

NOETHER SYMMETRIES AND INTEGRABILITY IN TIME-DEPENDENT HAMILTONIAN MECHANICS

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ABSTRACT. We consider Noether symmetries within Hamiltonian setting as transformations that preserve Poincaré–Cartan form, i.e., as symmetries of characteristic line bundles of nondegenerate 1-forms. In the case when the Poincaré–Cartan form is contact, the explicit expression for the symmetries in the inverse Noether theorem is given. As examples, we consider natural mechanical systems, in particular the Kepler problem. Finally, we prove a variant of the theorem on complete (non-commutative) integrability in terms of Noether symmetries of time-dependent Hamiltonian systems.

1. Introduction

1.1. Since Emmy Noether’s paper [27] on integrals related to invariant variational problems, there has been a lot of efforts on its generalization, geometrical formulation, as well as on the application in various concrete problems (e.g., see [23, 29]). For finite dimensional Lagrangian systems, Noether’s general statement [27] takes the following simple form.

Consider a Lagrangian system (Q, L) , where Q is a configuration space and $L(t, q, \dot{q})$ is a Lagrangian, $L: \mathbb{R} \times TQ \rightarrow \mathbb{R}$. Let $q = (q_1, \dots, q_n)$ be local coordinates on Q . The motion of the system is described by the Euler–Lagrange equations

$$(1.1) \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}, \quad i = 1, \dots, n.$$

One of the basic principles of classical mechanics, *the Hamiltonian principle of least action*, or *the principle of stationary action*, says that the solutions of the Euler–Lagrange equations are the critical points of the action integral

$$(1.2) \quad S_L(\gamma) = \int_a^b L(t, q, \dot{q}) dt$$

in a class of curves $\gamma: [a, b] \rightarrow Q$ with fixed endpoints $\gamma(a) = q_0$, $\gamma(b) = q_1$ (e.g., see [2, 9, 18, 21, 35]).

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Consider the action of an one-parameter group of diffeomorphisms g_s on $\mathbb{R} \times Q$ with the induced vector field $\nu = (\tau, \xi)|_{(t,q)} = \frac{d}{ds}|_{s=0} g_s(t, q)$. After prolongation to $\mathbb{R} \times TQ$, the induced vector field reads

$$(1.3) \quad \hat{\nu} = \tau(t, q) \frac{\partial}{\partial t} + \sum_i \xi^i(t, q) \frac{\partial}{\partial q_i} + \hat{\xi}^i(t, q, \dot{q}) \frac{\partial}{\partial \dot{q}_i},$$

where (e.g., see [4, 11])

$$\hat{\xi}^i = \frac{\partial \xi^i}{\partial t} - \dot{q}_i \frac{\partial \tau}{\partial t} + \sum_j \left(\frac{\partial \xi^i}{\partial q_j} \dot{q}_j - \dot{q}_i \frac{\partial \tau}{\partial q_j} \dot{q}_j \right).$$

The group g_s is a *Noether symmetry* of the Lagrangian system if it preserves the action functional (1.2), that is, if

$$(1.4) \quad \frac{\partial L}{\partial t} \tau + \sum_i \left(\frac{\partial L}{\partial q_i} \xi^i + \frac{\partial L}{\partial \dot{q}_i} \hat{\xi}^i \right) + L \left(\frac{\partial \tau}{\partial t} + \sum_j \frac{\partial \tau}{\partial q_j} \dot{q}_j \right) = 0.$$

The Noether theorem says that if g_s is a Noether symmetry then

$$I(t, q, \dot{q}) = \frac{\partial L}{\partial \dot{q}} (\xi - \tau \dot{q}) + L \tau = \sum_i \frac{\partial L}{\partial \dot{q}_i} (\xi^i - \tau \dot{q}_i) + L \tau$$

is the first integral of the Euler-Lagrange equations. More generally, if we have the invariance of (1.2) modulo the integral of df/dt , that is at the right hand side of (1.4) we have df/dt , for some function $f(t, q)$ (so called *gauge term*), then the integral is $I(t, q, \dot{q}) - f(t, q)$.

Two cases are of particular interest. If $\tau \equiv 0$ then (1.4) reduces to the condition that the Lagrangian L is invariant with respect to g_s ,

$$\sum_i \frac{\partial L}{\partial q_i} \xi^i + \frac{\partial L}{\partial \dot{q}_i} \hat{\xi}^i = 0,$$

and the Noether integral takes the basic form

$$I(t, q, \dot{q}) = \frac{\partial L}{\partial \dot{q}} (\xi) = \sum_i \frac{\partial L}{\partial \dot{q}_i} \xi^i.$$

In particular, when the Lagrangian does not depend on q_1 , then q_1 is *ignorable (cyclic) coordinate* and the integral is $\partial L / \partial \dot{q}_1$ (e.g., see [2, 35]).

Secondly, if the Lagrangian does not depend on time, we can take the translations in time: $g_s(t, q) = (t + s, q)$. Then the vector field (1.3) is simply $\partial / \partial t$ and the integral I becomes the energy of the system multiplied by -1 :

$$I = -E = - \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i + L.$$

1.2. The Noether theorem can be seen as a part of time-dependent mechanics that is studied and geometrically formulated both in the Lagrangian and Hamiltonian setting (e.g., see [1, 3, 6, 8, 10–13, 24–26, 31, 32] and references therein¹). In Sarlet and Cantrijn [4], one can find a review of various approaches on the Noether

¹This list is far away to be a complete list of contributions on the subject.

theorem in the Lagrangian framework for velocity dependent transformations, as well as a geometrical setting for the equivalence of the first integrals and symmetries of the Lagrangian system considered as a characteristic system of the two-form $d\alpha$ (α being Poincaré–Cartan form).

The aim of this paper is to present the problem through the perspective of contact geometry, continuing the study of the Maupertuis principle, isoenergetic, and partial integrability [18, 19]. We consider Noether symmetries as symmetries of characteristic line bundles of nondegenerate 1-forms (Theorems 2.1, 3.2). In the case of time-dependent Hamiltonian systems, Noether symmetries are transformations that preserve Poincaré–Cartan form (see Proposition 2.1), and, via Legendre transformation, this is equivalent to Crampin’s notion of symmetry of Lagrangian systems [10]. This will allow us to use contact geometry for the inverse Noether theorem in Section 4.

The notion of a weak Noether symmetry is also given and the relation with the Noether symmetries is established (Proposition 4.1). The Noether symmetry is a natural generalization of classical Noether symmetry described above (see Proposition 2.2), while the notion of the weak Noether symmetry corresponds to the classical Noether symmetry with the gauge term.

In the case when the Poincaré–Cartan form is contact, the explicit expression for the Noether symmetry for a given first integral without using the gauge term is given (Theorem 4.1). As examples, we consider natural mechanical systems (Corollary 4.1), in particular the Kepler problem (Example 4.3).

Finally, in Section 5, we obtain a variant of the complete (non-commutative) integrability in terms of Noether symmetries of time-dependent Hamiltonian systems (Theorem 5.1, Corollary 5.1).

2. Noether symmetries in the Hamiltonian formulation

2.1. Let (Q, L) be a Lagrangian system. The *Legendre transformation* $\mathbb{F}L : TQ \rightarrow T^*Q$ is defined by

$$(2.1) \quad \mathbb{F}L(t, q, \xi) \cdot \eta = \left. \frac{d}{ds} \right|_{s=0} L(t, q, \xi + s\eta) \iff p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad i = 1, \dots, n,$$

where $\xi, \eta \in T_qQ$ and $(q_1, \dots, q_n, p_1, \dots, p_n)$ are canonical coordinates of the cotangent bundle T^*Q . In order to have a Hamiltonian description of the dynamics we suppose that the Legendre transformation (2.1) is a diffeomorphism. The corresponding Lagrangian L is called *hyperregular* [9].

Let $L(t, q, \dot{q})$ be a hyperregular Lagrangian. We can pass from velocities \dot{q}_i to the momenta p_j by using the standard Legendre transformation (2.1). In the coordinates (q, p) of the cotangent bundle T^*Q , the equations of motion (1.1) read:

$$(2.2) \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad i = 1, \dots, n,$$

where the Hamiltonian function $H(t, q, p)$ is the *Legendre transformation* of L

$$(2.3) \quad \begin{aligned} H(t, q, p) &= E(t, q, \dot{q})|_{\dot{q}=\mathbb{F}L^{-1}(t, q, p)} \\ &= \mathbb{F}L(t, q, \dot{q}) \cdot \dot{q} - L(t, q, \dot{q})|_{\dot{q}=\mathbb{F}L^{-1}(t, q, p)}. \end{aligned}$$

Let $pdq = \sum_i p_i dq_i$ be the *canonical 1-form* and

$$\omega = d(pdq) = dp \wedge dq = \sum_{i=1}^n dp_i \wedge dq_i$$

the *canonical symplectic form* of the cotangent bundle T^*Q . The equations (2.2) are Hamiltonian, i.e., they can be written as $\dot{x} = X_H$, where the Hamiltonian vector field X_H is defined by

$$i_{X_H} \omega(\cdot) = \omega(X_H, \cdot) = -dH(\cdot).$$

2.2. Noether symmetries. Consider the *Poincaré–Cartan 1-form*

$$\alpha = pdq - Hdt$$

on the extended phase space $\mathbb{R} \times T^*Q(t, q, p)$, where $H: \mathbb{R} \times T^*Q \rightarrow \mathbb{R}$ is a Hamiltonian function. The phase trajectories of the canonical equations (2.2) are extremals of the action

$$(2.4) \quad A_H(\gamma) = \int_{\gamma} \alpha = \int_{\gamma} pdq - Hdt$$

in the class of curves $\gamma(t) = (t, q(t), p(t))$ connecting the subspaces $\{t_0\} \times T_{q_0}^*Q$ and $\{t_1\} \times T_{q_1}^*Q$ [2, 9] (Poincaré’s modification of the Hamiltonian principle of least action [30]). Obviously, we can replace $(T^*Q, dp \wedge dq)$ by an arbitrary exact symplectic manifold $(P, \omega = d\theta)$.

We shall say that the vector field

$$\zeta = \tau(t, q, p) \frac{\partial}{\partial t} + \sum_i \xi^i(t, q, p) \frac{\partial}{\partial q_i} + \eta^i(t, q, p) \frac{\partial}{\partial p_i},$$

i.e., the induced one-parameter group of diffeomorphisms g_s^ζ of $\mathbb{R} \times T^*Q$,

$$\zeta = \frac{d}{ds} g_s^\zeta|_{s=0}(t, q, p),$$

is a *Noether symmetry* of the Hamiltonian system (2.2) if the Poincaré–Cartan 1-form is preserved. Then, by the analogy with the Lagrangian formulation, g_s^ζ preserves the action functional (2.4). The above definition is suitable for a contact approach presented in Section 4.

We shall say that ζ is a *weak Noether symmetry* if we have the invariance of the perturbation of the Poincaré–Cartan 1-form by a closed 1-form β modulo the differential of the function f :

$$(2.5) \quad L_\zeta(pdq - Hdt + \beta) = df.$$

The function $f(t, q, p)$ plays a role of a gauge term in the classical formulation. The closed form β corresponds to the fact that the solutions of the canonical equation (2.2) are also extremal of the perturbed action

$$A_H(\gamma) = \int_{\gamma} pdq - Hdt + \beta.$$

THEOREM 2.1. *Let ζ be a weak Noether symmetry satisfying (2.5). Then*

i) *The function*

$$J = i_\zeta(pdq - H dt + \beta) - f = \sum_i p_i \xi^i - H\tau + \beta(\zeta) - f$$

is the first integral of the Hamiltonian equations (2.2).

ii) *The integral J is also preserved under the flow of g_s^ζ .*

iii) *The one-parameter group of diffeomorphisms g_s^ζ permutes the trajectories of the Hamiltonian equations in the extended phase $\mathbb{R} \times T^*Q$ space modulo reparametrization.*

The notion of weak Noether symmetries for $\beta = 0$ is equivalent, via Legendre transformation, to the symmetries of Lagrangian systems considered by Crampin [10], see also Sarlet and Catrijn [4, 5]. The proof of Theorem 2.1 is similar to the proofs presented in [4, 5, 10] and for the completeness of the exposition it will be given in the next section (see the proof of Theorem 3.2). Also, recently, a similar approach to the higher order Lagrangian problems is given in [14].

By definition, ζ is a Noether symmetry if and only if the Lie derivative of the Poincaré–Cartan 1-form vanish:

$$\begin{aligned} 0 &= L_\zeta \alpha \\ &= i_\zeta(d\alpha) + d(i_\zeta \alpha) \\ &= i_\zeta(dp \wedge dq - dH \wedge dt) + d(p\xi - H\tau) \\ &= \eta dq - \xi dp + \tau dH - dH(\zeta)dt + d(p\xi - H\tau) \\ &= \eta dq + pd\xi - Hd\tau - dH(\zeta)dt. \end{aligned}$$

Comparing the components with dp_j , dq_j , and dt we get the following statement.

PROPOSITION 2.1. *ζ is a Noether symmetry if and only if*

$$(2.6) \quad \sum_i p_i \frac{\partial \xi^i}{\partial p_j} - H \frac{\partial \tau}{\partial p_j} = 0,$$

$$(2.7) \quad \eta^j + \sum_i p_i \frac{\partial \xi^i}{\partial q_j} - H \frac{\partial \tau}{\partial q_j} = 0, \quad j = 1, \dots, n,$$

$$(2.8) \quad \sum_i \left(p_i \frac{\partial \xi^i}{\partial t} - \eta^i \frac{\partial H}{\partial p_i} - \xi^i \frac{\partial H}{\partial q_i} \right) - H \frac{\partial \tau}{\partial t} - \tau \frac{\partial H}{\partial t} = 0.$$

PROPOSITION 2.2. *If the Lagrangian L and the Hamiltonian H are related by the Legendre transformation (2.1), (2.3) and if ζ is a Noether symmetry of the Hamiltonian equation (2.2) with $\xi^i = \xi^i(t, q)$, $\tau = \tau(t, q)$, then the classical invariance condition (1.4) is satisfied.*

PROOF. Since ξ^i and τ do not depend on p , the conditions (2.6) is satisfied. By expressing η^i from (2.7) and substituting into (2.8) we get the following equation

$$(2.9) \quad \sum_j p_j \left(\frac{\partial \xi^j}{\partial t} + \sum_i \frac{\partial \xi^j}{\partial q_i} \frac{\partial H}{\partial p_i} \right) - H \left(\frac{\partial \tau}{\partial t} + \sum_j \frac{\partial \tau}{\partial q_j} \frac{\partial H}{\partial p_j} \right) - \sum_j \xi^j \frac{\partial H}{\partial q_j} - \tau \frac{\partial H}{\partial t} = 0.$$

On the other hand, (1.4) transforms to

$$(2.10) \quad \frac{\partial L}{\partial t} \tau + \sum_i \frac{\partial L}{\partial q_i} \xi^i + \sum_i \frac{\partial L}{\partial \dot{q}_i} \left(\frac{\partial \xi^i}{\partial t} + \sum_j \frac{\partial \xi^i}{\partial q_j} \dot{q}_j \right) - \left(\frac{\partial \tau}{\partial t} + \sum_i \frac{\partial \tau}{\partial q_i} \dot{q}_i \right) \left(\sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L \right) = 0.$$

Since the Legendre transformation (2.1), (2.3) implies well known identities

$$\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}, \quad \frac{\partial L}{\partial q_i} = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i},$$

the equations (2.9) and (2.10) are equivalent. \square

Therefore, we can consider the above definition of a Noether symmetry as a natural generalization of the classical one.

3. Noether symmetries of characteristic line bundles

3.1. Let (M, α) be a $(2n + 1)$ -dimensional manifold endowed with a *nondegenerate* 1-form α . This means that $d\alpha$ has the maximal rank $2n$. The kernel of $d\alpha$ defines one dimensional distribution

$$\mathcal{L} = \bigcup_x \mathcal{L}_x, \quad \mathcal{L}_x = \ker d\alpha|_x$$

of the tangent bundle TM called *characteristic line bundle*. Also, at every point $x \in M$ we have the *horizontal space* \mathcal{H}_x defined by

$$\mathcal{H}_x = \ker \alpha|_x.$$

In the case when $\alpha \neq 0$, $d\alpha \neq 0$ on M , then the collection of horizontal subspaces $\mathcal{H} = \bigcup_x \mathcal{H}_x = \bigcup_x \ker \alpha|_x$ is a nonintegrable distribution of TM , called *horizontal distribution*. If, in addition, $\alpha \wedge (d\alpha)^n \neq 0$, then α is a *contact form* and (M, α) is a *strictly contact manifold* [26]. The horizontal distribution \mathcal{H} is also referred as *contact distribution*.

The following variational statement is well known.

THEOREM 3.1. *The integral curves $\gamma: [a, b] \rightarrow M$ of the characteristic line bundle \mathcal{L} are extremals of the action functional*

$$A(\gamma) = \int_\gamma \alpha = \int_a^b \alpha(\dot{\gamma}) dt$$

in the class of variations $\gamma_s(t)$, such that $\delta\gamma(a)$ and $\delta\gamma(b)$ are horizontal vectors.

Here, a *variation* of a curve $\gamma: [a, b] \rightarrow M$ is a mapping: $\Gamma: [a, b] \times [0, \epsilon] \rightarrow M$, such that $\gamma(t) = \Gamma(t, 0)$, $t \in [a, b]$, $\delta\gamma(t) = \frac{\partial \Gamma}{\partial s}|_{s=0} \in T_{\gamma(t)}M$, and $\gamma_s(t) = \Gamma(t, s)$.

The proof is a direct consequence of Cartan's formula (e.g., see [16])

$$L_{\partial/\partial s} \Gamma^* \alpha|_{(t,0)} = \gamma^*(i_{\delta\gamma(t)} d\alpha) + d\gamma^*(\alpha(\delta\gamma(t))),$$

which implies

$$(3.1) \quad \frac{d}{ds} \left(\int_{\gamma_s} \alpha \right) \Big|_{s=0} = \int_a^b d\alpha(\delta\gamma(t), \dot{\gamma}(t)) dt + \alpha(\delta\gamma(b)) - \alpha(\delta\gamma(a)).$$

For $\delta\gamma(a)$ and $\delta\gamma(b)$ being horizontal, the expression (3.1) is equal to zero if and only if $\dot{\gamma}$ is in the kernel of the form $d\alpha$. That is, $\gamma(t)$ is an integral curve of the line bundle \mathcal{L} .

EXAMPLE 3.1. As an example we can take the extended phase space endowed with the Poincaré–Cartan 1-form

$$(3.2) \quad (\mathbb{R} \times T^*Q(t, q, p), pdq - Hdt)$$

(e.g., see [2]). The sections of $\ker d(pdq - Hdt)$ are of the form

$$Z_\mu = \mu Z,$$

where

$$(3.3) \quad Z = \frac{\partial}{\partial t} + \sum_i \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i}.$$

and $\mu = \mu(t, q, p)$ are smooth functions. Therefore, in this case, Theorem 3.1 implies Hamiltonian principle of least action in the extended phase space. Here, the vector space T_q^*Q , considered as a subspace of $T_{(t,q,p)}\mathbb{R} \times T^*Q$, is a subspace of the horizontal space

$$(3.4) \quad \mathcal{H}_{(t,q,p)} = \left\{ \tau \frac{\partial}{\partial t} + \sum_i \xi^i \frac{\partial}{\partial q_i} + \eta^i \frac{\partial}{\partial p_i} \mid \tau, \xi^i, \eta^i \in \mathbb{R}, \sum_i p_i \xi^i = \tau H(t, q, p) \right\}.$$

REMARK 3.1. The vector field (3.3) is determined by the conditions $i_Z(dp \wedge dq - dH \wedge dt) = 0$ and $dt(Z) = 1$. In other words, Z can be seen also as the Reeb vector field of the cosymplectic manifold $(\mathbb{R} \times T^*Q(t, q, p), dp \wedge dq - dH \wedge dt, dt)$. Recall that a cosymplectic manifold (M, ω, η) is a $(2n + 1)$ -dimensional manifold M endowed with a closed 2-form ω and a closed 1-form η , such that $\eta \wedge \omega^n$ is a

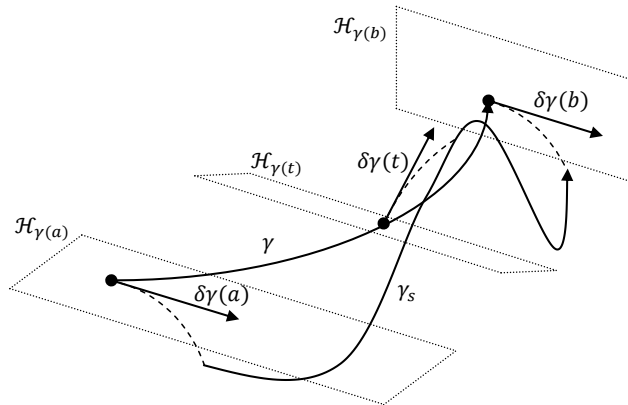


FIGURE 1. Variation of a curve γ , such that $\delta\gamma(a)$ and $\delta\gamma(b)$ are horizontal vectors.

volume form, which is a natural framework for the time-dependent Hamiltonian mechanics (see [1, 6, 7]).

3.2. Noether symmetries and integrals. Consider the equation

$$(3.5) \quad \dot{x} = Z,$$

where Z is a section of \mathcal{L} ($i_Z d\alpha = 0$).

We shall say that the vector field ζ , i.e., the induced one-parameter group of diffeomorphisms g_s^ζ , is a *Noether symmetry* of the equation (3.5) if it preserves the 1-form α . A similar definition for exterior differential systems is given in [16].

Note that the vector field Z is also a section of the characteristic line bundle of a nondegenerate 1-form $\alpha + \beta$, where β is arbitrary closed 1-form on M . We refer to the vector field ζ as a *weak Noether symmetry* if we have the invariance of the perturbation of the 1-form α by a closed 1-form β modulo the differential of the function f :

$$(3.6) \quad L_\zeta(\alpha + \beta) = df.$$

THEOREM 3.2. *Let ζ be a weak Noether symmetry that satisfies (3.6). Then:*

- i) *The function $J = i_\zeta(\alpha + \beta) - f$ is the first integral of (3.5).*
- ii) *The integral J is preserved under the flow of g_s^ζ as well: $dJ(\zeta) = 0$.*
- iii) *The commutator of vector fields $[Z, \zeta]$ is a section of \mathcal{L} , i.e., g_s^ζ permutes the trajectories of (3.5) modulo reparametrization.*

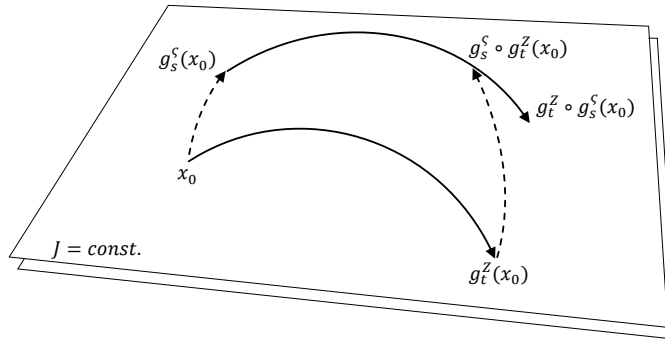


FIGURE 2. Illustration of Theorem 3.2.

PROOF. i) From the definition (3.6) and Cartan's formula $L_\zeta = i_\zeta \circ d + d \circ i_\zeta$, we have

$$(3.7) \quad i_\zeta d\alpha = -d(i_\zeta(\alpha + \beta)) + df = -dJ,$$

which proves the first assertion of the statement:

$$(3.8) \quad Z(J) = i_Z(-i_\zeta d\alpha) = d\alpha(Z, \zeta) = 0.$$

Alternatively, we have the variational interpretation of the first integral. Let $\gamma: [a, b] \rightarrow M$ be the trajectory of (3.5) and consider the variation $\gamma_s = g_s^\zeta(\gamma)$, $\delta\gamma(t) = \zeta|_{\gamma(t)}$. From (3.6), (3.1) we get, respectively,

$$\begin{aligned} \frac{d}{ds} \left(\int_{\gamma_s} \alpha + \beta \right) \Big|_{s=0} &= \int_{\gamma_s} df = f(b) - f(a), \\ \frac{d}{ds} \left(\int_{\gamma_s} \alpha + \beta \right) \Big|_{s=0} &= \int_a^b d(\alpha + \beta)(\zeta|_{\gamma(t)}, \dot{\gamma}(t)) dt + (\alpha + \beta)(\zeta|_b) - (\alpha + \beta)(\zeta|_a) \\ &= (\alpha + \beta)(\zeta|_b) - (\alpha + \beta)(\zeta|_a). \end{aligned}$$

Therefore, $(\alpha + \beta)(\zeta|_a) - f(a) = (\alpha + \beta)(\zeta|_b) - f(b)$.

ii) Similarly as the equation (3.8), (3.7) implies

$$\zeta(J) = i_\zeta(-i_\zeta d\alpha) = d\alpha(\zeta, \zeta) = 0.$$

iii) We need to prove that $[\zeta, Z]$ belongs to the kernel of $d\alpha$. We have

$$i_{[\zeta, Z]} d\alpha = L_\zeta(i_Z d\alpha) - i_Z(L_\zeta d\alpha) = -i_Z(i_\zeta d^2\alpha + d(i_\zeta(d\alpha))) = -i_Z d(dJ) = 0. \quad \square$$

3.3. It is clear that in the study of integrals of the Hamiltonian equations (2.2), we can consider arbitrary section $Z_\mu = \mu Z$ of \mathcal{L} , where $\mu(t, q, p)$ is a function that is almost everywhere different from zero. The normalization $dt(Z_\mu) = 1$ implies $\mu \equiv 1$. In the case when Z is transversal to the horizontal spaces (3.4):

$$(3.9) \quad \rho = i_Z(pdq - Hdt) = p \frac{\partial H}{\partial p} - H \neq 0,$$

there is another natural normalization $(pdq - Hdt)(Z_\mu) = 1$, $\mu = \rho^{-1}$,

$$(3.10) \quad Z_{\rho^{-1}} = \rho^{-1} \frac{\partial}{\partial t} + \rho^{-1} \sum_i \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right).$$

The condition (3.9) is equivalent to the property that (3.2) is a strictly contact manifold with the Reeb vector field (3.10) (see [26]). If the Hamiltonian H is obtained from the Lagrangian L under the Legendre transformation (2.1), (2.3), the function ρ is the Lagrangian $L(t, q, \dot{q})|_{\dot{q}=\dot{q}(t, q, p)}$. In [26] it is referred as an *elementary action*.

4. Inverse Noether theorem

A natural question is the converse of the Noether theorem (e.g., see [4]): if F is the integral of (2.2), is there a Noether symmetry ζ , such that $F = i_\zeta \alpha$?

A geometrical setting for the equivalence of the first integrals and weak symmetries can be found in [4, 10]. With the above notation, one should firstly construct a vector field ζ , such that $i_\zeta d\alpha = dF$. Then ζ is a weak Noether symmetry with $L_\zeta \alpha = df$, where $f = F + i_\zeta \alpha$.

It appears that the contact approach provides a simple explicit expression for the Noether symmetry for a generic Hamiltonian function.

4.1. Contact Hamiltonian vector fields. Let (M, α) be a strictly contact manifold. Then the contact distribution \mathcal{H} is transversal to the characteristic line bundle \mathcal{L} :

$$TM = \mathcal{L} \oplus \mathcal{H}.$$

A vector field X that preserves \mathcal{H} :

$$(g_t^X)_* \mathcal{H} = \mathcal{H} \iff L_X \alpha = \lambda \alpha, \quad \text{for some smooth function } \lambda$$

is called *contact vector field*. There is a distinguish contact vector field, the *Reeb vector field* Z , uniquely defined by

$$(4.1) \quad i_Z\alpha = 1, \quad i_Zd\alpha = 0.$$

For a given function f , one can associate the *contact vector field* Y_f with *Hamiltonian* f :

$$Y_f = fZ + \hat{Y}_f,$$

where \hat{Y}_f is a horizontal vector field defined by $i_{\hat{Y}_f}d\alpha = -(df - Z(f)\alpha)$ (e.g., see [26]). The mapping $f \mapsto Y_f$ is a bijection between smooth functions and contact vector fields on M . The inverse mapping is simply the contraction: $f = i_{Y_f}\alpha$. In particular, the Hamiltonian of the Reeb vector field is $f \equiv 1$.

Note that

$$\mathcal{L}_{Y_f}\alpha = Z(f)\alpha,$$

i.e., for the Reeb flow we have the inverse Noether theorem directly: Y_f is a Noether symmetry of the Reeb flow if and only if f is the integral of the Reeb flow.

4.2. Inverse Noether theorem. If the elementary action is different from zero (3.9), a Noether symmetry ζ of the Hamiltonian equation (2.2) is a contact vector field of (3.2) with the Hamiltonian function $J = i_\zeta(pdq - Hdt)$.

We say that H is a *generic Hamiltonian*, if the condition (3.9) hold for an open dense subset U_H of $\mathbb{R} \times T^*Q$. From now on, we assume that H is generic. Thus, we have:

THEOREM 4.1. *To every integral F of the Hamiltonian equation (2.2), we can associate unique Noether symmetry ζ on U_H , such that the corresponding Noether integral is equal to F : $i_\zeta(pdq - Hdt) = F$. The vector field ζ reads:*

$$\zeta = \tau(t, q, p) \frac{\partial}{\partial t} + \sum_i \xi^i(t, q, p) \frac{\partial}{\partial q_i} + \eta^i(t, q, p) \frac{\partial}{\partial p_i},$$

where the coefficient τ, ξ^i, η^i are given by

$$(4.2) \quad \tau = \rho^{-1}F - \rho^{-1} \sum_j \frac{\partial F}{\partial p_j} p_j,$$

$$(4.3) \quad \xi^i = \rho^{-1}F \frac{\partial H}{\partial p_i} - \rho^{-1} \sum_j \frac{\partial F}{\partial p_j} p_j \frac{\partial H}{\partial p_i} + \frac{\partial F}{\partial p_i},$$

$$(4.4) \quad \eta^i = -\rho^{-1}F \frac{\partial H}{\partial q_i} + \rho^{-1} \sum_j \frac{\partial F}{\partial p_j} p_j \frac{\partial H}{\partial q_i} - \frac{\partial F}{\partial q_i}, \quad i = 1, \dots, n.$$

In particular, if the invariant regular hypersurface $M_c = \{F(t, q, p) = c\}$ is a subset of U_H , the vector field ζ is well defined on the whole M_c .

PROOF. The required vector field ζ is the contact vector field with the Hamiltonian F :

$$\zeta = Y_F = FZ_{\rho^{-1}} + \hat{Y}_F,$$

where the Reeb vector field $Z_{\rho^{-1}}$ is given by (3.10) and the the coefficient a, b^i, c^i

of the horizontal vector field

$$\hat{Y}_F = a(t, q, p) \frac{\partial}{\partial t} + \sum_i b^i(t, q, p) \frac{\partial}{\partial q_i} + c^i(t, q, p) \frac{\partial}{\partial p_i},$$

are uniquely determined by the conditions:

$$(4.5) \quad i_{\hat{Y}_F}(pdq - Hdt) = \sum_i b^i p_i - Ha = 0,$$

$$(4.6) \quad i_{\hat{Y}_F}(dp \wedge dq - dH \wedge dt) = -(dF - Z_{\rho^{-1}}(F)(pdq - Hdt)) = -dF.$$

Here we used the fact that F is the integral of the Hamiltonian equations (2.2):

$$Z_{\rho^{-1}}(F) = \rho^{-1} \left(\frac{\partial F}{\partial t} + \sum_i \frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i} \right) = 0.$$

The left hand side of (4.6) is

$$\sum_i (c^i dq_i - b^i dp_i) + a \left(dH - \frac{\partial H}{\partial t} dt \right) - \sum_j \left(c^j \frac{\partial H}{\partial p_j} + b^j \frac{\partial H}{\partial q_j} \right) dt.$$

Therefore, by comparing the terms with dp_i , dq_i , and dt in (4.6), respectively, we obtain:

$$(4.7) \quad b^i - a \frac{\partial H}{\partial p_i} = \frac{\partial F}{\partial p_i},$$

$$(4.8) \quad c^i + a \frac{\partial H}{\partial q_i} = -\frac{\partial F}{\partial q_i}, \quad i = 1, \dots, n,$$

$$(4.9) \quad \sum_j \left(c^j \frac{\partial H}{\partial p_j} + b^j \frac{\partial H}{\partial q_j} \right) = \frac{\partial F}{\partial t}.$$

By multiplying (4.7) with p_i , and taking the sum of all i , from (4.5), we get

$$(4.10) \quad a = -\rho^{-1} \sum_j \frac{\partial F}{\partial p_j} p_j.$$

Next, by substitution of (4.10) into (4.7) and (4.8), we get, respectively:

$$(4.11) \quad b^i = -\rho^{-1} \sum_j \frac{\partial F}{\partial p_j} p_j \frac{\partial H}{\partial p_i} + \frac{\partial F}{\partial p_i},$$

$$(4.12) \quad c^i = \rho^{-1} \sum_j \frac{\partial F}{\partial p_j} p_j \frac{\partial H}{\partial q_i} - \frac{\partial F}{\partial q_i}, \quad i = 1, \dots, n.$$

Now, the equation (4.9) is equivalent to the property that F is the integral of the canonical equations (2.2). \square

The inverse Noether theorem for symmetries of k -th order Lagrangians is given recently in [14]. The set U_H corresponds to the set \mathcal{W} given there.

Note that in many well studied examples of natural mechanical systems, such as Kovalevskaya top (see [12, 33]²), the integrals are interpreted as Noether integrals with the gauge terms. Here we have the following statement.

COROLLARY 4.1. *Consider a natural mechanical system on T^*Q with the Hamiltonian of the form $H = T + V$, where $T = \frac{1}{2} \sum_{i,j} g^{ij}(t, q)p_i p_j$ is the positive definite kinetic energy and $V = V(t, q)$ is the potential. If the potential is bounded from the above, $\max_{(t,q) \in \mathbb{R} \times Q} V(t, q) < v$, then we can take the same system with the potential replaced by $V - c$, where $c > v$. Then every integral F of the Hamiltonian equations (2.2) is a Noether integral with the Noether symmetry $\zeta = Y_F$.*

PROOF. The elementary action $\rho = T - V$ is always greater then 0. Therefore (3.2) is a strictly contact manifold. \square

REMARK 4.1. If there exist a closed 1-form β ,

$$\beta = a dt + \sum_i b_i dq_i + c_i dp_i,$$

such that

$$(4.13) \quad \rho + a + \sum_i b_i \frac{\partial H}{\partial p_i} - c_i \frac{\partial H}{\partial q_i} \neq 0, \quad \text{for all } (t, q, p),$$

then the extended phase space $\mathbb{R} \times T^*Q$ is a strongly contact manifold with respect to the contact 1-form $pdq - Hdt + \beta$. To every integral F of the Hamiltonian equation (2.2), we can associate unique weak Noether symmetry ζ , the contact Hamiltonian flow of the integral F with respect to the contact form $pdq - Hdt + \beta$, such that the corresponding Noether integral is equal to F : $i_\zeta(pdq - Hdt + \beta) = F$. For example, the replacement of potential energy V by $V - c$ in the Corollary 4.1 corresponds to the closed form $\beta = a dt$, $a \equiv c$.

PROPOSITION 4.1. *Assume that ζ is a weak Noether symmetry that satisfies (2.5). Then, the associated Noether symmetry $\tilde{\zeta}$ on U_H with the same conserved quantity is*

$$\tilde{\zeta} = \zeta + \rho^{-1}(\beta(\zeta) - f)Z,$$

where Z is given by (3.3).

PROOF. According to Theorem 4.1, on U_H we have the Noether symmetry $\tilde{\zeta}$, such that

$$(4.14) \quad J = i_\zeta(pdq - Hdt + \beta) - f = i_{\tilde{\zeta}}(pdq - Hdt).$$

From (3.7), (4.14) we have

$$dJ = -i_\zeta(dp \wedge dq - dH \wedge dt) = -i_{\tilde{\zeta}}(dp \wedge dq - dH \wedge dt),$$

and, therefore,

$$\zeta - \tilde{\zeta} \in \ker(dp \wedge dq - dH \wedge dt).$$

²There, one can find the formulation of the Noether theorem in quasi-coordinates within Lagrangian setting, such that transformations of time and coordinates (t, q) depend on (t, q, \dot{q}) . Recently, this approach is extended to nonconservative systems in [28] as well.

Thus,

$$\tilde{\zeta} = \zeta + \nu Z,$$

for some function $\nu(t, q, p)$. Finally, by substitution of the above relation to (4.14), we get $\nu = \rho^{-1}(\beta(\zeta) - f)$. \square

EXAMPLE 4.1. *Linear integrals and energy.* Assume that

$$F = \sum_j \xi^j(q, t)p_j$$

is the first integral. Then

$$\sum_j \frac{\partial F}{\partial p_j} p_j = F,$$

and Theorem 4.1 gives the well known expression for the Noether symmetry

$$\zeta = \sum_i \xi^i \frac{\partial}{\partial q_i} - \sum_{ij} \frac{\partial \xi^j}{\partial q^i} p_j \frac{\partial}{\partial p_i}.$$

Next, if the Hamiltonian H does not depend on time it is the integral of the system. The Noether symmetry from Theorem 4.1, for the integral $F = -H$, takes the expected form:

$$\zeta = \frac{\partial}{\partial t}.$$

Note that the above vector fields have smooth extensions from U_H to $\mathbb{R} \times T^*Q$.

EXAMPLE 4.2. *Quadratic integrals of the geodesic flows.* Consider the geodesic flow with the Hamiltonian function $H = \frac{1}{2} \sum_{ij} g^{ij}(q)p_i p_j$. We have $\rho = H \neq 0$ outside the zero section of T^*Q . Assume that we have a quadratic first integral

$$F = \frac{1}{2} \sum_{ij} a^{ij}(q)p_i p_j.$$

Then

$$\sum_j \frac{\partial F}{\partial p_j} p_j = 2F$$

and outside the zero section of T^*Q we have the Noether symmetry

$$\zeta = -\frac{F}{H} \frac{\partial}{\partial t} + \sum_i \left(-\frac{F}{H} \frac{\partial H}{\partial p_i} + \frac{\partial F}{\partial p_i} \right) \frac{\partial}{\partial q_i} + \sum_i \left(\frac{F}{H} \frac{\partial H}{\partial q_i} - \frac{\partial F}{\partial q_i} \right) \frac{\partial}{\partial p_i}.$$

EXAMPLE 4.3. *Kepler problem.* The Noether symmetries associated to the Runge–Lenz vector in the Kepler problem are one of the basic examples for the inverse Noether theorem, see [4, 8, 32, 34]. Consider a planar motion of a unit mass particle in the central gravitational force field. We have $Q = \mathbb{R}^2 \setminus \{(0, 0)\}$, and the Hamiltonian is

$$H = T + V = \frac{1}{2}(p_1^2 + p_2^2) - \frac{\mu}{r}, \quad r = \sqrt{q_1^2 + q_2^2}, \quad \mu > 0$$

The system is superintegrable with well known integrals: the Hamiltonian H , the angular momentum $L = q_1 p_2 - q_2 p_1$, and the Runge–Lenz vector

$$A = (A_1, A_2) = \left(q_1 p_2^2 - q_2 p_1 p_2 - \mu \frac{q_1}{r}, q_2 p_1^2 - q_1 p_1 p_2 - \mu \frac{q_2}{r} \right).$$

The elementary action $\rho = T - V$ is greater than 0 and the Noether symmetries for the integrals H and L are already described in Example 4.1. We have

$$\frac{\partial A_k}{\partial p_j} p_j = 2A_k + 2\mu \frac{q_k}{r}, \quad k = 1, 2.$$

Therefore, the Noether symmetries of integrals A_1 and A_2 are

$$\zeta_k = \tau_k(t, q, p) \frac{\partial}{\partial t} + \xi_k^i(t, q, p) \frac{\partial}{\partial q^i} + \eta_k^i(t, q, p) \frac{\partial}{\partial p_i}, \quad k = 1, 2,$$

where the coefficient $\tau_k, \xi_k^i, \eta_k^i$, $k = 1, 2$, are given by

$$\begin{aligned} \tau_1 &= -\rho^{-1} \left(q_1 p_2^2 - q_2 p_1 p_2 + \mu \frac{q_1}{r} \right), \\ \xi_1^1 &= \tau_1 p_1 - q_2 p_2, \\ \xi_1^2 &= \tau_1 p_2 + 2q_1 p_2 - q_2 p_1, \\ \eta_1^1 &= -\tau_1 \mu \frac{q_1}{r^3} - p_2^2 - \mu \frac{q_1^2}{r^3} + \mu \frac{1}{r}, \\ \eta_1^2 &= -\tau_1 \mu \frac{q_2}{r^3} + p_1 p_2 - \mu \frac{q_1 q_2}{r^3}, \\ \tau_2 &= -\rho^{-1} \left(q_2 p_1^2 - q_1 p_1 p_2 + \mu \frac{q_2}{r} \right), \\ \xi_2^1 &= \tau_2 p_1 + 2q_2 p_1 - q_1 p_2, \\ \xi_2^2 &= \tau_2 p_2 - q_1 p_1, \\ \eta_2^1 &= -\tau_2 \mu \frac{q_1}{r^3} + p_1 p_2 - \mu \frac{q_1 q_2}{r^3}, \\ \eta_2^2 &= -\tau_2 \mu \frac{q_2}{r^3} - p_1^2 - \mu \frac{q_2^2}{r^3} + \mu \frac{1}{r}. \end{aligned}$$

5. Integrability by means of Noether symmetries

For $\rho \neq 0$, the Noether symmetries are contact vector fields and we can use the notion of complete integrability of contact systems (see [17, 19, 20]) to obtain a variant of the complete (non-commutative) integrability in terms of Noether symmetries of time-dependent Hamiltonian systems. It appears, however, that we do not need the contact assumption $\rho \neq 0$.

THEOREM 5.1. *Assume that Hamiltonian equations (2.2) have m independent Noether symmetries ζ_1, \dots, ζ_m , independent of the vector field (3.3), such that first r of them commute with all symmetries,*

$$[\zeta_i, \zeta_j] = 0, \quad i = 1, \dots, r, \quad j = 1, \dots, m,$$

and $2n = m + r$. Then

- (i) The Noether integrals $J_i = i_{\zeta_i}(pdq - Hdt)$ are independent and the equations (2.2) are locally solvable by quadratures.
- (ii) If the vector fields $Z, \zeta_1, \dots, \zeta_r$ are complete, then a connected regular component of the invariant variety in the extended phase space

$$(5.1) \quad M_c = \{(t, q, p) \in \mathbb{R} \times T^*Q \mid J_i = c_i, i = 1, \dots, m\}$$

is diffeomorphic to a cylinder $\mathbb{T}^l \times \mathbb{R}^{r+1-l}$, for some $l, 0 \leq l \leq r$, where \mathbb{T}^l is a l -dimensional torus. There exist coordinates $\varphi_1, \dots, \varphi_l, x_1, \dots, x_{r+1-l}$ of $\mathbb{T}^l \times \mathbb{R}^{r+1-l}$, which linearise the equation in the extended phase space:

$$\begin{aligned} \dot{\varphi}_i &= \omega_i = \text{const}, & i &= 1, \dots, l, \\ \dot{x}_j &= a_j = \text{const}, & j &= 1, \dots, r + 1 - l. \end{aligned}$$

Note that the above statement slightly differs from the Arnold–Liouville and non-commutative integrability of time-dependent Hamiltonian systems studied in [15, 31].

PROOF. (i) According to the item (iii) of Theorem 2.1, we have

$$[Z, \zeta_i] = \nu_i Z,$$

for some smooth functions $\nu_i = \nu_i(t, q, p), i = 1, \dots, m$. We need to find functions f_i , such that the vector fields

$$Z, \quad \tilde{\zeta}_i = \zeta_i - f_i Z$$

pairwise commute:

$$(5.2) \quad [Z, \tilde{\zeta}_i] = 0, \quad [\tilde{\zeta}_i, \tilde{\zeta}_j] = 0, \quad i, j = 1, \dots, r.$$

Let $\tau_i = dt(\zeta_i)$. Since $dt(Z) = 1$, we have

$$\nu_i = [Z, \zeta_i](dt) = Z(dt(\zeta_i)) - \zeta_i(dt(Z)) = Z(\tau_i) = \frac{\partial \tau_i}{\partial t} + \frac{\partial \tau_i}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial \tau_i}{\partial p} \frac{\partial H}{\partial q}.$$

Therefore, for $f_i = \tau_i = dt(\zeta_i)$, we have

$$[Z, \tilde{\zeta}_i] = [Z, \zeta_i - \tau_i Z] = Z(\tau_i)Z - Z(\tau_i)Z = 0.$$

Further,

$$\begin{aligned} [\tilde{\zeta}_i, \tilde{\zeta}_j] &= [\zeta_i - \tau_i Z, \zeta_j - \tau_j Z] \\ &= [\zeta_i, \zeta_j] - [\zeta_i, \tau_j Z] - [\tau_i Z, \zeta_j] + [\tau_i Z, \tau_j Z] \\ &= 0 - (\zeta_i(\tau_j)Z - \tau_j Z(\tau_i)Z) - (\tau_i Z(\tau_j)Z - \zeta_j(\tau_i)Z) \\ &\quad + (\tau_i Z(\tau_j) - \tau_j Z(\tau_i)Z) \\ &= (\zeta_j(\tau_i) - \zeta_i(\tau_j))Z. \end{aligned}$$

On the other hand, since $[\zeta_i, \zeta_j] = 0$, we have

$$[\zeta_i, \zeta_j](dt) = \zeta_i(\tau_j) - \zeta_j(\tau_i) = 0,$$

which proves (5.2).

Now, consider the invariant variety (5.1). At a generic point (t, q, p) , the differentials of the integrals J_i are independent. Indeed, in our case (3.7) reads

$$(5.3) \quad dJ_i = -i_{\zeta_i}d(pdq - Hdt), \quad i = 1, \dots, m.$$

Therefore, if there exist real parameters a_1, \dots, a_m , $a_1^2 + \dots + a_m^2 \neq 0$, such that

$$a_1dJ_1 + \dots + a_mdJ_m = 0,$$

then

$$i_{a_1\zeta_1 + \dots + a_m\zeta_m} \in \ker d(pdq - Hdt),$$

which implies that $Z, \zeta_1, \dots, \zeta_m$ are dependent at (t, q, p) . Thus, the integrals J_1, \dots, J_m are independent and the regular invariant levels sets (5.1) are $(r + 1)$ -dimensional submanifolds.

Since J_i are the integrals of the equations (2.2), the vector field Z is tangent to M_c . Further, we have

$$(5.4) \quad \begin{aligned} 0 &= i_{[\zeta_i, \zeta_j]}(pdq - Hdt) \\ &= L_{\zeta_i} \circ i_{\zeta_j}(pdq - Hdt) - i_{\zeta_j} \circ L_{\zeta_i}(pdq - Hdt) \\ &= \zeta_i(J_j), \quad i = 1, \dots, r, \quad j = 1, \dots, m. \end{aligned}$$

Thus, the commuting vector fields $\tilde{\zeta}_1 = \zeta_1 - \tau_1 Z, \dots, \tilde{\zeta}_r = \zeta_r - \tau_r Z$ are also tangent to M_c and, by the Lie theorem [22], the trajectories of (2.2) can be found locally by quadratures.

(ii) The proof of item (ii) is the same as the corresponding statement in the Arnold–Liouville theorem (see [2]). \square

If the Hamiltonian and Noether symmetries are periodic with respect to the time translation $(t, q, p) \mapsto (t + 1, q, p)$, we can consider $S^1 \times T^*Q(t, q, p)$, $S^1 = \mathbb{R}/\mathbb{Z}$, as an extended phase space.

With the above notation we have

COROLLARY 5.1. *The regular compact connected components of M_c/\mathbb{Z} are $(r + 1)$ -dimensional tori with quasi-periodic dynamics*

$$\dot{\varphi}_i = \omega_i = \text{const}, \quad i = 1, \dots, r + 1.$$

REMARK 5.1. Assume that the vector fields ζ_i are weak Noether symmetries

$$(5.5) \quad L_{\zeta_i}(pdq - Hdt + \beta) = df_i, \quad i = 1, \dots, m.$$

Then (5.2) still holds, the vector field Z is tangent to M_c , and (3.7) implies (5.3), where the Noether integrals are $J_i = i_{\zeta_i}(pdq - Hdt + \beta) - f_i$, $i = 1, \dots, m$.

Now,

$$\begin{aligned} 0 &= i_{[\zeta_i, \zeta_j]}(pdq - Hdt + \beta) \\ &= L_{\zeta_i} \circ i_{\zeta_j}(pdq - Hdt + \beta) - i_{\zeta_j} \circ L_{\zeta_i}(pdq - Hdt + \beta) \\ &= \zeta_i(J_j) + \zeta_i(f_j) - \zeta_j(f_i) \end{aligned}$$

and

$$0 = i_{[\zeta_i, \zeta_j]}d(pdq - Hdt)$$

$$\begin{aligned}
&= L_{\zeta_i} \circ i_{\zeta_j} d(pdq - Hdt) - i_{\zeta_j} \circ L_{\zeta_i} d(pdq - Hdt) \\
&= L_{\zeta_i}(dJ_j) - i_{\zeta_j}(i_{\zeta_i} d^2(pdq - Hdt) - d^2 J_i) \\
&= d(\zeta_i(J_j)), \quad i = 1, \dots, r, \quad j = 1, \dots, m.
\end{aligned}$$

Thus, we obtain $\zeta_i(J_j) = c_{ij} = \text{const}$. However, in order to have $c_{ij} = 0$, the additional assumptions $\zeta_i(f_j) = \zeta_j(f_i)$, $i = 1, \dots, r$, $j = 1, \dots, m$ should be added in Theorem 3.2.

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**НЕТЕРИНЕ СИМЕТРИЈЕ И ИНТЕГРАБИЛНОСТ У
ВРЕМЕНСКИ ЗАВИСНОЈ ХАМИЛТОНОВОЈ МЕХАНИЦИ**

РЕЗИМЕ. Разматрамо Нетерине симетрије, у оквиру Хамилтонове формулације механике, као трансформације које чувају Поенкаре–Картанову 1-форму, односно као симетрије карактеристичних линијских раслојења недегенеративних 1-форми. У случају када је Поенкаре–Картанова форма контактна, дају се експлицитни изрази за симетрије у инверзној Нетериној теорему. Као примери, посматрају се природни механички системи, посебно Кеплеров проблем. Коначно, доказује се варијанта теореме о комплетној (некомутативној) интегралности помоћу Нетериних симетрија временски-зависног Хамилтоновог система.

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