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BILLIARDS ON CONSTANT CURVATURE SPACES AND GENERATING FUNCTIONS FOR SYSTEMS WITH CONSTRAINTS

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To the memory of Professor Aleksandar Baskša (Александар Бакша) 1937–2016

Abstract. In this note we consider a method of generating functions for systems with constraints and, as an example, we prove that the billiard mappings for billiards on the Euclidean space, sphere, and the Lobachevsky space are sympletic. Further, by taking a quadratic generating function we get the skew-hodograph mapping introduced by Moser and Veselov, which relates the ellipsoidal billiards in the Euclidean space with the Heisenberg magnetic spin chain model on a sphere. We define analogous mapping for the ellipsoidal billiard on the Lobachevsky space. It relates the billiard with the Heisenberg spin model on the light-like cone in the Lorentz–Poincare–Minkowski space.

1. Introduction

The billiard within $(n-1)$-dimensional ellipsoid in the Euclidean space

$$\mathbb{Q}^{n-1} = \{x \in \mathbb{R}^n \mid \langle A^{-1}x, x\rangle = 1\}, \tag{1.1}$$

$A = \operatorname{diag}(a_1, \dots, a_n) > 0$, is a basic example of a discrete integrable system (e.g, see [6, 12, 19]). One of manifestations of integrability is the existence of the skew-hodograph mapping, which relates the system with the Heisenberg magnetic spin chain model, i.e., a discrete Neumann system [13, 16, 21].

In this note we consider a method of generating functions for constrained system and give another interpretation of the skew-hodograph mapping, allowing its formulation for ellipsoidal billiards on the Lobachevsky space. It is related to the Heisenberg spin model on light-like cones defined in [8].

Usually, the symplectic property of the billiard mapping for convex regions D in R^n is formulated for a space of lines that intersect the boundary ∂D (see [11, 19]). Alternatively, in Section 2, we use a method of generating functions for constrained

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systems (Theorem 2.1). For ellipsoidal billiards, we consider the variety M defined by constraints

$$M: \quad f_1 = \langle A^{-1}x, x\rangle = 1, \quad f_2 = \langle y, y\rangle = 1, \quad f_3 = \langle A^{-1}x, y\rangle \neq 0 \tag{1.2}$$

within the symplectic linear space $(\mathrm{R}^{2n}, dy \wedge dx)$, $dy \wedge dx = \sum_i dy_i \wedge dx_i$ (see Proposition 2.1). Next, we take a quadratic generating function

$$S(x, X) = \langle A^{-1/2}x, X\rangle \tag{1.3}$$

for a system given by constraints (1.2). The corresponding symplectic dynamics

$$\psi\colon M \to M$$

is actually the skew-hodorgraph mapping given by Moser and Veselov [13] (Subsection 2.3).

This observation motivated us to consider an analogous construction for ellipsoidal billiards on the sphere and the Lobachevsky space in Section 3. By using Theorem 2.1 we prove that the billiard mappings are symplectic (Lemma 3.1, Proposition 3.1), and taking the generating function (1.3) we define the skew-hodograph mapping Ψ for the later system (Theorem 3.1). Following [10], in the construction we use a correspondence between the ellipsoidal billiard on the sphere (Lobachevsky space) and the virtual billiard within the cone in the Euclidean (Lorentz–Poincare–Minkowski) space (see Lemma 3.2).

For the completeness of the exposition we included the notion of virtual billiards and the proof of Lemma 3.2 (Subsection 3.3). Note again that the symplectic properties of the ellipsoidal billiards (Propositions 2.1, 3.1) are well known, but the presented proofs are quite simple and can be applied for billiards within an arbitrary convex region. To the author knowledge, the observation about generating functions for systems with constraints (Theorem 2.1) and the construction of skew-hodograph mappings by the use of quadratic generating functions (Subsections 2.3, 3.4, Theorem 3.1) are not given in the literature. It would be interesting to study dynamics and symmetries related to some other discrete integrable systems with constraints by taking suitable generating functions (see also [17]).

2. Billiards and generating functions for systems with constraints

2.1. Generating functions. Recall that if a graph Γ_ϕ of the diffeomorphism $\phi\colon U \to V$, $U \subset \mathrm{R}^{2n}(p, x)$, $V \subset \mathrm{R}^{2n}(P, X)$ can be written in the form

$$p = \frac{\partial S(x, X)}{\partial x}, \qquad P = -\frac{\partial S(x, X)}{\partial X},$$

for a certain function $S(x, X)$, then ϕ is symplectic with respect to the canonical structure: $\phi^* dP \wedge dX = dp \wedge dx$. The function $S(x, X)$ is called a generating function of the mapping ϕ (the generating function S_1 in notation of [1]). The above set up allows generalisation to cotangent bundles T^*Q endowed with the standard symplectic structures (e.g., see [14]). Instead, we use redundant variables and constraints and have the following simple observation.

Consider $(2n-2k)$-dimensional submanifolds $M \subset \mathrm{R}^{2n}(p, x)$ and $N \subset \mathrm{R}^{2n}(P, X)$, defined by the constraints of the form

$$M: \qquad f_i(x)=0, \qquad f_{k+i}(p,x)=0, \qquad i=1,\dots,k,$$
$$N: \qquad F_i(X)=0, \qquad F_{k+i}(P,X)=0, \qquad i=1,\dots,k.$$

We assume that M and N are symplectic submanifolds, that is

$$\det(\{f_i,f_j\})|_M \neq 0, \qquad \det(\{F_i,F_j\})|_N \neq 0, \qquad i,j=1,\dots,2k$$

(e.g., see [16]).

THEOREM 2.1. *If a graph Γ_ϕ of the diffeomorphism $\phi\colon M\to N$ can be given by*

$$p=\frac{\partial S(x,X)}{\partial x}+\sum_{i=1}^k \lambda_i\frac{\partial f_i}{\partial x}, \qquad P=-\frac{\partial S(x,X)}{\partial X}-\sum_{i=1}^k \Lambda_i\frac{\partial F_i}{\partial X}, \tag{2.1}$$

for certain Lagrange multipliers λ_i,Λ_i, then ϕ is symplectic. Similarly, if (2.1) defines a diffeomorphism $\phi\colon M\to N$, then ϕ is symplectic.

PROOF. The equations (2.1) imply

$$pdx-PdX=dS+\sum_{i=1}^k \lambda_i\frac{\partial f_i}{\partial x}dx+\Lambda_i\frac{\partial F_i}{\partial X}dX|_{\Gamma_\phi},$$

and, therefore,

$$dp\wedge dx-dP\wedge dX=\sum_{i=1}^k d\lambda_i\wedge df_i+d\Lambda_i\wedge dF_i|_{\Gamma_\phi}. \tag{2.2}$$

Note that the right hand side of (2.2) is equal to zero for vectors ξ tangent to Γ_ϕ, since $df_i(\xi)=dF_i(\xi)=0$. Thus,

$$\phi^*\left(\sum_i dP_i\wedge dX_i|_N\right)=\sum_i dp_i\wedge dx_i|_M. \qquad \square$$

Theorem 2.1 allows a quite simple proof that the billiard mapping for the billiards within convex regions $D\subset \mathrm{R}^n$ is symplectic.

2.2. Billiards. Let $\phi:(y,x)\mapsto(Y,X)$ be the billiard mapping mapping

$$X=x+\mu y, \tag{2.3}$$
$$Y=y+\nu A^{-1}X \tag{2.4}$$

of a billiard system within the ellipsoid (1.1) (e.g., see [13]). The multipliers

$$\mu=-2\,\frac{\langle A^{-1}x,y\rangle}{\langle A^{-1}y,y\rangle}, \qquad \nu=2\,\frac{\langle A^{-1}X,Y\rangle}{\langle A^{-2}X,X\rangle} \tag{2.5}$$

are determined from the conditions that two successive impact points x and X belong to the ellipsoid ($\langle A^{-1}X,X\rangle=\langle A^{-1}x,x\rangle=1$) and that y and Y are unit vectors ($\langle Y,Y\rangle=\langle y,y\rangle=1$).

For $\mu>0$ (i.e., $\langle A^{-1}x,y\rangle<0$), the vectors y and Y are the outgoing velocities at two successive impact points x and X, while for $\mu<0$ (i.e., $\langle A^{-1}x,y\rangle>0$), the outgoing velocities are $-y$ and $-Y$. The function $\langle A^{-1}x,y\rangle$ is the first integral of the system and we have the decomposition of M on invariant subsets $M = M_+ \cup M_-$, where M_+ and M_- are domains with $\langle A^{-1}x, y\rangle < 0$ and $\langle A^{-1}x, y\rangle > 0$, respectively.

The billiard mapping ϕ commutes with the $\mathbb{Z}_2$-action $y \mapsto -y$ that interchange M_+ and M_-. Thus, ϕ defines the dynamics on $M/\mathbb{Z}_2$, which can be identified with the space of lines that intersect the ellipsoid $\mathbb{Q}^{n-1}$. Also, since

$$\{f_1, f_2\} = 4f_3 \neq 0|_M,$$

M is a symplectic submanifold of $(\mathrm{R}^{2n}(y,x), dy \wedge dx)$. Note that here we identified vectors and covectors in R^n by mean of the Euclidean scalar product.

The equations (2.1) for the constraints (1.2) and the generating function

$$S(x, X) = -|X - x| = -\sqrt{\langle X - x, X - x\rangle}$$

get the form

$$y = \frac{X - x}{|X - x|} + \lambda A^{-1}x, \tag{2.6}$$

$$Y = \frac{X - x}{|X - x|} - \Lambda A^{-1}X. \tag{2.7}$$

By taking $\lambda = 0$, $\Lambda = -\nu$, $\mu = |X - x|$, we obtain the billiard system (2.3), (2.4) within the domain M_+. Applying the $\mathbb{Z}_2$-symmetry, we get:

Proposition 2.1. *The billiard map $\phi\colon M \to M$ is symplectic.*

A similar proof can be applied for a billiard within an arbitrary convex region $D = \{f(x) \leq 0\} \subset \mathrm{R}^n$, by taking the constraints $f(x) = 0$, $\langle y, y\rangle = 1$, and replacing the normal vector $A^{-1}X$ at X in (2.4) and (2.7) and $A^{-1}x$ at x in (2.6) by $\nabla f|_X$ and $\nabla f|_x$, respectively.

Proposition 2.1 also follows from the fact that the skew-hodograph mapping $\psi\colon M \to M$ is symplectic and satisfies $\psi^2 = -\phi$ (see below). It is also convenient to consider the billiard as a discrete Lagrangian system on $\mathbb{Q}^{n-1}$: the billiard trajectories x_k, $k \in \mathbb{Z}$ are the extremals of the discrete action $\mathbf{S}[x_k] = \sum_k |x_{k+1} - x_k|$ (see [13]). An interesting dual Lagrangian formulation is recently given by Suris [17], representing the ellipsoidal billiard mapping as a symplectic mapping of the cotangent bundle of the sphere $\langle y, y\rangle = 1$.

2.3. Skew-hodograph mapping. Now, we take a quadratic generating function (1.3). The equations (2.1) become

$$y = A^{-1/2}X + \lambda A^{-1}x, \tag{2.8}$$

$$Y = -A^{-1/2}x - \Lambda A^{-1}X, \tag{2.9}$$

where $\langle A^{-1}x, x\rangle = 1$, $\langle A^{-1}X, X\rangle = 1$.

From the constraints $\langle y, y\rangle = 1$, $\langle Y, Y\rangle = 1$, we get that λ and Λ are solutions of the equations

$$2\lambda\langle A^{-1}x, A^{-1/2}X\rangle + \lambda^2\langle A^{-2}x, x\rangle = 0, \tag{2.10}$$

$$2\Lambda\langle A^{-1}X, A^{-1/2}x\rangle + \Lambda^2\langle A^{-2}X, X\rangle = 0. \tag{2.11}$$

We have four real solutions (λ,Λ) of (2.10), (2.11) given by

$$\lambda = 0 \quad \text{or} \quad \lambda = -2\langle A^{-1}x, A^{-1/2}X\rangle/\langle A^{-2}x, x\rangle,$$
$$\Lambda = 0 \quad \text{or} \quad \Lambda = -2\langle A^{-1}x, A^{-1/2}X\rangle/\langle A^{-2}X, X\rangle.$$

The cases when both (λ,Λ) are equal to zero, or different from zero, lead to the trivial dynamics:

$$\psi^2 = -\operatorname{Id}_M .$$

The cases when one of the multipliers is zero and the second one is not zero are equivalent. Let us take $\lambda = 0$, $\Lambda = -2\langle A^{-1}x, A^{-1/2}X\rangle/\langle A^{-2}X, X\rangle$. Then the relations (2.8), (2.9) define the symplectic mapping $\psi\colon M \to M$:

$$X = A^{1/2}y,$$
$$Y = -A^{-1/2}x - \mu A^{-1/2}y, \qquad \mu = -2\langle A^{-1}x, y\rangle/\langle A^{-1}y, y\rangle.$$

Note that $\langle A^{-1}x, y\rangle$ is the integral of ψ. Also, ψ coincides with the skew-hodograph mapping introduced by Moser and Veselov [13][1].

Remark 2.1. The skew-hodograph mapping has nice properties:

$$\phi\circ\psi = \psi\circ\phi, \qquad \psi^2 = -\phi,$$

i.e., it maps billiard trajectories into a billiard trajectories and it can be considered as a square root of the billiard dynamics [13]. Further, let (y_j, x_j) be its trajectory. Then the sequence y_j satisfies the equations

$$A^{1/2}y_{j+1} + A^{1/2}y_{j-1} = 2\langle A^{-1/2}y_j, y_{j-1}\rangle/\langle A^{-1}y_j, y_j\rangle y_j, \tag{2.12}$$

which are the equations of the Heisenberg model on a sphere $\mathbb{S}^{n-1}(y)$ with the action $\mathbf{S}[y_i] = \sum_i \langle y_i, A^{1/2}y_{i+1}\rangle$ (see [13, 16, 21]). Also, if y_j is a solution of the Heisenberg model (2.12), then $x_j = (-1)^j A^{1/2}y_{2j}$ is a sequence of impact points of the billiard trajectory within ellipsoid (1.1) (see [13]).

3. Billiards on the sphere and the Lobachevsky space

3.1. Definition of billiards. From now on we use the following notation:

$$\langle \xi, \eta\rangle = \xi_1\eta_1 + \dots + \xi_{n-1}\eta_{n-1} \pm \xi_n\eta_n,$$

for all $\xi, \eta \in \mathrm{R}^n$ and whenever we have ± 1 (∓ 1), this means $+1$ (-1) for the Euclidean space and -1 ($+1$) for the pseudo-Euclidean space of signature $(n-1, 1)$. We identify tangent $T\,\mathrm{R}^n = \mathrm{R}^{2n}(y,x)$ and cotangent bundle $T^*\,\mathrm{R}^n = \mathrm{R}^{2n}(p,x)$ by means of the metric $E = \operatorname{diag}(1,\dots,1,\pm 1)$: $p = Ey$, defining the symplectic structure

$$E dy \wedge dx = \sum_{i=1}^{n-1} dy_i \wedge dx_i \pm dy_n \wedge dx_n$$

on $\mathrm{R}^{2n}(y,x)$.

[1] More precisely, in order to have a skew-hodograph mapping given in [13], one should take $\Lambda = 0$, $\lambda = 0$ and that y is incoming velocity at $x \in \mathbb{Q}^{n-1}$ in the billiard mapping.

Next, we consider the sphere $\mathbb{S}^{n-1} = \mathbb{S}^{n-1}_+$ and the Lobachevsky space $\mathbb{H}^{n-1}$, realized as upper pseudo-sphere $\mathbb{S}^{n-1}_-$ in the the Lorentz–Poincare–Minkowski space, given by

$$\mathbb{S}^{n-1}_+ = \{x \in \mathrm{R}^n \mid \langle x, x\rangle = 1\}, \qquad \mathbb{S}^{n-1}_- = \{x \in \mathrm{R}^n \mid \langle x, x\rangle = -1 \mid x_n > 0\}.$$

The induced metrics on $\mathbb{S}^{n-1}_\pm$ is Riemannian of constant curvature ± 1, while geodesic lines are simply intersections of $\mathbb{S}^{n-1}_\pm$ with two-dimensional planes through the origin.

It is well known that the ellipsoidal billiards on the sphere and the Lobachevsky space are completely integrable [4,5,12,18,20]. In the above notation, the ellipsoid $\mathbb{Q}^{n-2} \subset \mathbb{S}^{n-1}_\pm$ can be defined as a intersection of a cone

$$\mathbb{K}^{n-1}: \qquad \langle A^{-1}x, x\rangle = 0$$

with $\mathbb{S}^{n-1}_\pm$ and the upper half-space $\{x_n > 0\}$, where

$$A = \operatorname{diag}(a_1, \dots, a_n), \quad 0 < a_1, a_2, \dots, a_{n-2}, a_{n-1} < \mp a_n.$$

Note that $A^{-1}x$ for $x \in \mathbb{Q}^{n-1}$ is normal to $T_x\mathbb{K}^{n-1}$ and belongs to $T_x\mathbb{S}^{n-1}_\pm$. Thus, $A^{-1}x$ is a normal of the quadric Q^{n-2} at x.

For a phase space we take $(2n-4)$-dimensional variety $M \subset \mathrm{R}^{2n}(y, x)$ defined by

$$\begin{aligned} M: \quad & f_1 = \langle x, x\rangle = \pm 1, \quad && f_3 = \langle A^{-1}x, x\rangle = 0, \quad && x_n > 0,\\ & f_2 = \langle x, y\rangle = 0, && f_4 = \langle y, y\rangle = 1, && f_5 = \langle A^{-1}x, y\rangle < 0. \end{aligned}$$

Note that now y denotes the outgoing velocity (tangent to $\mathbb{S}^{n-1}_\pm$) at the point $x \in \mathbb{Q}^{n-2}$.

LEMMA 3.1. *The billiard mapping* $\phi\colon M \to M$ *is given by*

$$X = \alpha x + \beta y, \tag{3.1}$$

$$Y = \mp\beta x + \alpha y + \gamma A^{-1}X, \tag{3.2}$$

where

$$\alpha = \frac{\langle A^{-1}y, y\rangle}{\sqrt{\langle A^{-1}y, y\rangle^2 \pm 4\langle A^{-1}x, y\rangle^2}}, \quad \beta = -2\alpha\frac{\langle A^{-1}x, y\rangle}{\langle A^{-1}y, y\rangle}, \quad \gamma = 2\frac{\langle A^{-1}x, y\rangle}{\langle A^{-2}X, X\rangle}.$$

PROOF. For $(y, x) \in M$, $(Y, X) = \phi(y, x)$ is determined in two steps. First, we consider the geodesic line $c(t)$, $c(0) = x$, $\dot c(0) = y$ and find its intersection $X = c(t_0)$ with $\mathbb{Q}^{n-2}$. Let $Y' = \dot c(t_0)$ be the incoming velocity at X. Then X and Y' are (pseudo-)Euclidean rotations of x and y:

$$X = \alpha x + \beta y, \qquad Y' = \mp\beta x + \alpha y, \qquad \alpha^2 \pm \beta^2 = 1, \qquad \alpha, \beta > 0,$$

implying the relations $\langle X, X\rangle = \pm 1$, $\langle Y', Y'\rangle = 1$, and $\langle Y', X\rangle = 0$.

From the condition $\langle A^{-1}X, X\rangle = 0$, it follows that the parameters α and β are related by the equation

$$\beta\langle A^{-1}y, y\rangle + 2\alpha\langle A^{-1}x, y\rangle = 0, \tag{3.3}$$

and after the substitution to $\alpha^2 \pm \beta^2 = 1$ we get $\alpha = 1/\sqrt{1 \pm 4\langle A^{-1}x, y\rangle^2/\langle A^{-1}y, y\rangle^2}$.

Now, let Y be the reflection of Y' with respect to $T_X\mathbb{Q}^{n-2}$ - the outgoing velocity at X. The difference of Y' and Y is normal to $\mathbb{Q}^{n-2}$: $Y = Y' + \gamma A^{-1}x$, for a certain multiplier γ. From the condition $\langle Y, Y\rangle = 1$, it follows

$$\gamma = -2\frac{\langle A^{-1}X, Y'\rangle}{\langle A^{-2}X, X\rangle}. \tag{3.4}$$

Next, as in the case of the billiard within an ellipsoid, $J = \langle A^{-1}x, y\rangle$ is the integral of the billiard mapping:

$$\begin{aligned}
\langle A^{-1}Y, X\rangle &= \langle A^{-1}Y', X\rangle + \gamma\langle A^{-2}X, X\rangle \\
&= -\langle A^{-1}X, Y'\rangle && (3.4) \\
&= -\langle A^{-1}(\alpha x + \beta y), (\mp\beta x + \alpha y)\rangle \\
&= -\alpha\beta\langle A^{-1}y, y\rangle - (\alpha^2 \mp \beta^2)\langle A^{-1}x, y\rangle \\
&= 2\alpha^2\langle A^{-1}x, y\rangle - (\alpha^2 \mp \beta^2)\langle A^{-1}x, y\rangle && (3.3) \\
&= \langle A^{-1}x, y\rangle,
\end{aligned}$$

which completes the proof. □

3.2. Generating function. Firstly, note that M is a symplectic submanifold of $(\mathrm{R}^{2n}, Edy \wedge dx)$. Indeed, let $D_{ij} = \{f_i, f_j\}$. We have

$$D_{12} = 2f_1, \quad D_{13} = 0, \quad D_{14} = 4f_2, \quad D_{23} = -2f_3, \quad D_{24} = 2f_4, \quad D_{34} = 4f_5.$$

Therefore, $\det D|_M = (D_{12}D_{34} - D_{13}D_{24} + D_{23}D_{14})^2|_M = 8f_5^2 \neq 0$ [2].

Proposition 3.1. *The billiard mapping (3.1), (3.2) is symplectic with the generating functions for the sphere and the Lobachevsky space given by*

$$S_+ = -\arccos\langle x, X\rangle \qquad \text{and} \qquad S_- = -\operatorname{arccosh}(-\langle x, X\rangle),$$

respectively.

Proof. Let $\theta = \arccos\langle x, X\rangle$, i.e., $\theta = \operatorname{arccosh}(-\langle x, X\rangle)$, and let us denote

$$\sin_+\theta = \sin\theta, \quad \cos_+\theta = \cos\theta, \quad \sin_-\theta = \sinh\theta, \quad \cos_-\theta = \cosh\theta.$$

Then

$$\frac{\partial S_\pm}{\partial x} = \frac{EX}{\sin_\pm\theta}.$$

Therefore, the relations (2.1) are

$$p = Ey = \frac{EX}{\sin_\pm\theta} + \lambda_1 Ex + \lambda_2 EA^{-1}x, \tag{3.5}$$

$$P = EY = -\frac{Ex}{\sin_\pm\theta} - \Lambda_1 EX - \Lambda_2 EA^{-1}X. \tag{3.6}$$

[2]Equivalently, it is well known that the submanifold $f_1 = 1$, $f_2 = 0$ is simplectomorphic to the cotangent bundle $P = T^*\mathbb{S}^{n-1}_\pm$. Denote the induced Poisson bracket on P by $\{\cdot,\cdot\}_P$. We have $\{f_3, f_4\}_P = \{f_3, f_4\} = 4f_5 \neq 0|_M$. Thus, M is a symplectic submanifold of P, that is a symplectic submanifold of $(\mathrm{R}^{2n}, Edy \wedge dx)$.

Let us take $\lambda_2 = 0$. From (3.5) we get

$$X = \sin_\pm \theta y - (\lambda_1 \sin_\pm \theta)x = \alpha x + \beta y,$$

for $\alpha = -\lambda_1 \sin_\pm \theta = \cos_\pm \theta$, $\beta = \sin_\pm \theta$, $\lambda_1 = -\cos_\pm \theta / \sin_\pm \theta = -\alpha/\beta$. Further, from (3.6) we get

$$\begin{aligned} Y &= -\frac{x}{\beta} - \Lambda_1(\alpha x + \beta y) - \Lambda_2 A^{-1} X \\ &= \Big(-\frac{\alpha^2 \pm \beta^2}{\beta} - \Lambda_1 \alpha \Big) x - \Lambda_1 \beta y - \Lambda_2 A^{-1} X \\ &= \mp \beta x + \alpha y - \Lambda_2 A^{-1} X, \end{aligned}$$

for $\Lambda_1 = -\alpha/\beta$. Thus, the above relations imply the mapping (3.1), (3.2) with $\Lambda_2 = -\gamma$. □

As in Subsection 2.2, a similar proof can be applied for a billiard within an arbitrary convex region $D \subset \mathbb{S}^{n-1}_\pm$. The billiard mappings for convex domains on the sphere $\mathbb{S}^2_+$ and the Lobachevsky plane $\mathbb{S}^2_-$ are recently studied in [2, 15, 22].

3.3. Virtual billiards within cones. There are alternative descriptions of the billiards on $\mathbb{S}^{n-1}_\pm$ (see [10, 20]). As for the ellipsoidal billiards, consider the mapping:

$$X = x + \mu y, \tag{3.7}$$

$$Y = y + \nu A^{-1} X, \tag{3.8}$$

where the multipliers (2.5) are now determined from the conditions $\langle A^{-1}X, X\rangle = 0$, $\langle Y, Y\rangle = \langle y, y\rangle$, i.e., for the phase space we take

$$N = \mathrm{R}^n \times \mathbb{K}^{n-1}(y, x) \backslash \{\langle A^{-1}x, y\rangle = 0,\ \langle A^{-1}y, y\rangle = 0\}.$$

We refer to the mapping Φ defined above as the virtual billiard mapping (see [9, 10][3]). Note that $J = \langle A^{-1}x, y\rangle$ is an invariant of Φ, so if $\langle A^{-1}x, y\rangle \neq 0$, then $\langle A^{-1}X, Y\rangle \neq 0$ as well. On the other hand, if $\langle A^{-1}Y, Y\rangle = 0$, then the flow stops by definition.

Let $(y_{j+1}, x_{j+1}) = \Phi(y_j, x_j)$, $j \in \mathrm{Z}$. The Hamiltonian $H = \frac{1}{2}\langle y_j, y_j\rangle$ is an invariant. Therefore, the segments $x_j x_{j+1}$ of a given virtual billiard trajectory are of the same type: they are all either space-like ($H > 0$), time-like ($H < 0$) or light-like ($H = 0$)[4]. For a fixed value of the Hamiltonian $H = h$, the corresponding mapping is a symplectic transformation of $(N_h, Edy \wedge dx|_{N_h})$, $N_h = H^{-1}(h) \subset N$ (see Theorem 2.1, [10][5]).

[3] The matrix A used here corresponds to the matrix EA used in [10].

[4] The segments $x_{j-1}x_j$ and $x_j x_{j+1}$ could be either on the same side (the usual billiard reflection in the pseudo-Euclidean space [7, 11]) or on the opposite sides of the tangent plane $T_{x_j}\mathbb{K}^{n-1}$ (in the three-dimensional Euclidean case, Darboux referred to such reflection as the *virtual reflection*, e.g., see [6, Ch. 5]).

[5] In [10, Theorem 2.1] a direct proof in terms of the induced Dirac–Poison brackets on N_h is given for quadrics $\langle A^{-1}x, x\rangle = 1$, but the same proof applies for the virtual billiards within cones $\langle A^{-1}x, x\rangle = 0$.

We have the following statement (see [10, Lemma 5.1]).

LEMMA 3.2. *Let* $\hat{x}_j$ *be a sequence of the impact points of a virtual billiard trajectory* $(\hat{y}_{j+1}, \hat{x}_{j+1}) = \Phi(\hat{y}_j, \hat{x}_j)$, $j \in \mathbb{Z}$. *Then*

$$x_j = \operatorname{span}\{\hat{x}_j\} \cap \mathbb{S}^{n-1}_{\pm}$$

is the sequence of the impact points of a billiard trajectory within ellipsoid $\mathbb{Q}^{n-2}$ *on the (pseudo-)sphere* $\mathbb{S}^{n-1}_{\pm}$.

PROOF. First, we note that Φ defines the dynamics of planes

$$\Delta\colon \operatorname{span}\{\hat{y}, \hat{x}\} \to \operatorname{span}\{\hat{Y}, \hat{X}\}, \qquad (\hat{Y}, \hat{X}) = \Phi(\hat{y}, \hat{x}).$$

Indeed, we have that the planes spanned by $(\hat{Y}, \hat{X}) = \hat{\phi}(\hat{y}, \hat{x})$ and $(\hat{Y}', \hat{X}') = \hat{\phi}(\hat{y}', \hat{x}')$ coincide, where

$$\hat{x}' = a\hat{x}, \quad \hat{y}' = b\hat{x} + c\hat{y}, \quad a, b, c \in \mathbb{R}, \quad a, c \neq 0.$$

(see the proof of [10, Lemma 5.1]). Also, the billiard mapping (3.1), (3.2) determines the dynamics

$$\delta\colon \operatorname{span}\{y, x\} \to \operatorname{span}\{Y, X\}, \qquad (Y, X) = \phi(y, x),$$

and we need to prove the identity $\Delta = \delta$, which is sketched in the proof of [10, Lemma 5.1]. For the completeness of the exposition, here we will derive it.

Let us take $(\hat{y}, \hat{x}) = (y, x)$, where $(y, x) \in M \subset N$. Then from (3.1) and (3.2) we get

$$\begin{aligned} Y \wedge X &= (\mp\beta x + \alpha y + \gamma A^{-1}X) \wedge (\alpha x + \beta y) \\ &= (\alpha^2 \pm \beta^2) y \wedge x + \gamma A^{-1}X \wedge X \\ &= y \wedge x + 2\,\frac{\langle A^{-1}x, y\rangle}{\langle A^{-2}X, X\rangle} A^{-1}X \wedge X, \end{aligned}$$

while from (3.7) and (3.8) it follows

$$\begin{aligned} \hat{Y} \wedge \hat{X} &= (y + \hat{\nu} A^{-1}\hat{X}) \wedge (x + \hat{\mu} y) \\ &= y \wedge x + 2\,\frac{\langle A^{-1}\hat{X}, \hat{Y}\rangle}{\langle A^{-2}\hat{X}, \hat{X}\rangle} A^{-1}\hat{X} \wedge \hat{X} \\ &= y \wedge x + 2\,\frac{\langle A^{-1}x, y\rangle}{\langle A^{-2}X, X\rangle} A^{-1}X \wedge X, \end{aligned}$$

where we used that J is the integral of Φ and that X and $\hat{X}$ are proportional. Therefore, $\Delta = \delta$. The statement is proved. $\square$

3.4. Skew-hodograph mapping. Let $\mathbb{L}^{n-1} = \{y \in \mathbb{R}^n \mid \langle y, y\rangle = 0\}$ be a light-like cone. By an analogy with the Subsection 2.3, we consider the symplectic submanifold

$$N_0 = \mathbb{L}^{n-1} \times \mathbb{K}^{n-1}(y, x) \backslash \{\langle A^{-1}x, y\rangle = 0,\ \langle A^{-1}y, y\rangle = 0\}$$

of $(\mathbb{R}^{2n}(y, x), E dy \wedge dx)$ and the generating function (1.3). For $p = Ey$, the equations (2.1) become

$$Ey = EBX + \lambda EA^{-1}x, \tag{3.9}$$

$$EY = -EBx - \Lambda EA^{-1}X, \tag{3.10}$$

where $\langle A^{-1}x, x\rangle = 0$, $\langle A^{-1}X, X\rangle = 0$. We choose the following Lagrange multipliers:

$$\lambda = 0, \qquad \Lambda = -2\langle A^{-1}x, A^{-1/2}X\rangle/\langle A^{-2}X, X\rangle.$$

Then the relations (3.9), (3.10) define the symplectic mapping $\Psi\colon N_0 \to N_0$:

$$X = A^{1/2}y,$$
$$Y = -A^{-1/2}x - \mu A^{-1/2}y, \qquad \mu = -2\langle A^{-1}x, y\rangle/\langle A^{-1}y, y\rangle.$$

We have the following analogy with the skew-hodograph mapping for the billiards within ellipsoid (compare with Remark 2.1).

THEOREM 3.1. (i) *The mapping Ψ commutes with the virtual billiard mapping for the light-like trajectories within the cone $\mathbb{K}^{n-1}$*

$$\Psi \circ \Phi|_{N_0} = \Phi|_{N_0} \circ \Psi$$

and, moreover,

$$\Psi^2 = -\Phi|_{N_0}.$$

(ii) *Let (y_j, x_j), $j \in \mathbb{Z}$ be a trajectory of Ψ. Then the sequence of light-like vectors $y_j \in \mathbb{L}^{n-1}$ satisfies the equations*

$$A^{1/2}y_{j+1} + A^{1/2}y_{j-1} = 2\langle A^{-1/2}y_j, y_{j-1}\rangle/\langle A^{-1}y_j, y_j\rangle y_j, \tag{3.11}$$

which are the equations of the Heisenberg model on a light-like cone $\mathbb{L}^{n-1}$ with the action functional $\mathbf{S}[y_i] = \sum_i \langle y_i, A^{1/2}y_{i+1}\rangle$.

(iii) *If y_j is a solution of the Heisenberg model (3.11), then*

$$x_j = \operatorname{span}\{(-1)^j A^{1/2}y_{2j}\} \cap \mathbb{S}^{n-1}_{-}$$

is a sequence of impact points of the billiard within the quadric $\mathbb{Q}^{n-2}$ on the Lobachevsky space $\mathbb{S}^{n-1}_{-}$.

PROOF. (Sketch) The proof of items (i) and (ii) is straightforward. Item (iii) follows from items (i), (ii), and Lemma 3.2. □

The Heisenberg model on pseudo-spheres and light-likes cones in pseudo-Euclidean spaces is studied in [8]. The system on light-like cones is an example of a discrete contact integrable system.

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БИЛИЈАРИ НА ПРОСТОРИМА КОНСТАНТНЕ КРИВИНЕ И ГЕНЕРАТОРНЕ ФУНКЦИЈЕ ЗА СИСТЕМЕ СА ВЕЗАМА

Резиме. У овом прилогу разматрамо метод генераторних функција за системе са везама и, као пример, показујемо да су билијарна пресликавања за билијаре у Еуклидком простору, сфери и простору Лобачевског симплектичка. Даље, узимањем квадратне генераторне функције, добијамо косо-ходографско прескликавање Мозера и Веселова, које повезује билијар унутар елипсоида у Еуклидском простору са дискретним Хајзенберговим системом на сфери. Дефинишемо одговарајуће пресликавање за билијаре унутар елипсоида у простору Лобачевског. Оно повезује билијар са дискретним Хајзенберговим системом на светлосном конусу у простору Лоренца–Поенкареа–Минковског.

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