THEORETICAL AND APPLIED MECHANICS
Volume 44 (2017) Issue 1, 115–132 DOI: https://doi.org/10.2298/TAM170601006M

EXTENDED LAGRANGIAN FORMALISM FOR RHEONOMIC SYSTEMS WITH VARIABLE MASS

Djordje Mušicki

Abstract. In this paper the extended Lagrangian formalism for the rheonomic systems (Dj. Mušicki, 2004), which began with the modification of the mechanics of such systems (V. Vujičić, 1987), is extended to the systems with variable mass, with emphasis on the corresponding energy relations. This extended Lagrangian formalism is based on the extension of the set of chosen generalized coordinates by new quantities, suggested by the form of nonstationary constraints, which determine the position of the frame of reference in respect to which these generalized coordinates refer.

As a consequence, an extended system of the Lagrangian equations is formulated, accommodated to the variability of the masses of particles, where the additional ones correspond to the additional generalized coordinates. By means of these equations, the energy relations of such systems have been studied, where it is demonstrated that here there are four types of energy conservation laws. The obtained energy laws are more complete and natural than the corresponding ones in the usual Lagrangian formulation for such systems.

It is demonstrated that the obtained energy laws, are in full accordance with the energy laws in the corresponding vector formulation, if they are expressed in terms of the quantities introduced in this formulation of mechanics.

The obtained results are illustrated by an example: the motion of a rocket, which ejects the gasses backwards, while this rocket moves up a straight line on an oblique plane, which glides uniformly in a horizontal direction.

1. Introduction

The modification of the analytical mechanics of rheonomic systems began with the papers of V. Vujičić [1–3], in the aim to study more thoroughly the energy relations of such systems in the frames of Lagrangian formulation, based on the usage of generalized coordinates, where in the usual Lagrangian formulation for the rheonomic systems the corresponding energy laws do not comprehead all the factors which influence on the energy relations. By introduction of one additional generalized coordinate, suggested by the form of nonstationary constraints, the main general principles of mechanics are formulated and the associated Lagrangian

2010 *Mathematics Subject Classification:* 70H03; 70F20; 70G10.

Key words and phrases: extended Lagrangian formalism, rheonomic systems with variable mass, generalized Emmy Noether's theorem.

and Hamiltonian equations are obtained. By aid of them, the corresponding energy relations are studied and, inter alia, an energy conservation law is obtained in an unexpected form, which contains an additional term, arising from the nonstationary constraints.

Another approach to this problem was given by the author himself (Dj. Mušicki, [4–6]) in the form of a parametric formulation of mechanics, which is based on the substitution of time as parameter by a new parameter, which is taken as an additional generalized coordinate. A few years ago, this formulation is transformed, also by the author (Dj. Mušicki, [7–9]) to so–called extended Lagrangian formalism, which is based on the extension of the set of chosen generalized coordinates by some new characteristic quantities $q^a = \varphi_a(t)$, which as given functions of time figure in the nonstationary constraints $f_\mu[\vec{r}_\nu, \varphi_a(t)] = 0$. It is demonstated that these quantities q^a have an important kinematic meaning: they determine the position of the frame of reference to which these generalized coordinates refer with respect to an immobile frame of reference. In this way, starting from the corresponding d'Alembert–Lagrange's principle, the extended Lagrangian equations are formulated, and by aid of them the corresponding energy relations are examined, where is demonstrated that there are four types of the energy change law $d\mathcal{E}/dt$ and the corresponding conservation laws with the introduced characteristic quantities.

The study of the energy relations for the rheonomic systems in the usual Lagrangian formulation was recently effectuated in an indirect way by means of the generalized Noether's theorem, obtained form transformed d'Alembert–Lagrange's principle by B.Vujanović and Dj.Djukić [10–12] and extended to the systems with variable mass by L. Cvetićanin [13, 14]. But these results can be obtained in a direct way as well, generalizing the usual proof of Noether's theorem for the systems with permanent and with variable mass (Dj. Mušicki [15,16]). Their method of finding energy integrals is more general, but in the case of rheonomic systems essentially different from here presented method.

2. Basic ideas of extended Lagrangian formalism

Let us consider the motion of a mechanical system of N particles under the influence of arbitrary forces, limited by k nonstationary holonomic constraints

$$f_\mu[\vec{r}_\nu, \varphi_a(t)] = 0 \quad (\mu = 1, 2, \dots, k), \tag{2.1}$$

where we supposed that time always appears through some functions $\varphi_a(t)$. Let us determine the position of this mechanical system by a set of generalized coordinates q^i $(i = 1, 2, \dots, n)$, where $n = 3N - k$, with respect to a suitable frame of reference, which due to the form of constraints (2.1) is always moving in the course of time.

The fundamental idea of this formulation of mechanics (Dj. Mušicki, [7–9]) is based on the introduction of new quantities, which are suggested by the form of nonstationary constraints and which change according to the law $\tau^a = \varphi_a(t)$, and on the extension of the set of chosen generalized coordinates by these quantities $q^a = \varphi_a(t)$. It has been demonstrated that these quantities determine the position of the frame of reference to which these generalized coordinates refer in respect to an immobile frame of reference. In this way, the complete set of generalized coordinates will be

$$q^\alpha = \{q^i \ (i = 1, 2, \dots, n),\ q^a(t) \ (a = n+1, \dots, n+A)\}, \tag{2.2}$$

where the first n generalized coordinates q^i determine the position of the mechanical system with respect to the corresponding associated moving frame of reference and the additional ones q^a, the most often only one, the position of this reference frame with respect to an inertial one. In distinction from the usual generalized coordinates, the dependence of these additional ones q^a on time is given in advance and determined by the form of nonstaionary constraints.

Since in this case $\vec{r}_\nu = \vec{r}_\nu[q^i, q^a(t)]$, the velocity of any particle can be presented as

$$\vec{v}_\nu = \frac{d\vec{r}_\nu}{dt} = \frac{\partial \vec{r}_\nu}{\partial q^i}\dot{q}^i + \frac{\partial \vec{r}_\nu}{\partial q^a}\dot{q}^a = \frac{\partial \vec{r}_\nu}{\partial q^\alpha}\dot{q}^\alpha,$$

where the summation over the repeated index is understood, and the kinetic energy of the system in this formulation of mechanics is always a homogeneous quadratic function of the generalized velocities

$$T = \frac{1}{2} m_\nu \vec{v}_\nu^{\,2} = \frac{1}{2} a_{\alpha\beta}\dot{q}^\alpha \dot{q}^\beta, \tag{2.3}$$

where

$$a_{\alpha\beta} = m_\nu \frac{\partial \vec{r}_\nu}{\partial q^\alpha} \cdot \frac{\partial \vec{r}_\nu}{\partial q^\beta} \tag{2.4}$$

These coefficients $a_{\alpha\beta}$ represent the components of the metric tensor of the corresponding extended configuration space.

An important characteristic of this extended Lagrangian formalism concerns the elementary work of the ideal forces of constraints $\vec{R}_\nu^{\text{id}} \cdot \delta\vec{r}_\nu$, where $\vec{R}_\nu^{\text{id}} = -\lambda_\mu(\partial f_\mu / \partial \vec{r}_\nu)$ and $\delta\vec{r}_\nu$ the virtual displacements of ν-th particle. Here we must extend the notion of the variation of positions so that the additional generalized coordinates $q^a = \tau^a$ also must be varied, and then by varying the constraints (2.1) in so extended sense we obtain the conditions which must be satisfied by the virtual displacements $\delta\vec{r}_\nu$ in the form

$$\frac{\partial f_\mu}{\partial \vec{r}_\nu} \cdot \delta\vec{r}_\nu + \frac{\partial f_\mu}{\partial \tau^a}\delta\tau^a = 0 \quad (\mu = 1, 2, \dots, n)$$

On the base of these results this work will be presented as

$$d'A = \vec{R}_\nu^{\text{id}} \cdot \delta\vec{r}_\nu = R_a^0 \delta\tau^a, \tag{2.5}$$

where

$$R_a^0 = -\lambda_\mu \frac{\partial f_\mu}{\partial \tau^a} = \vec{R}_\nu^{\text{id}} \cdot \frac{\partial \vec{r}_\nu}{\partial q^a} \tag{2.6}$$

So defined quantities R_a^0 arise from the nonstationary constraints, i.e. from the additional generalized coordinates and the property that this work $\vec{R}_\nu^{\text{id}} \cdot \delta\vec{r}_\nu$ is different from zero is one of the essential characteristics of this formulation of mechanics of rheonomic systems, in distinction from the usual Lagrangian formulation, where $\vec{R}_\nu^{\text{id}} \cdot \delta\vec{r}_\nu = 0$.

In this formulation of the rheonomic systems with permanent mass an extended set of the Lagrangian equations is formulated, where the additional Lagrangian equations correspond to the additional generalized coordinates. By means of these equations the corresponding energy relations for such systems are analyzed, and it is demonstrated (Dj. Mušicki [7]) that here there are two types of the energy integrals. Under certain conditions the energy conservative law has the form

$$\mathcal{E} = T + \mathcal{U} = const.,$$

in contrast to the corresponding Painlevé's energy integral $\mathcal{E} = T_2 - T_0 + \mathcal{U} = const.$ in the usual Lagrangian formulation for the rheonomic systems, and if the expression $R^0_a dq^a$ can be presented as a total differential of some function, it has an unexpect form

$$\mathcal{E}^{ex} = T + \mathcal{U} + \mathcal{P} = const., \tag{2.7}$$

where the additional term $\mathcal{P}$ arises from the influence of the nonstationary constraints and which coincides with the result obtained by V. Vujičić [1].

The obtained results (2.2)–(2.6) are independent from any supposition about the masses of particles, therefore they remain valid for the systems with variable mass as well. However, the energy relations, which are a consequence of the corresponding Lagrangian equations for the systems with variable mass will be more general than for the systems with permanent mass.

The main advantage of such formulation of the rheonomic systems is that the extended generalized coordinates determine the position of considered system in respect to the same frame of reference, to which all the dynamical quantities and the energy laws refer. Due to this property the energy laws in this formulation will be more complete and natural, including all the factors which influence on the energy relations.

3. D'Alembert–Lagrange's principle and Lagrangian equations

Let us suppose that, except the influence of the active forces and the forces of constraints on the particles of the system, there occurs a simultaneous process of the separation and annexation of some mass concerning each particle of this system. The influence of this process on ν-th particle can be described by so-called Meschchersky's reactive force, which is given by (see e.g. Kosmodem'janski, [17, p. 62])

$$\vec{\phi}_\nu = \frac{dm_{\nu 1}}{dt}(\vec{u}_{\nu_1} - \vec{v}_\nu) + \frac{dm_{\nu 2}}{dt}(\vec{u}_{\nu_2} - \vec{v}_\nu) \equiv \frac{dm_{\nu\sigma}}{dt}(\vec{u}_{\nu\sigma} - \vec{v}_\nu). \tag{3.1}$$

Here $dm_{\nu 1}/dt$ and $dm_{\nu 2}/dt$ are the changes of the mass of ν-th particle per time unit in the process of separation and annexation respectively, $\vec{u}_{\nu 1}$ and $\vec{u}_{\nu 2}$ are the corresponding velocities of separation and annexation and $\vec{v}_\nu$ the velocity of ν-th particle in the instant t.

In order to obtain the corresponding differential equations of motion in the Lagrangian from, let us start from d'Alembert–Lagrange's principle

$$(\vec{F}^{\rm tot}_\nu - m_\nu \vec{a}_\nu) \cdot \delta \vec{r}_\nu = 0,$$

generalizing Kosmodem'janski's proof [17] of the corresponding Lagrangian equations in the usual Lagrangian formulation, accommodated to this formulation of rheonomic systems with variable mass. In this case $\vec{F}_\nu^{\rm tot} = \vec{F}_\nu + \vec{R}_\nu^{\rm id} + \vec{R}_\nu^* + \vec{\Phi}_\nu$, where we have decomposed the forces of constraints into the ideal and nonideal ones, denoted by $\vec{R}_\nu^*$, and where the Meschchersky's reactive force $\vec{\phi}_\nu$ is added to the other ones. Then d'Alembert–Lagrange's principle (3.1) obtains the form

$$(\vec{F}_\nu + \vec{R}_\nu^{\rm id} + \vec{R}_\nu^* + \vec{\phi}_\nu - m_\nu \vec{a}_\nu) \cdot \delta\vec{r}_\nu = 0$$

and if we substitute $\vec{R}_\nu^{\rm id} \cdot \delta\vec{r}_\nu$ by the expression (2.5), we can write the previous relation in the form

$$(\vec{F}_\nu + \vec{R}_\nu^* + \vec{\phi}_\nu - m_\nu \vec{a}_\nu) \cdot \delta\vec{r}_\nu = -R_a^0 \delta\tau^a,$$

where R_a^0 is given by (2.6). This relation represents d'Alembert–Lagrange's principle for the considered systems with variable mass in this formulation of rheonomic systems.

We can this principle express in the extended generalized coordinates, putting $\delta\vec{r}_\nu = (\partial\vec{r}_\nu/\partial q^\alpha)\delta q^\alpha$ and $R_a^0\delta\tau^a = R_a^0\delta q^a$, whereby we get

$$\Big(\vec{F}_\nu \cdot \frac{\partial\vec{r}_\nu}{\partial q^\alpha} + \vec{R}_\nu^* \cdot \frac{\partial\vec{r}_\nu}{\partial q^\alpha} + \vec{\phi}_\nu \cdot \frac{\partial\vec{r}_\nu}{\partial q^\alpha} - m_\nu \vec{a}_\nu \cdot \frac{\partial\vec{r}_\nu}{\partial q^\alpha}\Big)\delta q^\alpha = -R_a^* \delta q^a,$$

what can be written concisely in the form

$$\Big(Q_\alpha + \mathcal{R}_\alpha^* + \Phi_\alpha + \delta_\alpha^a R_a^0 - m_\nu \vec{a}_\nu \cdot \frac{\partial\vec{r}_\nu}{\partial q^\alpha}\Big)\delta q^\alpha = 0, \tag{3.2}$$

where

$$Q_\alpha = \vec{F}_\nu \cdot \frac{\partial\vec{r}_\nu}{\partial q^\alpha}, \quad \mathcal{R}_\alpha^* = \vec{R}_\nu^* \cdot \frac{\partial\vec{r}_\nu}{\partial q^\alpha}, \quad \Phi_\alpha = \vec{\phi}_\nu \cdot \frac{\partial\vec{r}_\nu}{\partial q^\alpha} \tag{3.3}$$

Let us now transform the last term in parentheses in the following way

$$\begin{aligned} m_\nu \vec{a}_\nu \cdot \frac{\partial\vec{r}_\nu}{\partial q^\alpha} &= \frac{d}{dt}\Big(m_\nu \vec{v}_\nu \cdot \frac{\partial\vec{r}_\nu}{\partial q^\alpha}\Big) - \frac{dm_\nu}{dt}\vec{v}_\nu \cdot \frac{\partial\vec{r}_\nu}{\partial q^\alpha} - m_\nu \vec{v}_\nu \cdot \frac{d}{dt}\frac{\partial\vec{r}_\nu}{\partial q^\alpha} \\ &= \frac{d}{dt}\Big(m_\nu \vec{v}_\nu \cdot \frac{\partial\vec{v}_\nu}{\partial \dot{q}_\alpha}\Big) - \frac{dm_\nu}{dt}\vec{v}_\nu \cdot \frac{\partial\vec{r}_\nu}{\partial q^\alpha} - m_\nu \vec{v}_\nu \cdot \frac{\partial\vec{v}_\nu}{\partial q^\alpha} \end{aligned}$$

where we have put $\partial\vec{r}_\nu/\partial q^\alpha = \partial\vec{v}_\nu/\partial\dot{q}^\alpha$ and $\frac{d}{dt}(\partial\vec{r}_\nu/\partial q^\alpha) = \partial\vec{v}_\nu/\partial q^\alpha$, or

$$m_\nu \vec{a}_\nu \cdot \frac{\partial\vec{r}_\nu}{\partial q^\alpha} = \frac{d}{dt}\frac{\partial T}{\partial\dot{q}^\alpha} - \frac{dm_\nu}{dt}\vec{v}_\nu \cdot \frac{\partial\vec{r}_\nu}{\partial q^\alpha} - \frac{\partial T}{\partial q^\alpha},$$

where $T = \frac{1}{2}m_\nu\vec{v}_\nu^2$ is the kinetic energy of the system.

If we insert this expression into the relation (3.2) and substitute Φ_α by (3.3) and $\vec{\phi}_\nu$ by the expression (3.1), we obtain

$$\begin{aligned} \Big[Q_\alpha + \mathcal{R}_\alpha^* + \frac{dm_{\nu\sigma}}{dt}(\vec{u}_{\nu\sigma} - \vec{v}_\nu) \cdot \frac{\partial\vec{r}_\nu}{\partial q^\alpha} \\ + \delta_\alpha^a R_a^0 - \frac{d}{dt}\frac{\partial T}{\partial\dot{q}^\alpha} + \frac{dm_\nu}{dt}\vec{v}_\nu \cdot \frac{\partial\vec{r}_\nu}{\partial q^\alpha} + \frac{\partial T}{\partial q^\alpha}\Big]\delta q^\alpha = 0 \end{aligned}$$

or, by grouping the third and the sixth term

$$\Big(Q_\alpha + \mathcal{R}^*_\alpha + \frac{dm_{\nu\sigma}}{dt}\vec{u}_{\nu\sigma}\cdot\frac{\partial\vec{r}_\nu}{\partial q^\alpha} + \delta^a_\alpha R^0_a - \frac{d}{dt}\frac{\partial T}{\partial\dot{q}^\alpha} + \frac{\partial T}{\partial q^\alpha}\Big)\delta q^\alpha = 0$$

Since the variations (3.4) δq^α are arbitrary, this relation will be always satisfied only if the expression in parentheses is equal to zero

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{q}^a} - \frac{\partial T}{\partial q^\alpha} = Q_\alpha + \mathcal{R}^*_\alpha + P_\alpha + \delta^a_a R^0_a \quad (\alpha = 1, 2, \ldots, n + A), \tag{3.4}$$

where

$$P_\alpha = \frac{dm_{\nu\sigma}}{dt}\vec{u}_{\nu\sigma}\cdot\frac{\partial\vec{r}_\nu}{\partial q^\alpha}, \quad R^0_a = -\lambda_\mu\frac{\partial f_\mu}{\partial q^a} = \vec{R}^{\rm id}_\nu\cdot\frac{\partial\vec{r}_\nu}{\partial q^a} \tag{3.5}$$

These are the Lagrangian equations in the first form for the considered systems in this formulation of rheonomic systems. If we still decompose the generalized forces into the potential and nonpotential ones

$$Q_\alpha = Q^{\rm pot}_\alpha + Q^{\rm npot}_\alpha = -\frac{\partial V}{\partial q^\alpha} + \frac{d}{dt}\frac{\partial V}{\partial\dot{q}^\alpha} + Q^*_\alpha,$$

where V is the generalized potential and the nonpotential forces are denoted by Q^*_α, Lagrangian equations (3.4) by grouping the similar terms obtain the form

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha} = \tilde{Q}^*_\alpha + P_\alpha + \delta^a_\alpha R^0_a, \quad (\alpha = 1, 2, \ldots, n + A), \tag{3.6}$$

where

$$L(q^\alpha, \dot{q}^\alpha, t) = T - V, \quad \tilde{Q}^*_\alpha = Q^*_\alpha + \mathcal{R}^*_\alpha \tag{3.7}$$

These are the corresponding Lagrangian equations in the second, usual form and they can be decomposed into two groups of equations

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial L}{\partial\dot{q}^i} - \dfrac{\partial L}{\partial q^i} = \tilde{Q}^*_i + P_i & (i = 1, 2, \ldots, n) \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial\dot{q}^a} - \dfrac{\partial L}{\partial q^a} = \tilde{Q}^*_a + P_a + R^0_a & (a = n + 1, \ldots, n + A) \end{cases}$$

The first n Lagrangian equations coincide with the corersponding ones in the usual Lagrangian formulations, obtained by A. Kosmodem'janski 1951 [18], therefore they give the same equations of motion $q^i = q^i(t)$ $(i = 1, 2, \ldots, n)$ as in the usual formulation of mechanics. But, the additional Lagrangian equations (the most often only one) are specific for this formulation and they together with the first ones will give more complete energy laws.

4. Energy relations in this formulation of mechanics

In order to examine the energy relations for the systems with variable mass in this formulation of mechanics, let us start from the Lagrangian equations (3.6), multiply them by $dq^\alpha = \dot{q}^\alpha dt$ and sum over the repeated index

$$d\Big(\frac{\partial L}{\partial\dot{q}^\alpha}\Big)\dot{q}^\alpha - \frac{\partial L}{\partial q^\alpha}dq^\alpha = (\tilde{Q}^*_\alpha + P_\alpha + \delta^a_\alpha R^0_a)dq^\alpha$$

If we transform the expression on the left–hand side in the following way

$$d\Big(\frac{\partial L}{\partial \dot q^\alpha}\dot q^\alpha\Big) - \frac{\partial L}{\partial \dot q^\alpha}d\dot q^\alpha - \frac{\partial L}{\partial q^\alpha}dq^\alpha = (\tilde Q^*_\alpha + P_\alpha)dq^\alpha + R^0_a dq^a$$

and since in this case the Lagrangian depends on the masses as well

$$dL = \frac{\partial L}{\partial q^\alpha}dq^\alpha + \frac{\partial L}{\partial \dot q^\alpha}d\dot q^\alpha + \frac{\partial L}{\partial m_\nu}dm_\nu + \frac{\partial L}{\partial t}dt,$$

the previous relation becomes

$$d\Big(\frac{\partial L}{\partial \dot q^\alpha}\dot q^\alpha\Big) - \Big(dL - \frac{\partial L}{\partial t}dt - \frac{\partial L}{\partial m_\nu}dm_\nu\Big) = \tilde Q^*_\alpha dq^\alpha + P_\alpha dq^\alpha + R^0_a dq^a$$

If we still group two similar terms

$$d\Big(\frac{\partial L}{\partial \dot q^\alpha}\dot q^\alpha - L\Big) = -\frac{\partial L}{\partial t}dt - \frac{\partial L}{\partial m_\nu}\frac{dm_\nu}{dt}dt + \tilde Q^*_\alpha dq^\alpha + P_\alpha dq^\alpha + R^0_a dq^a$$

and divide this relation by dt, bearing in mind that the expression in parentheses represents the generalized energy of the system, it can be written in the form

$$\frac{d\mathcal{E}}{dt} = \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot q^\alpha}\dot q^\alpha - L\Big) = -\Big(\frac{\partial L}{\partial t}\Big)^* + \tilde Q^*_\alpha \dot q^\alpha + P_\alpha \dot q^\alpha + R^0_a \dot q^a, \tag{4.1}$$

where

$$\Big(\frac{\partial L}{\partial t}\Big)^* = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial m_\nu}\frac{dm_\nu}{dt} \tag{4.2}$$

This relation represents the general energy change law for the systems with variable mass in this formulation of mechanics of rheonomic systems. So defined $(\partial L/\partial t)^*$ represents the specific dependence of the Lagrangian on time, its possible explicit dependance as well as implicit one through the masses of particles, and the term P_α expresses the influence of the variability of these masses on the energy relations.

Here, in dependence of the structure of the two last terms, there are four possible cases, which give the different types of the energy change law and the corresponding conservation laws.

a) In the general case, the energy change law is given by the general relation (4.1). The expression in parentheses, the corresponding generalized energy, if all the generalized forces are potential, by application of Euler's theorem for homogeneous functions will be

$$\mathcal{E} = \frac{\partial T}{\partial \dot q^\alpha}\dot q^\alpha - L = 2T - (T - U) = T + U,$$

Then, from (4.1) we immediately conclude that if $(\partial L/\partial t)^* = 0$, $\tilde Q^*_\alpha \dot q^\alpha = 0$, $P_\alpha \dot q^\alpha = 0$ and $R^0_a \dot q^a = 0$, the energy conservation law will be valid in the form

$$\mathcal{E} = \frac{\partial L}{\partial \dot q^\alpha}\dot q^\alpha - L = T + U = const.,$$

which differs essentially from the corresponding Painlevé's energy integral $\mathcal{E} = T_2 - T_o + U = const.$ in the usual Lagrangian formulation for the rheonomic systems.

b) If the quantities P_α can be presented in the form

$$P_\alpha = -\frac{\partial \mathcal{P}}{\partial q^\alpha} \quad \Leftrightarrow \quad P_\alpha dq^\alpha = -d\mathcal{P}(q^\alpha), \tag{4.3}$$

the energy change law (4.1) by grouping the similar terms, can be written as

$$\begin{aligned} \frac{d\mathcal{E}^{\text{ext}}}{dt} &= \frac{d}{dt}\Big(\frac{\partial L'}{\partial \dot{q}^\alpha}\dot{q}^\alpha - L'\Big) = -\Big(\frac{\partial L}{\partial t}\Big)^* + \tilde{Q}^*_\alpha \dot{q}^\alpha + R^0_a \dot{q}^a, \\ & L'(q^\alpha, \dot{q}^\alpha, t) = L - \mathcal{P} = T - (U + \mathcal{P}) \end{aligned} \tag{4.4}$$

This generalized energy, bearing in mind that $\mathcal{P}_\alpha$ is independent from $\dot{q}^\alpha$, by application of the Euler's theorem will be

$$\mathcal{E}^{\text{ext}} = \frac{\partial T}{\partial \dot{q}^\alpha}\dot{q}^\alpha - L' = 2T - (T - U - \mathcal{P}) = T + U + \mathcal{P}$$

and from (4.4) we see that if $(\partial L/\partial t)^* = 0$, $\tilde{Q}^*_\alpha \dot{q}^\alpha = 0$ and $R^0_a \dot{q}^\alpha = 0$, the energy conservation law will have the form

$$\mathcal{E}^{\text{ext}} = \frac{\partial L'}{\partial \dot{q}^\alpha}\dot{q}^\alpha - L' = T + U + \mathcal{P} = const.$$

So defined quantity $\mathcal{P}$, can be named the mass potential, since from here we see that it has an important role in the energy laws.

c) If the quantities R^0_a can be presented in the form

$$R^0_a = -\frac{\partial \mathcal{R}}{\partial q^a} \quad \leftrightarrow \quad R^0_a dq^a = -d\mathcal{R}(q^a) \tag{4.5}$$

the energy change law will be

$$\begin{aligned} \frac{d\mathcal{E}^{\text{ext}}_1}{dt} &= \frac{d}{dt}\Big(\frac{\partial L_1}{\partial \dot{q}^\alpha}\dot{q}^\alpha - L_1\Big) = -\Big(\frac{\partial L}{\partial t}\Big)^* + Q^*_\alpha \dot{q}^\alpha + P_\alpha \dot{q}^\alpha, \\ & L_1 = L - \mathcal{R} = T - (U + \mathcal{R}) \end{aligned}$$

In a similar way we can see that this generalized energy is $\mathcal{E}^{\text{ext}} = T + U + \mathcal{R}$, from where follows that if $(\partial L/\partial t)^* = 0$, $\tilde{Q}^*_\alpha \dot{q}^\alpha = 0$ and $P_\alpha \dot{q}^\alpha = 0$, the energy conservation law will be valid in the form

$$\mathcal{E}^{\text{ext}}_1 = \frac{\partial L_1}{\partial \dot{q}^\alpha}\dot{q}^\alpha - L_1 = T + U + \mathcal{R} = const.$$

So introduced quantity $\mathcal{R}$ is named the rheonomic potential (V. Vujičić [1]), it coincides with the quantity P in the relation (2.7) in the mechanics of particles with permanent mass.

d) If both quantities P_α and R^0_a can be presented in these forms (4.3) and (4.5), the corresponding energy change law will be presented as

$$\begin{aligned} \frac{d\bar{\mathcal{E}}^{\text{ext}}}{dt} &= \frac{d}{dt}\Big(\frac{\partial L'_1}{\partial \dot{q}^\alpha}\dot{q}^\alpha - L'_1\Big) = -\Big(\frac{\partial L}{\partial t}\Big)^* + \tilde{Q}^*_\alpha \dot{q}^\alpha, \\ & L'_1 = L - (\mathcal{P} + \mathcal{R}) = T - (U + \mathcal{P} + \mathcal{R}) \end{aligned} \tag{4.6}$$

and if $(\partial L/\partial t)^* = 0$ and $Q^*_\alpha \dot{q}^\alpha = 0$, the corresponding energy conservation law will be

$$\bar{\mathcal{E}}^{\text{ext}} = \frac{\partial L'_1}{\partial \dot{q}^\alpha}\dot{q}^\alpha - L'_1 = T + U + \mathcal{P} + \mathcal{R} = const. \tag{4.7}$$

So obtained energy laws differ essentially from the corresponding ones in the usual Lagrangian formulation for such systems, where except the unnatural Painlevé's energy integral the quantities $\mathcal{P}$ and $\mathcal{R}$ do not exist. This means that in the usual Lagrangian formulation, in contrast to the corresponding vector formulation, the influence of nonstationary constraints is absent, and the explicit appearance of the variability of the masses is also absent. These differences arise from the fact that in the mechanics of rheonomic systems the chosen generalized coordinates always refer to a moving frame of reference, in distinction from all the dynamical quantities and the energy laws themselves, which refer to an immobile frame of reference. This inconsistency is removed by introduction of this formulation, where all the quantities, including extended generalized coordinates refer to a same frame of reference. For that, the obtained energy laws in this formulation are more complete and natural than the corresponding ones in the usual Lagrangian formulation of such systems, including the influence of nonstationary constraints. This will be affirmed by the fact that this formulation of rheonomic systems with variable mass is in full accordance with the corresponding vector formulation of such systems, where all the quantities also refer to a same frame of reference (see section 5).

However, let us remark that because of very strong conditions for the validity of the energy conservation law only in rare cases this conservation law is valid, except some trivial or artificial cases. But, in the absence of this conservation law these energy change laws can give some information how the energy of the considered mechanical system changes in the course of time.

5. Connection with the vector formulation of mechanics

Let us examine how these energy laws refer to the corresponding ones in the usual vector formulation of mechanics, and in this aim start from the corresponding kinetic energy law, which is independent from the type of the formulation of mechanics. In order to demonstrate this law for the systems with variable mass, let us depart from the corresponding fundamental equation of dynamics

$$m_\nu \frac{d\vec{v}_\nu}{dt} = \vec{F}_\nu + \vec{R}_\nu + \vec{\Phi}_\nu \quad (\nu = 1, 2, \dots, N)$$

and multiply it by $d\vec{r}_\nu = \vec{v}_\nu dt$, which gives

$$m_\nu \vec{v}_\nu \cdot d\vec{v}_\nu = \vec{F}_\nu \cdot d\vec{r}_\nu + \vec{R}_\nu \cdot d\vec{r}_\nu + \vec{\Phi}_\nu \cdot d\vec{r}_\nu$$

Since in this case we have

$$d\Big(\frac{1}{2} m_\nu \vec{v}_\nu^{\,2}\Big) = \frac{1}{2}\vec{v}_\nu^{\,2} dm_\nu + m_\nu \vec{v}_\nu \cdot d\vec{v}_\nu,$$

by inserting $m_\nu \vec{v}_\nu \cdot d\vec{v}_\nu$ from this relations into previous one we obtain

$$dT = d\Big(\frac{1}{2} m_\nu v_\nu^2\Big) = \vec{F}_\nu \cdot d\vec{r}_\nu + \vec{R}_\nu \cdot d\vec{r}_\nu + \vec{\Phi}_\nu \cdot d\vec{r}_\nu + \frac{1}{2} dm_\nu \vec{v}_\nu^{\,2} \tag{5.1}$$

and this represents the kinetic energy law for the systems with variable mass (see Kosmodem'janski, [17, p. 112–114]).

Let us transform this law, with the aim to obtain the corresponding energy change law, expressed through the quantities introduced in this formulation of mechanics. Let us suppose that the components of metric tensor do not depend explicitly on time, that the masses of particles are dependant only on time and the potential forces are usual. Since in this case

$$d\mathcal{U} = \frac{\partial \mathcal{U}}{\partial \vec{r}_\nu} d\vec{r}_\nu + \frac{\partial \mathcal{U}}{\partial m_\nu} dm_\nu + \frac{\partial \mathcal{U}}{\partial t} dt,$$

the potential part of the first term in the relation (5.1), the elementary work of the potential forces will be

$$\vec{F}_\nu^{\text{pot}} \cdot d\vec{r}_\nu = -\frac{\partial \mathcal{U}}{\partial \vec{r}_\nu} \cdot d\vec{r}_\nu = -d\mathcal{U} + \frac{\partial \mathcal{U}}{\partial t} dt + \frac{\partial \mathcal{U}}{\partial m_\nu} dm_\nu,$$

The fundamental part of the second term, the elementary work of the ideal forces of constraints can be found bearing in mind that the elementary displacements $d\vec{r}_\nu$ are limited by the conditions

$$\frac{\partial f_\mu}{\partial \vec{r}_\nu} \cdot d\vec{r}_\nu + \frac{\partial f_\mu}{\partial t} dt = 0 \quad (\mu = 1, 2, \dots, k),$$

obtained by differentiating the nonstationary constraints $f_\mu(\vec{r}_\nu, t) = 0$. Then this elementary work of the ideal forces of constraints $\vec{R}_\nu^{\text{id}} = \lambda_\mu(\partial f_\mu / \partial \vec{r}_\nu)$ can be presented as

$$\vec{R}_\nu^{\text{id}} \cdot d\vec{r}_\nu = \lambda_\mu \frac{\partial f_\mu}{\partial \vec{r}_\nu} \cdot d\vec{r}_\nu = -\lambda_\mu \frac{\partial f_\mu}{\partial t} dt$$

and the coefficient next to dt, according to (2.6) and the terminology introduced in this formulation of mechanics can be interpreted as the quantity R_a^0 for $q^a = t$

$$\vec{R}_\nu^{\text{id}} \cdot d\vec{r}_\nu = R^0_{a(\tau_a = t)} dt, \quad R_a^0 = -\Big(\lambda_\mu \frac{\partial f_\mu}{\partial \tau_a}\Big)_{\tau_a = t}$$

The sum of the last two terms in (5.1), if we write explicitly the first term according to (3.1), can be presented as

$$\vec{\phi}_\nu \cdot d\vec{r}_\nu + \frac{1}{2} dm_\nu v_\nu^2 = \frac{dm_{\nu\sigma}}{dt} (\vec{u}_{\nu\sigma} - \vec{v}_\nu) \cdot d\vec{r}_\nu + \frac{1}{2} \frac{dm_\nu}{dt} \vec{v}_\nu \cdot d\vec{r}_\nu = \frac{dm_{\nu\sigma}}{dt} \Big(\vec{u}_{\nu\sigma} - \frac{1}{2}\vec{v}_\nu\Big) \cdot d\vec{r}_\nu$$

In this manner, by adding of the elementary work of the nonpotential active forces and the nonideal forces of constraints, the kinetic energy law (5.1) obtains the form

$$\begin{aligned} dT = &- d\mathcal{U} + \frac{\partial \mathcal{U}}{\partial t} dt + \frac{\partial \mathcal{U}}{\partial m_\nu} dm_\nu + R^0_{a(\tau^a = t)} dt \\ &+ (\vec{F}_\nu^* + \vec{R}_\nu^*) \cdot d\vec{r}_\nu + \frac{dm_{\nu\sigma}}{dt} \Big(\vec{u}_{\nu\sigma} - \frac{1}{2}\vec{v}_\nu\Big) \cdot d\vec{r}_\nu \end{aligned} \tag{5.2}$$

This law can be presented in the generalized coordinates as well, putting $d\vec{r}_\nu = (\partial \vec{r}_\nu / \partial q^\alpha) dq^\alpha$ and introducing a new added generalized coordinates $q^a = t$. Then the fifth term will be transformed as

$$(\vec{F}_\nu^* + \vec{R}_\nu^*) \cdot d\vec{r}_\nu = \Big(\vec{F}_\nu^* \cdot \frac{\partial \vec{r}_\nu}{\partial q^\alpha} + \vec{R}_\nu^* \frac{\partial \vec{r}_\nu}{\partial q^\alpha}\Big) dq^\alpha = \tilde{Q}_\alpha^* dq^\alpha,$$

with $\tilde{Q}^*_\alpha$ given by (3.7) and if we put in the fourth term $dt = dq^a$, the relation (5.2) passes to

$$d(T+\mathcal{U}) = \frac{\partial \mathcal{U}}{\partial t}dt + \frac{\partial \mathcal{U}}{\partial m_\nu}dm_\nu + R^0_{a(q^a=t)}dq^a + \tilde{Q}^*_\alpha dq^\alpha + \frac{dm_{\nu\sigma}}{dt}\Big(\vec{u}_{\nu\sigma} - \frac{1}{2}\vec{v}_\nu\Big)\cdot d\vec{r}_\nu$$

If we still transform the sum of second term and the second part of the last term in the following way

$$\frac{\partial \mathcal{U}}{\partial m_\nu}dm_\nu - \frac{dm_\nu}{dt}\cdot\frac{1}{2}\vec{v}_\nu\cdot d\vec{r}_\nu = \frac{dm_\nu}{dt}\Big(\frac{\partial \mathcal{U}}{\partial m_\nu} - \frac{1}{2}\vec{v}^{\,2}_\nu\Big)dt = \frac{dm_\nu}{dt}\Big(\frac{\partial \mathcal{U}}{\partial m_\nu} - \frac{\partial T}{\partial m_\nu}\Big)dt = -\frac{\partial L}{\partial m_\nu}\frac{dm_\nu}{dt}dt$$

and the previous relation gets the form

$$d(T+\mathcal{U}) = \frac{\partial \mathcal{U}}{\partial t}dt - \frac{\partial L}{\partial m_\nu}\frac{dm_\nu}{dt}dt + R^{0*}_{a(q^a=t)}dq^a + \tilde{Q}^*_\alpha dq^\alpha + \frac{dm_{\nu\sigma}}{dt}\vec{u}_{\nu\sigma}\cdot\frac{\partial \vec{r}_\nu}{\partial q^\alpha}dq^\alpha \tag{5.3}$$

Let us now divide this expression by dt and bear in mind that in the vector formulation of mechanics the kinetic energy does not depend explicitly on time, therefore $\partial L/\partial t = -\partial\mathcal{U}/\partial t$. So obtained relation can be written in the form

$$\frac{d\mathcal{E}}{dt} = \frac{d}{dt}(T+\mathcal{U}) = -\Big(\frac{\partial L}{\partial t}\Big)^* + \tilde{Q}^*_\alpha\dot{q}^\alpha + P_\alpha\dot{q}^\alpha + R^0_{a(q^a=t)}\dot{q}^a,$$

where in accordance with (3.5) and (4.2)

$$P_\alpha = \frac{dm_{\nu\sigma}}{dt}\vec{u}_{\nu\sigma}\cdot\frac{\partial\vec{r}}{\partial q^\alpha},\qquad \Big(\frac{\partial L}{\partial t}\Big)^* = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial m_\nu}\frac{dm_\nu}{dt},$$

This relation represents the corresponding energy change law, obtained from the kinetic energy law in the vector formulation of mechanics, it is in full accordance with the corresponding law (4.1), in this formulation of mechanics.

If P_α and $R^a_{a(q^a=t)}$ can be presented as the partial derivatives of some functions with respect to q^α and $q^a = t$ respectively, we can interprete these functions, in accordance with (4.3) and (4.5), as the quantities $\mathcal{P}$ and $\mathcal{R}_(q^a = t)$

$$P_\alpha = -\frac{\partial P}{\partial q^\alpha},\qquad R^0_{a(q^a=t)} = -\frac{d\mathcal{R}_{(q^a=t)}}{dq^a}$$

Then the sum of two last terms in the relation (5.3) will be, putting again $q^a = t$

$$P_\alpha\dot{q}^\alpha + R^0_{a(q^a=t)}\dot{q}^a = -\frac{\partial P}{\partial q^\alpha}\dot{q}^\alpha - \frac{d\mathcal{R}_{(q^a=t)}}{dq^a}\dot{q}^a = -\frac{d}{dt}[P + \mathcal{R}_{(q^a=t)}]$$

and the energy change law (5.3) by grouping the similar terms obtains the form

$$\frac{d\mathcal{E}^{\rm ext}}{dt} = \frac{d}{dt}\big[T + \mathcal{U} + \mathcal{P} + \mathcal{R}^0_{a(q^a=t)}\big] = -\Big(\frac{\partial L}{\partial t}\Big)^* + \tilde{Q}^*_\alpha\dot{q}^\alpha$$

From here follows: if $(\partial L/\partial t)^*$=0 and $\tilde{Q}^*_\alpha \dot{q}^\alpha = 0$, the energy corresponding law is valid in the form

$$\mathcal{E}^{\text{ext}} = T + \mathcal{U} + \mathcal{P} + \mathcal{R}_{a(q^a=t)} = const.,$$

what coincides with (4.7).

Therefore, the energy change laws, obtained in the vector formulation of mechanics for rheonomic systems and expressed through the terms introduced in this extended Lagrangian formalism, are in full accordance with the corresponding laws in this formulation of mechanics.

6. An example

Let us consider the motion of a rocket, which moves upwards in the Earth's gravitation field along an inclined straight line on an oblique plane, gliding in the horizontal direction uniformly according to the given law $x_A(t) = Vt$ (Fig. 1). Simultaneously, this rocket ejects the gasses backwards, in the same direction, but opposite sense, with the constant relative velocity $u = -u_0$ with respect to the frame of reference $Ax'y'z'$, with the origin in the moving point A, so that the mass of this rocket decreases according to the law $m = m_0 e^{-\beta t}$.

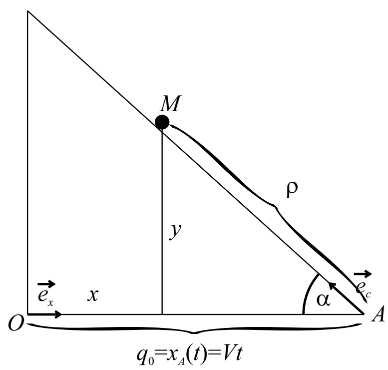

FIGURE 1. The particle, ejecting the gasses on a gliding inclined line

Treating this rocket as a mass point, in each instant it is necessary to be

$$\operatorname{tg}\alpha = \frac{y}{x_A(t) - x},$$

so that this motion is limited by a nonstationary constraint

$$f_1(\vec{r}, t) \equiv (x_A(t) - x)\sin\alpha - y\cos\alpha = 0, \quad f_2(\vec{r}, t) \equiv z = 0$$

This motion has only one degree of freedom and the position of the rocket can be determined by the generalized coordinate $q^1 = \rho$, presented on the figure. As the additional generalized coordinate it is necessary to take $q^2 \equiv q^0 = x_A(t) = Vt$, since it is such quantity, suggested by the form of nonstationary constraint, which changes according to the law $\tau = x_A(t) = Vt$. Therefore, the complete set of the generalized coordinates in this case will be

$$q^\alpha = \{q^1 = \rho, \quad q^0 = x_A = Vt\},$$

where the dependance on time of the additional generalized coordinates is given in advance.

The relations between the rectangular and these generalized coordinates are

$$x = q^0 - \rho\cos\alpha, \quad y = \rho\sin\alpha, \quad z = 0 \tag{6.1}$$

and if we denote the unit vectors of the x-axis and ρ-axis by $\vec{e}_x$ and $\vec{e}_\rho$, the position vector and velocity of this rocket will be

$$\vec{r} = q^0\vec{e}_x + \rho\vec{e}_\rho, \quad \vec{v} = \dot{q}^0\vec{e}_x + \dot{\rho}\vec{e}_\rho \tag{6.2}$$

The absolute velocity of the ejection of gasses, as the sum of the corresponding relative and transport velocity is

$$\vec{u} = -u_0\vec{e}_\rho + \dot{q}^0\vec{e}_x \tag{6.3}$$

Then, according to (6.1), the kinetic energy of this rocket will be, since $\vec{e}_\rho \cdot \vec{e}_x = -\cos\alpha$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m[\dot{\rho}^2 - 2\dot{\rho}\dot{q}^0\cos\alpha + (\dot{q}^0)^2]$$

and its Lagrangian, whereas $\mathcal{U} = mgy$

$$L = T - \mathcal{U} = \frac{1}{2}m\left[\dot{\rho}^2 - 2\dot{\rho}\dot{q}^0\cos\alpha + (\dot{q}^0)^2\right] - mg\rho\sin\alpha \tag{6.4}$$

Let us find the characteristics of variability of masses, the quantities $\mathcal{P}_\alpha$ defined by (3.5). Since the mass of this rocket changes according to the law $m = m_0e^{-\beta t}$, from where follows $dm/dt = -\beta m$, the quantities $\mathcal{P}_\rho$ and $\mathcal{P}_{q^0}$, according to (6.2) and (6.3) and since $\vec{e}_x \cdot \vec{e}_\rho = -\cos\alpha$ will be

$$P_\rho = \frac{dm}{dt}\vec{u} \cdot \frac{\partial\vec{r}}{\partial\rho} = -\beta m(-u_0\vec{e}_\rho + \dot{q}^0 \cdot \vec{e}_x) \cdot \vec{e}_\rho$$

$$P_{q^0} = \frac{dm}{dt}\vec{u} \cdot \frac{\partial\vec{r}}{\partial q^0} = -\beta m(-u_0\vec{e}_\rho + \dot{q}^0\vec{e}_x) \cdot \vec{e}_x$$

i.e.,

$$P_\rho = \beta m(u_0 + \dot{q}^0\cos\alpha), \quad P_{q^0} = -\beta m(u_0\cos\alpha + \dot{q}^0) \tag{6.5}$$

The quantity R_a^0 can be found either according to its definition (2.6) or by means of the corresponding Lagrangian equations (3.6). We shall here choose the second manner and in this aim let us form the corresponding Lagrangian equations, according to (6.4) and (6.5).

$$\begin{aligned} &\frac{d}{dt}(m\dot{\rho} - m\dot{q}^0\cos\alpha) + mg\sin\alpha = \beta m(u_0 + \dot{q}^0\cos\alpha) \\ &\frac{d}{dt}(m\dot{q}^0 - m\dot{\rho}\cos\alpha) = -\beta m(u_0\cos\alpha + \dot{q}^0) + R_0 \end{aligned} \tag{6.6}$$

In order to find R_0, let us resolve the first Lagrangian equation (6.6) with respect to $\frac{d}{dt}(m\dot{\rho})$, bearing in mind that $\dot{q}^0 = V = const.$ and $d\dot{q}^0/dt = 0$

$$\frac{d}{dt}(m\dot{\rho}) = \frac{dm}{dt}\dot{q}^0 \cos\alpha - mg\sin\alpha + \beta m(u_0 + \dot{q}^0\cos\alpha)$$

and insert this expression into the second one, putting $dm/dt = -\beta m$

$$\frac{dm}{dt}\dot{q}^0 - \cos\alpha\Big(\frac{dm}{dt}\dot{q}^0\cos\alpha - mg\sin\alpha + \beta m(u_0 + \dot{q}^0\cos\alpha)\Big) = -\beta m(u_0\cos\alpha + \dot{q}^0) + R_0$$

Since in this case $dm/dt = -\beta m$, this relation passes to

$$-\beta m\dot{q}^0 - \cos\alpha(-\beta m\dot{q}^0\cos\alpha - mg\sin\alpha + \beta m u_0 + \beta m\dot{q}^0\cos\alpha)$$
$$= -\beta m(u_0\cos\alpha + \dot{q}^0) + R_0,$$

from where follows

$$R_0 = mg\sin\alpha\cos\alpha$$

From here we see that the process of separation has not any influence on R_0, which expresses the influence of nonstationary constraints to energy relations.

Let us now study the energy relations for this rocket, and in this aim form the corresponding expression $P_\alpha dq^\alpha$ according to (6.5),

$$P_\alpha dq^\alpha = P_\rho d\rho + P_{q^0}dq^0 = \beta m(u_0 + \dot{q}^0\cos\alpha)d\rho - \beta m(u_0\cos\alpha + \dot{q}^0)dq^0 \equiv A\,d\rho + B\,dq$$

This is a total differential, since the necessary condition $\partial A/\partial q^0 = \partial B/\partial\rho$ is satisfied, whereas the coefficients A and B do not depend on ρ and q^0. Therefore, the mass potential exists, and according to (4.3) it is given by

$$\mathcal{P} = -\int P_\alpha dq^\alpha \equiv -\int \beta m(u_0 + \dot{q}^0\cos\alpha)d\rho + \int \beta m(u_0\cos\alpha + \dot{q}^0)dq^0$$

i.e.,

$$\mathcal{P} = -\beta m[(u_0 + \dot{q}^0\cos\alpha)\rho - (u_0\cos\alpha + \dot{q}^0)q^0] \tag{6.7}$$

Let us still form the expression $R^0_a q^a = R_0 q^0)$

$$R_0 dq^0 = mg\sin\alpha\cos\alpha\, dq^0 = d(mg\sin\alpha\cos\alpha q^0),$$

from where we see that it is a total differential. Therefore the quantity $\mathcal{R}$ also exists and it is determined by (4.5)

$$\mathcal{R} = -\int R_0 dq^0 = -mg\sin\alpha\cos\alpha\, q^0 \tag{6.8}$$

In this case both the expressions $P_\alpha dq^\alpha$ and $R_0 dq^0$ are the total differentials of some functions, therefore the conditions for the existence of the energy change law (4.6) are satisfied (but not for the existence of the corresponding energy integral (4.7) as well), this energy change law has the form

$$\frac{d\bar{\mathcal{E}}^{\text{ext}}}{dt} = \frac{d}{dt}\Big(\frac{\partial \mathcal{L}_1'}{\partial \dot{q}^\alpha}\dot{q}^\alpha - \mathcal{L}_1'\Big) = -\Big(\frac{\partial L}{\partial t}\Big)^* + \tilde{Q}^*_\alpha \dot{q}^\alpha$$

In this case the generalized energy according to (6.4), (6.7) and (6.8) is given by

$$(6.9)\quad \bar{\mathcal{E}}^{\text{ext}} = \frac{\partial L_1'}{\partial \dot{q}^\alpha}\dot{q}^\alpha - L_1' = T + \mathcal{U} + \mathcal{P} + \mathcal{R} = \frac{1}{2}m\big[\dot{\rho}^2 - 2\dot{\rho}\dot{q}^0\cos\alpha + (\dot{q}^0)^2\big]$$
$$+\, mg\rho\sin\alpha - \beta m[(u_0 + \dot{q}^0\cos\alpha)\rho - (u_0\cos\alpha + \dot{q}^0)q^0] + mg\sin\alpha\cos\alpha$$

and since in this case $\tilde{Q}^*_\alpha = 0$, the expression on the right–hand side according to (6.4) is equal to

$$\Big(\frac{\partial L}{\partial t}\Big)^* = \frac{\partial L}{\partial m}\frac{dm}{dt} = \Big\{\frac{1}{2}\big[\dot{\rho}^2 - 2\dot{\rho}\dot{q}^0 + (\dot{q}^0)^2\big] - g\rho\sin\alpha\Big\}(-\beta m)$$

and the energy change law (6.9) obtains the form

$$(6.10)\quad \frac{d\bar{\mathcal{E}}^{\text{ext}}}{dt} = \frac{d}{dt}(T + U + \mathcal{P} + \mathcal{R})$$
$$= -\beta\Big\{\frac{1}{2}m\Big[\dot{\rho}^2 - \frac{1}{2} - 2\dot{\rho}\dot{q}^0\cos\alpha + (\dot{q}^0)^2\Big] - mg\rho\sin\alpha\Big\}$$

In order to analyze the possibility of existence of conservation law of generalized energy $\mathcal{E}^{\text{ext}} = T + \mathcal{U} + \mathcal{P} + \mathcal{R}$ and the behavior of this generalized energy in the course of time, let us depart from the first Lagrangian equation (6.6) putting $\dot{q}^0 = V$ and $dm/dt = -\beta m$

$$\frac{d}{dt}(m\dot{\rho}) = \frac{dm}{dt}V\cos\alpha - mg\sin\alpha + \beta m(u_0 + V\cos\alpha)$$
$$= -m_0 e^{-\beta t}[\beta V\cos\alpha + g\sin\alpha - \beta(u_0 + V\cos\alpha)]$$

or more concisely

$$\begin{cases} \dfrac{d}{dt}(m\dot{\rho}) = -m_0 a e^{-\beta t} \\ a = \beta V\cos\alpha + g\sin\alpha - \beta(u_0 + V\cos\alpha) \end{cases}$$

By integration with respect to time we obtain

$$m\dot{\rho} = -m_0 a\int e^{-\beta t}dt = \frac{m_0 a}{\beta}e^{-\beta t} + c_1$$

and for $t = 0$ we have

$$m_0(\dot{\rho})_0 = m_0 v_0' = \frac{m_0 a}{\beta} + c_1$$

from where $c_1 = m_0 v_0' - \frac{m_0 a}{\beta}$. Then the previous relation will be

$$m\dot{\rho} = mv' = -\frac{m_0 a}{\beta}(1 - e^{-\beta t}) + m_0 v_0'$$

i.e.,

$$\dot{\rho} = -\frac{m_0 a}{\beta m}(1 - e^{-\beta t}) + \frac{m_0 v_0'}{m} = -\frac{a}{\beta}e^{\beta t}(1 - e^{-\beta t}) + v_0' e^{\beta t}$$

Thus, the relative velocity of the rocket with respect to the frame of reference $Ax'y'z'$ attached to the pole A is

$$(6.11)\quad v' = \dot{\rho} = e^{\beta t}\Big(v_0' - \frac{a}{\beta}\Big) + \frac{a}{\beta}$$

From here we see that the rocket move more quickly in the course of time.

Let us return now to the formula (6.10), where we must distinguish two cases, in dependance on the initial velocity v_0' of the rocket.

a) Case of the relatively small initial velocities v_0' - In the neighborhood of the initial position of the rocket, when the values of ρ are small, the potential energy of the rocket is smaller than its kinetic energy, from where according to (6.10) follows

$$mg\rho\sin\alpha < \frac{1}{2}m\big[\dot{\rho}^2 - 2\dot{\rho}\dot{q}^0\cos\alpha + (\dot{q}^0)^2\big] \quad \text{for} \quad 0<\rho<\rho_0 \quad \Rightarrow \quad \frac{d\bar{\mathcal{E}}^{\text{ext}}}{dt} < 0,$$

then the generalized energy $\bar{\mathcal{E}}^{\text{ext}} = T + \mathcal{U} + \mathcal{P} + \mathcal{R}$ decreases. When ρ increases, the potential energy of the rocket also increases and for some value $\rho = \rho_1$ becomes equal to its kinetic energy, then it will be $d\bar{\mathcal{E}}^{\text{ext}}/dt = 0$.

With further increasing of ρ the potential energy of the rocket will be greater than its kinetic energy, what gives as a consequence according to (6.10)

$$mg\rho\sin\alpha > \frac{1}{2}m\big[\dot{\rho}^2 - 2\dot{\rho}\dot{q}^0\cos\alpha + (\dot{q}^0)^2\big] \quad \text{for} \quad \rho_1<\rho<\rho_2 \quad \Rightarrow \quad \frac{d\bar{\mathcal{E}}^{\text{ext}}}{dt} > 0,$$

in this interval of ρ the generalized energy $\bar{\mathcal{E}}^{\text{ext}}$ increases. At the same time the kinetic energy of the rocket increases more rapidly than its potential energy. For some value $\rho = \rho_2$ it will be again equal to its potential energy, when $d\bar{\mathcal{E}}^{\text{ext}}/dt = 0$, and when ρ further increases, the kinetic energy of the rocket will be greater than its potential energy

$$mg\rho\sin\alpha < \frac{1}{2}m\big[\dot{\rho}^2 - 2\dot{\rho}\dot{q}^0\cos\alpha + (\dot{q}^0)^2\big] \quad \text{for} \quad \rho>\rho_2 \quad \Rightarrow \quad \frac{d\bar{\mathcal{E}}^{\text{ext}}}{dt} < 0.$$

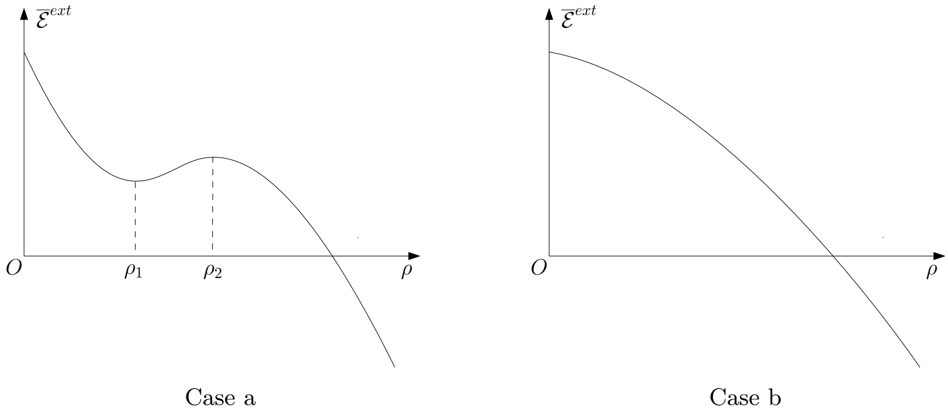

FIGURE 2

Therefore, in the case of relatively small values of v_0' the generalized energy $\bar{\mathcal{E}}^{\text{ext}}$ firstly decreases, then increases and finally again decreases, tending asymptotically to $\mathcal{E} = -\frac{\beta m_0 v_0'}{2}\rho + const.$ that is presented in Fig. 2.

b) Case of the big relative velocity v_0'. - In this case in the neighborhood of the initial position of the rocket, when the values of ρ are small, the potential energy of the rocket will be also smaller than its kinetic energy. However, because of the great values of v_0', the kinetic energy of the rocket now according to (6.11) will be very great, considerably greater than in the previous case. For that reason, in distinction from the previous case, when ρ increases the potential energy of the rocket never can overtake and surpass its kinetic energy. Thus, in the total domain of ρ the potential energy of the rocket will be smaller than its kinetic energy, from where follows according (6.10)

$$mg\rho\sin\alpha < \frac{1}{2}m\big[\dot\rho^2 - 2\dot\rho\dot q^0\cos\alpha + (\dot q^0)^2\big] \text{ for all values of } \rho \quad \Rightarrow \quad \frac{d\bar{\mathcal{E}}^{\text{ext}}}{dt} < 0.$$

Therefore, in the case of big values of v_0' the generalized energy $\bar{\mathcal{E}}^{\text{ext}}$ with the increasing of ρ always decreases, tending asymptotically to $\mathcal{E} = -\frac{\beta m_0 v_0'}{2}\rho + const.$ that is presented also in the Fig. 2.

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ПРОШИРЕНИ ЛАГРАНЖЕВ ФОРМАЛИЗАМ ЗА РЕОНОМНЕ СИСТЕМЕ ПРОМЕНЉИВЕ МАСЕ

Резиме. У овом раду проширени Лагранжев формализам за реономне системе (Ђ. Мушицки 2004), који потиче од модификације таквих система у (В. Вујичић 1987), проширен је за системе са променљивом масом, са нагласком на одговарајуће енеретске односе. Конструкција је базирана на проширењу изабраних уопштених координата новим величинама, које одговарају нестационарним везама.

Као последица, формулисан је проширени систем Лагранжевих једначина, где додате једначине одговарају додатим уопштеним координатама, који укључује и промену масе. Коришћењем датих једначина добијена су четири типа закона очувања енергије. Добијени закони су комплетнији и природнији него одговарајући закони у уобичајеној Лагранжевој формулацији таквих система.

Показано је да су закони промене енергије у потпуној сагласности са одговарајућом векторском формулацијом проблема.

Добијени резултати су илустровани примером: кретање ракете која избацује гасове и креће се дуж покретне стрме равни.

Faculty of Physics
University of Belgrade
Belgrade
Serbia
and
Mathematical Institute SANU
Belgrade
Serbia
musicki@mi.sanu.ac.rs

(Received 01.06.2017.)
(Revised 28.06.2017.)
(Available online 29.06.2017.)