$$(\dot{x}, \dot{y}, \dot{z})^T = \nabla H_1 \times \nabla H_2 - G\nabla H_1.$$

where the Hamiltonian functions are $H_1 = x + y + z$, *and* $H_2 = yz - \gamma y - \beta z$, *and the metric tensor is*

$$G = \begin{pmatrix} ax^3 - bx^2 & 0 & 0 \\ 0 & dx^2 & 0 \\ 0 & 0 & -px \end{pmatrix}.$$

6. Oregonator model

The Oregonator model was developed in [16] to illustrate the mechanism of the Belousov-Zhabotinsky oscillatory reaction. The model can be expressed in terms of coupled three ordinary differential equations

$$\begin{cases} \dot{x} = \frac{1}{\epsilon}(x + y - qx^2 - xy) \\ \dot{y} = -y + 2hz - xy \\ \dot{z} = \frac{1}{p}(x - z). \end{cases} \tag{6.1}$$

that describe the complex dynamics of the reaction process. In the physical model, all the parameters ϵ, q, p, h are positive. However, from a purely mathematical point of view, allowing the parameters to be negative, we obtain a first integral

$$I = e^{2t}(x + y + z),$$

for the parameters $q = 0$,$\epsilon = p = -1$ and $h = -\frac{3}{2}$ as may be easily verified.

We put the Oregonator model in the form of Hamilton's equations as follows. At first, we change the coordinates according to

$$u = xe^{2t}, \quad v = ye^{2t}, \quad w = e^{2t}z.$$

This enables us to write the system (6.1) as the following non-autonomous form

$$\begin{cases} \dot{u} = u - v + uve^{-2t} \\ \dot{v} = v - 3w - uve^{-2t} \\ \dot{w} = 3w - u \end{cases}$$

which is in the Hamiltonian form

$$(\dot{u}, \dot{v}, \dot{w})^T = N\nabla H$$

generated by $H = u + v + w$. Here the Poisson matrix is defined by

$$N = \begin{pmatrix} 0 & uve^{-2t} - v & u \\ v - uve^{-2t} & 0 & -3w \\ -u & 3w & 0 \end{pmatrix}.$$

7. Conclusions

In this paper, we have pointed out some theoretical details on the $3D$ polynomial systems such as integrability, Hamiltonian realization, and metriplectic formulation. We proposed a method to write a time-dependent integral of a system using its Darboux polynomials. Then, we have derived the first integrals of the reduced three-wave interaction model, the Rabinovich system, the Hindmarsh–Rose model, see the propositions 3.1, 4.1, and 5.1, respectively. We have achieved to exhibit metriplectic realizations of these systems, cf. propositions 3.2, 4.2, and 5.2, respectively. In the last section, we have derived a first integral of the Oregonator model and exhibited its Hamiltonian character.

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О ИНТЕГРАЛИМА, ХАМИЛТОНОВОЈ И МЕТРИПЛЕКТИЧКОЈ ФОРМУЛАЦИЈИ ПОЛИНОМИЈАЛНИХ СИСТЕМА У 3D

Резиме. Користећи методу Дарбуових полинома, изведени су први интеграли редукованог проблема троталасне интеракције, Рабиновичевог система, Хиндмарш - Розе и Орегонатор модела. Показано је да се редуковани проблем троталасне интеракције, Рабиновичев система и Хиндмарш - Розе модел могу представити у би-Хамилтоновој/Намбу метриплектичкој форми.

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