A THEORY OF STRAIN-GRADIENT PLASTICITY WITH EFFECT OF INTERNAL MICROFORCE

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ABSTRACT. This paper develops a theory of strain gradient plasticity for isotropic bodies undergoing small deformation in the absence of plastic spin. The proposed theory is based on a system of microstresses which include a microstress vector consistent with microforce balance; the mechanical form of the second law of thermodynamics which includes work performed by the microstresses during plastic flow; and a constitutive theory that allows the free energy to depend on the elastic strain $\mathbf{E}^e$, divergence of plastic strain $\operatorname{div} \mathbf{E}^p$ and the Burgers tensor $\mathbf{G}$. Substitution of the constitutive relations into the microforce balance leads to a nonlinear partial differential equation in the plastic strain known as flow rule which captures the presence of an additional energetic length scale arising from the accounting of microstress vector. In addition to the flow rule, nonstandard boundary conditions are obtained, and as an aid to finite element solution a variational formulation of the flow rule is deduced. Finite element solution is obtained of one-dimensional problem of viscoplastic simple shearing under gravity force, where it is shown that for a fixed dissipative length scale, increase in the energetic length scales will result in decrease in the plastic strain.

1. Introduction

The inability of the classical theory of plasticity to model materials between the nano and micro length scales has led to the attention of theories that could account for size-effects through dependence on the strain gradients [5,6,12,14,16,17]. The classical theory has also not been able to adequately define key concepts such as strain hardening, yield points and failure stresses. The comprehensive treatment of these related concepts would definitely provide useful information as regards what happens during failure of materials [2]. At the moment, it has been shown in [3] that the failure of material starts when material begins to experience yielding (i.e. material begins to undergo plastic behavior). Thus, processes such as size-effects, dislocations etc. during strain hardening should be explained through dependence of the stresses and free energy on some internal variables accounting for irreversible

restructuring of material during plastic behaviour [10]. However, the efforts of the classical theory to account for these processes, for which experiments have shown their necessities continue to meet with challenges [11]. In particular, in addition to the issue of differences in the experimental results and the classical theory, it has been difficult to obtain a sound extension of the classical theory of plasticity which incorporates dependence on quantities such as strain gradients that could account for size-effect phenomena [13].

There have been treatments of the well-posedness of the initial-boundary value problems of strain gradient theories. Among these treatments are the work in [4], introducing into the free energy function a term involving energy due to isotropic hardening in addition to the energy due to elastic strain and the Burgers tensor. Treatment of well-posedness based on the Gurtin–Anand model [9] in which the free energy is extended to include the divergence of the plastic strain has been studied in [15]. For this case, it has been shown that there exists a unique solution of the flow rule on the boundary of the set of admissible functions when the flow rule is formulated as a variational inequality. However, the inclusion of the divergence of the plastic strain in the constitutive relations in [15] has been from mathematical and not from physical consideration. This is with the purpose of establishing uniqueness results of the flow rule.

It is noteworthy that since each internal variable has its corresponding energy conjugate, then the divergence of the plastic strain also has an energy conjugate that would be represented by a vector. We shall refer to this vector as microstress vector or internal microforce. The effect of this vector would be investigated on the microforce balance and the flow rule.

Motivated by the works in [9, 10, 15], the purpose of this work is to obtain an extension of the flow rule of the Gurtin–Anand model by accounting for an energetic internal microforce conjugate to the divergence of the plastic strain in the absence of the plastic spin and examine the effect of the internal microforce on the flow rule.

2. Notations

The inner product of second order tensors $\mathbf{T}$ and $\mathbf{E}$ is denoted by $\mathbf{T} : \mathbf{E}$ and defined by

$$\mathbf{T} : \mathbf{E} = T_{ij}E_{ij}.$$

Let $\mathbf{T}_o$ denotes the deviatoric part of $\mathbf{T}$ defined by

$$\mathbf{T}_o = \mathbf{T} - \frac{1}{3}(\operatorname{tr}\mathbf{T})\mathbf{I}, \quad \left(T_{oij} = T_{ij} - \frac{1}{3}T_{kk}\delta_{ij}\right).$$

The symmetric and the skew parts of $\mathbf{E}$ are denoted by $\operatorname{sym}\mathbf{E}$ and $\operatorname{skw}\mathbf{E}$ respectively and are defined by

$$(\operatorname{sym}\mathbf{E})_{ij} = \frac{1}{2}(E_{ij} + E_{ji}), \quad (\operatorname{skw}\mathbf{E})_{ij} = \frac{1}{2}(E_{ij} - E_{ji}).$$

The symmetric-deviatoric part of $\mathbf{E}$ would be denoted by $\operatorname{sym}_o\mathbf{E}$ which is defined by

$$(\operatorname{sym}_o\mathbf{E})_{ij} = \frac{1}{2}(E_{ij} + E_{ji}) - \frac{1}{3}\delta_{ij}E_{kk}.$$

The divergence, curl, and laplacian of a tensor field $\mathbf{E}$ are defined respectively by

$$(\operatorname{div}\mathbf{E})_i = E_{ij,j}, \quad (\operatorname{curl}\mathbf{E})_{ij} = \epsilon_{ipq}E_{jq,p}, \quad (\Delta\mathbf{E})_{ij} = E_{ij,kk}$$

where $(\cdot)_{,i} = \frac{\partial}{\partial x_i}$ defines derivative with respect to the spatial coordinates x_i. Also $(\cdot)_{,ij} = \frac{\partial^2}{\partial x_i \partial x_j}$.

In this work, we will also make use of third-order tensors, where we define inner product of third order tensors $\mathbb{K}$ and $\mathbb{R}$ as

$$\mathbb{K}\!:\!\mathbb{R} = K_{ijk}R_{ijk}.$$

The notations $\operatorname{div}\mathbb{K}$ and $\operatorname{sym}_o\mathbb{K}$ are the divergence of $\mathbb{K}$ and the part of $\mathbb{K}$ which is symmetric-deviatoric in its first two subscripts defined by

$$(\operatorname{div}\mathbb{K})_{ij} = K_{ijk,k}, \quad (\operatorname{sym}_o\mathbb{K})_{ijk} = \frac{1}{2}(K_{ijk}+K_{jik}) - \frac{1}{2}\delta_{ij}K_{nnk}.$$

3. Basic Kinematics

Let $\mathbf{u}(\mathbf{x},t)$ denotes the displacement of an arbitrary point $\mathbf{x}$ in a region B. The classical theory of isotropic plastic solids undergoing small deformation is based on the decomposition;

(3.1) $$\nabla\mathbf{u} = \mathbf{H}^e + \mathbf{H}^p, \quad \operatorname{tr}\mathbf{H}^p = 0.$$

$\mathbf{H}^e$ represents elastic stretching and rotation which can be decomposed as $\mathbf{H}^e = \mathbf{E}^e + \mathbf{W}^e$, where $\mathbf{E}^e$ denotes elastic strain and $\mathbf{W}^e$ is the elastic rotation. $\mathbf{H}^p$ is the plastic distortion characterizing the evolution of dislocations and other defects through the structure. $\mathbf{H}^p$ can be decomposed as the sum of the plastic strain $\mathbf{E}^p$ and the plastic spin $\mathbf{W}^p$. The elastic and plastic strains are defined by

$$\mathbf{E}^e = \frac{1}{2}(\mathbf{H}^e + \mathbf{H}^{eT}) \quad \text{and} \quad \mathbf{E}^p = \frac{1}{2}(\mathbf{H}^p + \mathbf{H}^{pT}),$$

so that $\mathbf{E}^e$ and $\mathbf{E}^p$ are symmetric. The elastic and plastic rotations are defined by

$$\mathbf{W}^e = \operatorname{skw}\mathbf{H}^e \quad \text{and} \quad \mathbf{W}^p = \operatorname{skw}\mathbf{H}^p.$$

It would be assumed that the plastic spin denoted by the skew part of $\mathbf{H}^p$ is absent so that equation (3.1) reduces to

$$\nabla\mathbf{u} = \mathbf{H}^e + \mathbf{E}^p; \quad \operatorname{tr}\mathbf{E}^p = 0.$$

Thus the basic kinematic variables are $\mathbf{u}, \mathbf{H}^e$ and $\mathbf{E}^p$. Clearly the kinematic variables are not independent.

4. Internal and external power expenditure

Let P be a small part of a body B with the outward unit normal $\mathbf{n}$ on the boundary ∂P of P. The internal microstress, polar microstress and microstress vector would be denoted by $\mathbf{T}^p$, $\mathbb{K}^p$ and $\vec{\chi}$ which are rank two, rank three and rank one tensors respectively. $\mathbf{T}$ would be called the elastic stress. It would be assumed that:

- An internal microforce per unit area or microstress vector $\vec{\chi}$, is power-conjugate to $\operatorname{div}\dot{\mathbf{E}}^p$.

– An elastic macrostress $\mathbf{T}$, is power-conjugate to $\dot{\mathbf{H}}^e$.
– A plastic microstress $\mathbf{T}^p$, is power-conjugate to $\dot{\mathbf{E}}^p$.
– A (third-order) polar plastic microstress $\mathbb{K}^p$, is power-conjugate to $\nabla\dot{\mathbf{E}}^p$.
– An external microtraction $\mathbf{K}(\mathbf{n})$ on ∂P is power-conjugate to $\dot{\mathbf{E}}^p$.
– An external body force $\mathbf{b}$ on P is power-conjugate to the velocity vector $\dot{\mathbf{u}}$.
– An external macrotraction $\mathbf{t}(\mathbf{n})$ on ∂P is power-conjugate to $\dot{\mathbf{u}}$.

Based on these assumptions, the internal and external power in global form are given respectively as

$$W_{\mathrm{int}}(P) = \int_P \big[\mathbf{T} : \dot{\mathbf{H}}^e + \vec{\chi} \cdot \mathrm{div}\,\dot{\mathbf{E}}^e + \mathbf{T}^p : \dot{\mathbf{E}}^p + \mathbb{K}^p \vdots \nabla\dot{\mathbf{E}}^p\big]\,dV,$$

$$W_{\mathrm{ext}}(P) = \int_P \big[\mathbf{t}(\mathbf{n}) \cdot \dot{\mathbf{u}} + \mathbf{K}(\mathbf{n}) : \dot{\mathbf{E}}^p\big]\,dA + \int_{\partial P} \mathbf{b} \cdot \dot{\mathbf{u}}\,dV,$$

where dV and dA are the volume and area of an infintesimal portion of P. Since $\dot{\mathbf{E}}^p$ is symmetric deviatoric, without loss of generality, we require that $\mathbf{T}^p$ is also symmetric deviatoric and that $\mathbb{K}^p$ be symmetric deviatoric in its first two subscripts. $\mathbf{K}(\mathbf{n})$ is also symmetric deviatoric. The internal power would be balanced by power expended externally by tractions on ∂P and body forces acting within P.

Using the principle of frame-indifference as applied to small deformation we have as a consequence [8,9] that $\mathbf{T}$ is symmetric and thus we have the internal power given by

$$W_{\mathrm{int}}(P) = \int_P \big[\mathbf{T} : \dot{\mathbf{E}}^e + \vec{\chi} \cdot \mathrm{div}\,\dot{\mathbf{E}}^p + \mathbf{T}^p : \dot{\mathbf{E}}^p + \mathbb{K}^p \vdots \nabla\dot{\mathbf{E}}^p\big]\,dV.$$

Assume that, at some arbitrary fixed time, the fields $\mathbf{u}$, $\mathbf{H}^e$ and $\mathbf{E}^p$ are known. Consider the fields $\dot{\mathbf{u}}$, $\dot{\mathbf{H}}^e$ and $\dot{\mathbf{E}}^p$ as virtual velocities specified in a manner consistent with

$$\nabla\dot{\mathbf{u}} = \dot{\mathbf{H}}^e + \dot{\mathbf{E}}^p; \quad \mathrm{tr}\,\dot{\mathbf{E}}^p = 0.$$

This implies that we can denote the virtual fields by $\tilde{\mathbf{u}}$, $\tilde{\mathbf{E}}^e$ and $\tilde{\mathbf{E}}^p$, which requires that

$$(4.1) \qquad \nabla\tilde{\mathbf{u}} = \tilde{\mathbf{H}}^e + \tilde{\mathbf{E}}^p; \quad \mathrm{tr}\,\tilde{\mathbf{E}}^p = 0.$$

The generalized virtual velocity is defined to be the list

$$\nu = (\tilde{\mathbf{u}}, \tilde{\mathbf{E}}^e, \tilde{\mathbf{E}}^p).$$

The internal and external virtual power expenditure can be written as

$$(4.2) \qquad W_{\mathrm{int}}(P,\nu) = \int_P \big[\mathbf{T} : \tilde{\mathbf{E}}^e + \vec{\chi} \cdot \mathrm{div}\,\tilde{\mathbf{E}}^p + \mathbf{T}^p : \tilde{\mathbf{E}}^p + \mathbb{K}^p \vdots \nabla\tilde{\mathbf{E}}^p\big]\,dV$$

$$W_{\mathrm{ext}}(P,\nu) = \int_{\partial P} \big[\mathbf{t}(\mathbf{n}) \cdot \tilde{\mathbf{u}} + \mathbf{K} : \tilde{\mathbf{E}}^p\big]\,dA + \int_P \mathbf{b} \cdot \tilde{\mathbf{u}}\,dV.$$

By principle of virtual power, it is required that

$$W_{\mathrm{int}}(P,\nu) = W_{\mathrm{ext}}(P,\nu) \ \text{ for all virtual velocity.}$$

4.1. Macroscopic and Microscopic force balances. Since we are at liberty in choosing ν in a manner consistent with (4.1), we can assume that $\tilde{\mathbf{E}}^p = \mathbf{0}$, so that through the virtual power principle, we have

$$(4.3) \qquad \int_{\partial P} [\mathbf{t}(\mathbf{n}) \cdot \tilde{\mathbf{u}}] dA + \int_P \mathbf{b} \cdot \tilde{\mathbf{u}} \, dV = \int_P [\mathbf{T} : \nabla \tilde{\mathbf{u}}] dV.$$

Using the divergence theorem, (4.3) reduces to the macroforce balance

$$\operatorname{div} \mathbf{T} + \mathbf{b} = 0$$

and the macrotraction condition is given by

$$\mathbf{t}(\mathbf{n}) = \mathbf{Tn}.$$

Next consider a generalized virtual velocity for which $\tilde{\mathbf{u}} = \mathbf{0}$, so that $\nabla \tilde{\mathbf{u}} = \mathbf{0} = \tilde{\mathbf{H}}^e + \tilde{\mathbf{E}}^p$, and thus we have $\tilde{\mathbf{H}}^e = -\tilde{\mathbf{E}}^p$. Thus from the principle of virtual power we have

$$\int_{\partial P} [\mathbf{K}(\mathbf{n}) : \tilde{\mathbf{E}}^p] dA = \int_P \left[(\mathbf{T}^p - \mathbf{T}) : \tilde{\mathbf{E}}^p + \vec{\chi} \cdot div \tilde{\mathbf{E}}^p + \mathbb{K}^p \vdots \nabla \tilde{\mathbf{E}}^p \right] dV.$$

By divergence theorem, we have

$$\int_{\partial P} [\mathbf{K}(\mathbf{n}) : \tilde{\mathbf{E}}^p] dA = \int_P \left[\mathbf{T}^p - \mathbf{T} - \nabla \vec{\chi}^T - \operatorname{div} \mathbb{K}^p \right] : \tilde{\mathbf{E}}^p dV$$
$$+ \int_{\partial P} \vec{\chi} \cdot \tilde{\mathbf{E}}^p \cdot \mathbf{n} \, dA + \int_{\partial P} \mathbb{K}^p \mathbf{n} : \tilde{\mathbf{E}}^p \, dA.$$

Taking $\vec{\chi} \otimes \mathbf{n}$ as the dyad whose component is $\chi_i n_j$ we have

$$\int_{\partial P} \left(\mathbf{K}(\mathbf{n}) - \vec{\chi} \otimes \mathbf{n} - \mathbb{K}^p \mathbf{n} \right) : \tilde{\mathbf{E}}^p dA = \int_P \left[\mathbf{T}^p - \mathbf{T} - \nabla \vec{\chi}^T - \operatorname{div} \mathbb{K}^p \right] : \tilde{\mathbf{E}}^p dV.$$

Since $\mathbf{K}$, $\mathbf{T}^p$ and $\tilde{\mathbf{E}}^p$ are deviatoric and symmetric, we have the microforce balance given as

$$(4.4) \qquad \mathbf{T}_o = \mathbf{T}^p - \operatorname{sym}_o(\nabla \vec{\chi}) - \operatorname{div} \mathbb{K}^p.$$

The microtraction condition is given as

$$\mathbf{K}(\mathbf{n}) = \operatorname{sym}_o(\vec{\chi} \otimes \mathbf{n}) - \mathbb{K}^p \mathbf{n}.$$

In component form, we have

$$T_{oij} = T_{ij}^p - \left[\frac{1}{2}(\chi_{i,j} + \chi_{j,i}) - \frac{1}{3}\delta_{ij}\chi_{k,k} \right] - K_{ijk,k}^p$$
$$K_{ij} = \frac{1}{2}(\chi_i n_j + \chi_j n_i) - \frac{1}{3}\delta_{ij}\chi_k n_k + K_{ijk}^p n_k.$$

5. Free energy Imbalance

The second law of thermadynamics requires that the rate of increase of free energy ψ of P is less or equal to power expended on P and this is written as

$$\frac{d}{dt} \int_P \psi \, dV \leqslant W_{\text{ext}}(P).$$

By first law of thermodynamics it is required that $W_{\text{ext}} = W_{\text{int}}$, so that the local free-energy imbalance has the form

$$(5.1) \qquad \dot{\psi} - \mathbf{T} : \dot{\mathbf{E}}^e - \vec{\chi} \cdot \operatorname{div} \dot{\mathbf{E}}^p - \mathbf{T}^p : \dot{\mathbf{E}}^p - \mathbb{K}^p \vdots \nabla \dot{\mathbf{E}}^p \leqslant 0.$$

6. Constitutive relations

We will focus our attention on a constitutive theory that allows dependency on the divergence of $\mathbf{E}^p$ in addition to the dependency on the Burgers tensor $\mathbf{G} = \operatorname{curl} \mathbf{E}^p$. Following the work in [10,15], the free energy is assumed to take the form;

$$\psi = \hat{\psi}(\mathbf{E}^e, \operatorname{div} \mathbf{E}^p, \mathbf{G}).$$

$\mathbf{G}$ is a measure of dislocation structure in the body. $\operatorname{div} \mathbf{E}^p$ has been introduce to measure some defect from the action or reaction of an internal microforce on or from the polycrystalline structure of a material. In this work, it will be assumed that the microforce is purely energetic. Clearly, we have

$$\dot{\psi} = \frac{\partial \hat{\psi}}{\partial \mathbf{E}^e} : \dot{\mathbf{E}}^e + \frac{\partial \hat{\psi}}{\partial \operatorname{div} \mathbf{E}^p} \cdot \operatorname{div} \dot{\mathbf{E}}^p + \frac{\partial \hat{\psi}}{\partial \mathbf{G}} : \dot{\mathbf{G}},$$

$$\dot{\psi} = \frac{\partial \hat{\psi}}{\partial \mathbf{E}^e} : \dot{\mathbf{E}}^e + \frac{\partial \hat{\psi}}{\partial \operatorname{div} \mathbf{E}^p} \cdot \operatorname{div} \dot{\mathbf{E}}^p + \frac{\partial \hat{\psi}}{\partial G_{ij}} \epsilon_{ipq} \dot{E}^p_{jq,p}.$$

Define $\mathbb{P}$ by its component as

$$P_{jqp} = \frac{\partial \hat{\psi}}{\partial G_{ij}} \epsilon_{ipq},$$

so that we have

$$\frac{\partial \hat{\psi}}{\partial \mathbf{G}} : \dot{\mathbf{G}} = \mathbb{P} \vdots \nabla \dot{\mathbf{E}}^p.$$

Thus, we have

$$\dot{\psi} = \frac{\partial \hat{\psi}}{\partial \mathbf{E}^e} : \dot{\mathbf{E}}^e + \frac{\partial \hat{\psi}}{\partial \operatorname{div} \mathbf{E}^p} \cdot \operatorname{div} \dot{\mathbf{E}}^p + \mathbb{P} \vdots \nabla \dot{\mathbf{E}}^p.$$

But $\nabla \dot{\mathbf{E}}^p$ is symmetric deviatoric in its first two subscripts, so that we can define

$$\mathbb{K}^p_{en} = \operatorname{sym}_o \mathbb{P},$$

and we have

$$\mathbb{P} \vdots \nabla \dot{\mathbf{E}}^p = \mathbb{K}^p_{en} \vdots \nabla \dot{\mathbf{E}}^p.$$

Incorporating this into energy imbalance (5.1), we have

$$\left(\mathbf{T} - \frac{\partial \hat{\psi}}{\partial \mathbf{E}^e} \right) : \dot{\mathbf{E}}^e + \left(\vec{\chi} - \frac{\partial \hat{\psi}}{\partial \operatorname{div} \mathbf{E}^p} \right) \cdot \operatorname{div} \dot{\mathbf{E}}^p + \mathbf{T}^p : \dot{\mathbf{E}}^p + (\mathbb{K}^p - \mathbb{K}^p_{en}) \vdots \nabla \dot{\mathbf{E}}^p \geqslant 0.$$

This inequality will be used to develop suitable constitutive equations for $\mathbf{T}$, $\mathbb{K}^p$, $\mathbf{T}^p$ and $\vec{\chi}$. The standard constitutive relation for $\mathbf{T}$ is given by;

$$(6.1) \qquad \qquad \mathbf{T} = \frac{\partial \hat{\psi}}{\partial \mathbf{E}^e}.$$

Consisitent with the energy imbalance and coupled with the assumption that the free-energy is separable in the sense

(6.2) $$\psi = \hat{\psi}(\mathbf{E}^e, \operatorname{div} \mathbf{E}^p, \mathbf{G}) = \psi^e(\mathbf{E}^e) + \psi_1^p(\operatorname{div} \mathbf{E}^p) + \psi_2^p(\mathbf{G}),$$

we can assume that

(6.3) $$\vec{\chi} = \frac{\partial \hat{\psi}}{\partial \operatorname{div} \mathbf{E}^p}.$$

Define a polar microstress $\mathbb{K}_{\text{dis}}^p$ through the decomposition

(6.4) $$\mathbb{K}^p = \mathbb{K}_{\text{dis}}^p + \mathbb{K}_{en}^p.$$

Thus the dissipation inequality reduces to

$$\mathbf{T}^p : \dot{\mathbf{E}}^p + \mathbb{K}_{\text{dis}}^p \vdots \nabla \dot{\mathbf{E}}^p \geqslant 0.$$

$\mathbb{K}_{\text{dis}}^p$ is also symmetric in its first two indices.

6.1. Constitutive theory for microstresses. The constitutive assumptions of the microstresses proposed in [9] and consistent with free-energy imbalance are given by

(6.5) $$\mathbf{T}^p = S\left(\frac{d^p}{d_o}\right)^m \frac{\dot{\mathbf{E}}^p}{d^p}, \quad \mathbb{K}_{\text{dis}}^p = l^2 S_Y \frac{d^p}{d_o} \frac{\nabla \dot{\mathbf{E}}^p}{d^p},$$

$$\dot{S} = H(S)d^p, \quad S(\mathbf{x}, 0) = S_Y.$$

S is an internal state variable with the same dimension as the stress and it characterizes the current resistance to plastic flow, S_Y is a constant denoting the initial yield strength, $H(S)$ is a hardening function, d^p is the average flow rate defined by

$$d^p = \sqrt{|\dot{\mathbf{E}}^p|^2 + l^2 |\nabla \dot{\mathbf{E}}^p|^2},$$

where $|\cdot|$ is the modulus operation defined on tensors of any rank. l is called the dissipative length scale arising from the presence of the dissipative polar microstress, while $d_o > 0$ is a constant called the reference flow rate and m is a constant called rate-sensitivity parameter.

7. Quadratic free energy and the flow rule

Consisitent with (6.2) that the free energy is separable, we will assume that the free energy take the quadratic form based on [9];

$$\psi = \mu|\mathbf{E}_o^e|^2 + \frac{1}{2}\kappa|\operatorname{tr}\mathbf{E}^e|^2 + \frac{1}{2}\mu Q^2|\operatorname{div}\mathbf{E}^p|^2 + \frac{1}{2}\mu L^2|\mathbf{G}|^2,$$

where μ and κ are the elastic shear and the bulk modulus respectively. $L > 0$ and $Q \geqslant 0$ are energetic length scales. It is clear from (6.1) and (6.3) that

(7.1) $$\mathbf{T} = 2\mu\mathbf{E}_o^e + \kappa\operatorname{tr}(\mathbf{E}^e)\mathbf{I} \text{ and } \vec{\chi} = \mu Q^2 \operatorname{div}\mathbf{E}^p.$$

From

$$P_{jqp} = \frac{\partial \hat{\psi}}{\partial G_{ij}} \epsilon_{ipq}$$

we have

$$P_{jqp} = \mu L^2 G_{ij}\epsilon_{ipq} = \mu L^2\big(E^p_{jq,p} - E^p_{jp,q}\big).$$

This implies that

(7.2) $$(\mathbb{K}^p_{en})_{jpq} = \mu L^2\Big[E^p_{jq,p} - \frac{1}{2}\big(E^p_{jp,q} + E^p_{qp,j}\big) + \frac{1}{3}\delta_{jq}E^p_{rp,r}\Big].$$

The microforce balance (4.4) can be written as

(7.3) $$\mathbf{T}_o + \operatorname{div}\mathbb{K}^p_{en} + \operatorname{sym}_o(\nabla\vec{\chi}) = \mathbf{T}^p - \operatorname{div}\mathbb{K}^p_{\mathrm{dis}}.$$

Incorporating (6.5), (7.1) and (7.2) in (7.3) we have the consequent flow rule

(7.4) $$\mathbf{T}_o + \mu\big[L^2\Delta\mathbf{E}^p + (Q^2 - L^2)\operatorname{sym}_o(\nabla\operatorname{div}\mathbf{E}^p)\big]$$
$$= S\Big(\frac{d^p}{d_o}\Big)^m\frac{\dot{\mathbf{E}}^p}{d^p} - l^2 S_Y\operatorname{div}\Big(\Big(\frac{d^p}{d_o}\Big)^m\frac{\nabla\dot{\mathbf{E}}^p}{d^p}\Big).$$

REMARK 7.1.
- The term $-\mu\big[L^2\Delta\mathbf{E}^p + (Q^2 - L^2)\operatorname{sym}_o(\nabla\operatorname{div}\mathbf{E}^p)\big]$ of the left hand side of (7.4) are the energetic back stress
- The terms in the right hand side of (7.4) describe dissipative hardening
- (7.4) is a non-linear partial differential equation in $\mathbf{E}^p$
- If $Q = L$, then (7.4) reduces to

$$\mathbf{T}_o + \mu L^2\Delta\mathbf{E}^p = S\Big(\frac{d^p}{d_o}\Big)^m\frac{\dot{\mathbf{E}}^p}{d^p} - l^2 S_Y\operatorname{div}\Big(\Big(\frac{d^p}{d_o}\Big)^m\frac{\nabla\dot{\mathbf{E}}^p}{d^p}\Big).$$

If $Q = 0$, then (7.4) reduces to [9], equation (7.3).

7.1. Microscopic boundary conditions. By assuming null-expenditure of power on the boundary ∂B of B we have

$$\int_{\partial B}\Big(\mathbb{K}^p\mathbf{n}:\dot{\mathbf{E}}^p + \operatorname{sym}_o(\chi\vec{\otimes}\mathbf{n}):\dot{\mathbf{E}}^p\Big)dA = 0,$$

so that we can assume the boundary conditions

(7.5) $$\mathbb{K}^p\mathbf{n} + \operatorname{sym}_o(\vec{\chi}\otimes\mathbf{n}) = 0 \text{ on } \Gamma_{\mathrm{free}} \text{ and } \dot{\mathbf{E}}^p = \mathbf{0} \text{ on } \Gamma_{\mathrm{hard}}.$$

Γ_{free} and Γ_{hard} represent the respective microscopically free and hard boundaries of the surface ∂B of the body B. Γ_{free} and Γ_{hard} are complementary subsurfaces in the sense that their intersection is a closed curve on ∂B.

8. Variational formulation

Given the microboundary condition (7.5), consider the microscopic virtual power (4.2) with $\tilde{\mathbf{E}}^p = \mathbf{F}$, where the boundary term is omitted, to obtain

(8.1) $$\int_B[(\mathbf{T}^p - \mathbf{T}):\mathbf{F} + \vec{\chi}\cdot\operatorname{div}\mathbf{F} + \mathbb{K}^p\dot{:}\nabla\mathbf{F}]dV = 0,$$

where $\mathbf{F}$ is a test field which is admissible in the sense of (4.1). Using divergence theorem, (8.1) becomes

(8.2) $$\int_B\big[\mathbf{T}^p - \mathbf{T} - \nabla\vec{\chi}^T - \operatorname{div}\mathbb{K}^p\big]:\mathbf{F}\,dV + \int_{\partial B}(\mathbb{K}^p\mathbf{n} + \operatorname{sym}_o(\vec{\chi}\otimes\mathbf{n})):\mathbf{F}\,dA = 0.$$

The underlined surface integral in (8.2) is on Γ_{free}. Thus, it is required that the flow rule (7.4) and the microfree boundary condition $\mathbb{K}^p\mathbf{n} + \text{sym}_o(\vec{\chi} \otimes \mathbf{n}) = \mathbf{0}$ on Γ_{free} are together equivalent to the requirement that (8.1) be satisfied for all fields $\mathbf{F}$.

9. A simple one-dimensional visco-plastic problem

Assume an infinite slab with thickness h undergoes a pure shear T_{12} along the thickness of the slab such that the only nonzero component of the displacement vector $\mathbf{u}$ is u_1 given as a function of the variable x_2 along the thickness of the slab. The shear strain E_{12} in this case is defined by

$$E_{12} = \frac{1}{2}\frac{\partial u_1}{\partial x_2}.$$

Let $E_{12}^p = E$, $T_{12} = T$ and $F = \left(\frac{d^p}{d_o}\right)^m \frac{1}{d^p}$ then the flow rule (7.4) reduces to

$$T + \mu L^2 \frac{\partial^2 E}{\partial x_2^2} + \frac{\mu(Q^2 - L^2)}{2}\frac{\partial^2 E}{\partial x_2^2} = SF\dot{E} - l^2 S_Y \frac{\partial}{\partial x_2}\left(F\frac{\partial \dot{E}}{\partial x_2}\right).$$

Assume the body force is the force due to gravity given by $b_1(x_2, t) = \rho g$, where ρ is the density of the body and g is the acceleration due to gravity. Thus the macroscopic force balance is given by

$$(9.1) \qquad\qquad\qquad \frac{\partial T}{\partial x_2} + \rho g = 0,$$

with the simple boundary condition given by

$$(9.2) \qquad\qquad\qquad T(h, t) = 0.$$

The flow rule can be written in the form suitable for us to obtain a weak formulation as

$$0 = -\frac{\mu(L^2 + Q^2)}{2}\frac{\partial^2 E}{\partial x_2^2} - l^2 S_Y \frac{\partial}{\partial x_2}\left(F\frac{\partial \dot{E}}{\partial x_2}\right) + SF\dot{E} - \rho g(h - x_2).$$

The solution of (9.1) and (9.2) is $T = \rho g(h - x_2)$. The associated boundary and intial conditions of the flow rule are given by

$$\dot{E}(0, t) = 0, \quad \left[\frac{\mu(L^2 + Q^2)}{2}\frac{\partial E}{\partial x_2} + l^2 S_Y F \frac{\partial \dot{E}}{\partial x_2}\right]_{x_2 = h} = 0 \quad \text{and} \quad E(x_2, 0) = 0.$$

The weak formulation of the flow rule is given by

$$0 = \int_0^h \left[\frac{\mu(L^2 + Q^2)}{2}\frac{\partial E}{\partial x_2}\frac{\partial w}{\partial x_2} + l^2 S_Y F \frac{\partial \dot{E}}{\partial x_2}\frac{\partial w}{\partial x_2} + SF\dot{E}w - \rho g(h - x_2)w\right]dx_2$$

$$- \left[w\left[\frac{\mu(L^2 + Q^2)}{2}\frac{\partial E}{\partial x_2}\right] + l^2 S_Y F \frac{\partial \dot{E}}{\partial x_2}\right]_{x_2=0}^{x_2=h}.$$

If $H(S) = 0$ and $m = 1$, then the visco-plastic problem becomes a linear problem in the plastic strain. The finite element solutions using Maple 12 are illustrated in the following Figures 1 to 3 where the following parameters have been used (Assume $x_2 = y$); $\rho = 7800\,\text{kg/m}^3$, $g = 9.81\,\text{m/s}^2$, $d_o = 0.1\,\text{s}^{-1}$, $S_Y = 207\,\text{MPa}$, $\mu = 207\,\text{GPa}$.
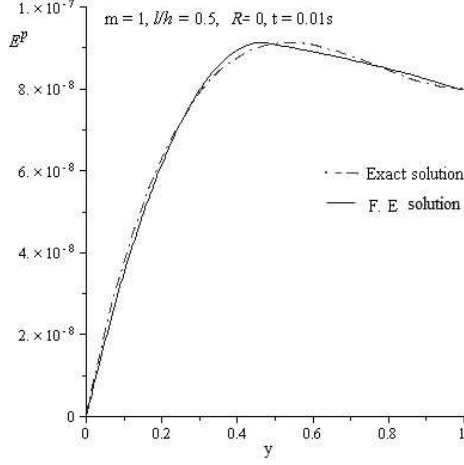
FIGURE 1. A comparison of the Exact and the Finite element (F. E.) solutions at $t = 0.01\,\mathrm{s}$ and $L = Q = 0$.
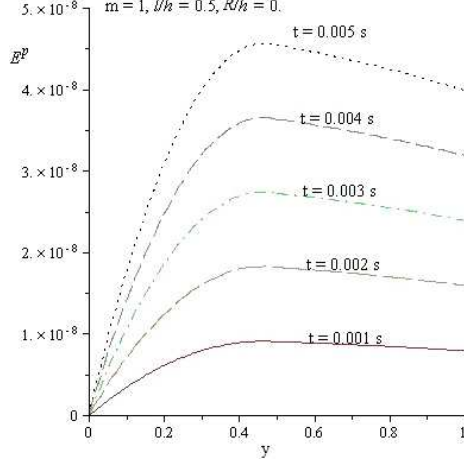
FIGURE 2. Finite element solution of a 1-dimensional viscoplastic problem for varying time t and $L = Q = 0$.
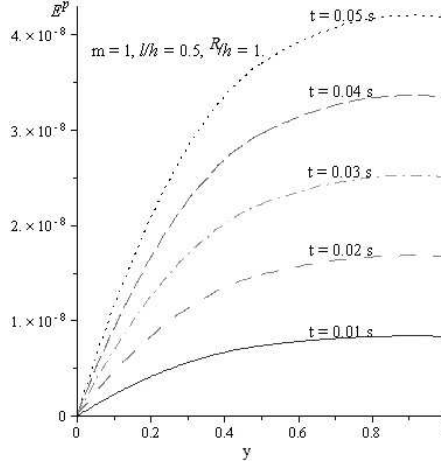
FIGURE 3. Finite element solution of a 1-dimensional viscoplastic problem for L/h=1 and Q=0.

REMARK 9.1. Clearly from Figure 1 the finite element method is a good approximate to the exact solution. Thus the finite element method used is reliable method for obtaining solutions of the problem at hand. In the absence of the energetic length scales L and Q, the plastic strain is higher compared to cases where there are presence of energetic length scales for fixed dissipative length scales l as shown in Figures 2 and 3. This observation can be clearly seen in the Table 1 below (where $M^2 = (L^2 + Q^2)/2$).

TABLE 1. F.E solutions for varying energetic length scales at $t = 0.1$ s and $l/h = 0.5$

y	$E(M/h = 0)$	$E(M/h = 0.2)$	$E(M/h = 0.4)$	$E(M/h = 0.6)$	$E(M/h = 0.8)$	$E(M/h = 1.0)$
0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.25	7.17×10^{-7}	4.51×10^{-7}	2.21×10^{-7}	1.21×10^{-7}	7.38×10^{-8}	4.93×10^{-8}
0.50	9.08×10^{-7}	6.17×10^{-7}	3.20×10^{-7}	1.79×10^{-7}	1.10×10^{-7}	7.40×10^{-8}
0.75	8.60×10^{-7}	6.34×10^{-7}	3.47×10^{-7}	1.97×10^{-7}	1.23×10^{-7}	8.26×10^{-8}
1.00	7.97×10^{-7}	6.14×10^{-7}	3.46×10^{-7}	1.98×10^{-7}	1.24×10^{-7}	8.36×10^{-8}

The results obtained thus far will be equivalent to the solution of the Gurtin–Anand model whenever $Q = 0$ for $H(S) = 0$ and $m = 1$.

10. Concluding remarks

The existence of an internal microforce and its conjugate with the divergence of the plastic strain has been obtained. The resulting model justifies the inclusion of an extra internal microforce in the theory here-in proposed. So far, we have introduced the length scale Q distinct from those introduced in [9], which are L and l. The length scales L and Q correspond to energetic length scales associated with the Burgers tensor and the divergence of the plastic strain respectively, while the length scale l corresponds to the dissipative effects asociated with the gradient of the plastic strain. The parameters (L, Q, l) show dimension consistency: they are expected to be respectively determined by fitting the theory to a particular experiment.

One dimensional problem of visco-plastic simple shearing shows that for a fixed dissipative length scale, the plastic shear strain decreases with increase in the energetic length scales, though in this case it does not seems that there is distinction between the two energetic length scales Q and L. However, for a two dimensional problem the effect of these two energetic length scales may become obvious. Thus the development of this theory and analysis of the obtained results for specific material behaviour applications and geometry are on-going.

There are other open questions: what happens in the case of large deformation? What is the character of propagating waves in such a plastic material, distinct from the case of elastic media [1]?

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ТЕОРИЈА ГРАДИЈЕНТА ДЕФОРМАЦИЈЕ ПЛАСТИЧНОСТИ СА ЕФЕКТИМА УНУТРШЊИХ МИКРОСИЛА

Резиме. У раду се развија теорија градијента деформације пластичности за изотропна тела које пролазе кроз малу деформацију у одсуству пластичног спина. Предложена теорија се заснива на систему микронапона који укључује: вектор микронапона у складу са равнотежом микросила; механички облик другог закона термодинамике који укључује рад микронапона током пластич- ног течења; конститутивну теорију која омогућава да слободна енергија зависи од еластичне деформације $\mathbf{E}^e$, дивергенције пластичне деформације div $\mathbf{E}^p$ и Бургерсовог тензора $\mathbf{G}$. Замена конститутивних релација у једначину равно- теже микросила доводи до нелинеарне парцијалне диференцијалне једначине за пластичне деформације познате као закон течења, која обухвата присуство додатне енергетске скале дужине која проистиче из урачунатог вектора ми- кронапона. Поред закона течења, добијени су нестандардни гранични услови и изведена је варијациона формулација закона. Методом коначних елемената добијено је решење једнодимензионог проблема вископластичног простог сми- цања под дејством гравитационе силе, где је показано да за фиксну дисипа- тивну скалу дужине, повећање енергетске скале дужине доводи до смањења пластичне деформације.

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