# ON THE ENTIRE AND FINITE VALUED SOLUTIONS OF THE THREE-BODY PROBLEM

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ABSTRACT. A full list of entire and finite valued solutions of the three body problem in the form of functions, depending on a time variable, is established. All the entire and finite valued solutions are among the solutions of Euler and Lagrange.

## 1. Introduction

This article was inspired by the recent result of Šuvakov and Dmitrašinović [1] and its main goal is to suggest a new method of analytic investigation of the three body problem. We hope, that the suggested method will turn out to be useful for further studying of this problem, supplementing the existing ones (see, for example, [2, 3] and the bibliography in [4–6]).

Broadly speaking, properties of solutions of differential equations with an analytic right hand side to a great extent depend on singular points, lying on the complex plane of time. Thus, for instance, the classical solution of S. Kovalevskaya [7] of the problem of a heavy rigid body's movement was found on the way to finding single valued solutions. A systematic investigation of singular points of solutions together with a compactification of the flow, defined by the Euler–Poisson equations [8], allow not only to find partial entire and finite valued solutions with some specified properties of their singular points, but also to study global properties of infinite valued solutions [9]. In this case applying the same approach to another problem of classical mechanics, particularly–the three body problem, seems to be quite natural.

The main point of the suggested method lies in getting a compact holomorphic manifold with a structure of a one-dimensional $\mathbb{C}$-foliation $\mathcal{F}$ with singular points as a result of a factorization of the flow on the phase space. Such a reduction of the problem causes the loss of the initial parametrization of solutions of the original differential equations. However this loss is not essential, as in case of getting an

effective representation for fibres of the foliation $\mathcal{F}$ the original parametrization can be found by means of a single integration.

At the same time the singularities of the foliation $\mathcal{F}$ correspond to the singularities of complex solutions of the original problem and can be effectively investigated. This fact allows us to get a full classification of asymptotic forms of singular points of these solutions. In turn, that fact gives us an opportunity to define a class of functions, containing the solutions, and select some subclasses from there, that are simple enough to find exact solutions of the problem. In fact, this means to act similarly to what S. Kovalevskaya did.

Moreover, a compactification of the problem (to put it differently, an enclosure of all the trajectories of solutions in a compact manifold) is an extremely strong restriction on a behaviour of the solutions at infinity. In the classical problem of a heavy rigid body's movement this compactification allows [8] to get a full list of entire solutions, though, broadly speaking, this problem is not integrable at all. For the three body problem a similar consideration also allows us to get a full list of entire solutions, at that all these solutions are expressed by circular functions as well as for the problem of a heavy rigid body's movement.

Furthermore, for the three body problem it proved to be possible to get a full list of finite valued solutions, that still remains to be a problem, not solved completely, for the problem of a heavy rigid body's movement. All the entire and finite valued solutions of the three body problem are confined in the list of already known solutions of Euler [10] and Lagrange [11], which means that all the new solutions, including the solutions, found later by Moore, Broucke, Henon, Hadjidemetriou, Šuvakov, Dmitrašinović (references see in [1]) are infinite valued. Besides, this explains the complexity of finding them using analytic methods only.

As it would be natural to suppose, some results, got in the context of the considered approach, are already known in a slightly different form. Thus, for example, a characteristic system for the $n$-body problem is almost equivalent to the system, describing central configurations of $n$ bodies [10,11] and the only difference is that the characteristic system does not sustain homogeneous stretches. Moreover, an asymptotic form of singular points of solutions is almost an asymptotic form of the bodies collision [12].

Note also, that the considered approach turns out to be similar to the ones, used in the investigation of non-integrability problems in the theory of Hamiltonian systems [13–15].

Nevertheless we suppose, that a clear understanding of an analytic nature of the exact solutions of the three body problem gives us a hope to get some new interesting results, concerning investigation of this famous problem in future.

## 2. Preliminaries

We consider the problem (see, for example, [2]) of $n$ bodies $(m_1, r_1), \ldots (m_n, r_n)$, $m_i \in \mathbb{R}^+, r_i \in \mathbb{R}^3$, which move according to the law of gravity. Kinetic and potential energies are correspondingly equal to

$$T = \frac{1}{2} \sum m_i \dot{r}_i^2, \quad U = -\frac{G}{2} \sum_{j,k} \frac{m_j m_k}{|r_j - r_k|}$$

Lagrangian $L = T - U$ determines the following system of differential equations:

(2.1) $$m\ddot{r}_i = -Gm_i \sum_j m_j \frac{r_i - r_j}{|r_i - r_j|^3}.$$

In the Hamiltonian form

$$\mathcal{H} = \sum_i \dot{r}_i \frac{\partial L}{\partial \dot{r}_i} - L = \frac{1}{2}\sum m_i \dot{r}_i^2 - \frac{G}{2}\sum_{j,k} \frac{m_j m_k}{|r_j - r_k|},$$

canonical coordinates are

$$q_i = r_i, \quad p_i = \frac{\partial L}{\partial \dot{r}_i} = m_i \dot{r}_i,$$

and a Hamiltonian system has the next form:

(2.2) $$\begin{cases} \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} = \frac{p_i}{m_i} \\ \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} = -Gm_i \sum_j m_j \frac{q_i - q_j}{|q_i - q_j|^3}. \end{cases}$$

The first integrals of the system (2.2) are

$$\mathcal{H}, \quad \mathcal{I} = \sum m_i \dot{q}_i, \quad \mathcal{M} = \sum q_i \times p_i.$$

In the classic notation (2.1), (2.2) of the $n$-body problem in the right-hand side there is the module which is a local real-analytic function. As we want to consider the $n$-body problem for a complex time, the right-hand side of the differential equations has to be complex-analytical. Therefore we consider a module function for the vector $q \in \mathbb{C}^3$ as a complex-analytical function which is determined by the formula:

$$|q| = \sqrt{q_1^2 + q_2^2 + q_3^2}.$$

We use the next notation for a norm of the vector:

$$\|q\| = \sqrt{q_1 \bar{q}_1 + q_2 \bar{q}_2 + q_3 \bar{q}_3}.$$

Two main theorems will be proved in this article

THEOREM 2.1. *All the entire solutions of the differential equations*

(2.3) $$\begin{cases} \dot{x}_1 = mz_1 - m_1 \sum_\sigma z_1, \sigma \\ \dot{z}_1 = -\frac{x_1}{|x_1|^3}, \sigma \end{cases}$$

*satisfy the following condition:* $|x_1(t)| = |x_2(t)| = |x_3(t)| = const.$

The theorem 2.1 is proved in the §7.

REMARK 2.1. The system of differential equations (2.3) describes a motion of the three bodies in a relative coordinate system and is equivalent to the original system (2.2) (see (4.1)–(4.4)) from the viewpoint of integrability. In other words, if we have a solution of the system (2.3) we can also find a solution of the classical three body problem (2.2). At the same time any property of one of these systems is not always true for the other system[1]. For our convenience we shall also call

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[1] see remark 4.1 and §7

the problem (2.3) "the three body problem", however in all our statements we shall make more precise notes about what system is particularly meant.

THEOREM 2.2. *All the entire and finite-valued solutions of* (2.3) *lead to the partial cases of the Euler's* [10] *and Lagrange's* [11] *solutions of the classical three body problem* (2.2).

The theorem 2.2 is proved in the §7.

## 3. The factorization of the flow of the $n$-body problem

Let $p_i(t), q_i(t)$, $i = 1, 2, 3$ be a solution of the $n$-body problem. Then, $\alpha p_i(\alpha^3 t)$, $\alpha^{-2} q_i(\alpha^3 t)$ is a solution too. This fact gives us a possibility for a factorization on the set of trajectories of the system (2.2).

REMARK 3.1. The flow of the problem (2.2) allows the factorization by the action of the orthogonal group $\mathrm{So}(3, \mathbb{C})$ [16]

$$p_i \to A p_i, \quad q_i \to A q_i, \quad A \in \mathrm{So}(3, \mathbb{C}),$$

but now we do not use this fact in our research.

PROPOSITION 3.1. *Let $\mathbb{C}$ act as a transformation group on $\mathbb{C}^n$ in the following way:*

$$\alpha \colon (z_1, \ldots, z_n) \to (\alpha^{k_1} z_1, \ldots, \alpha^{k_n} z_n),$$

$k = (k_1, \ldots, k_n) \in \mathbb{N}^n$. *Then the quotient-space* $\mathbf{P}_k^{n-1} = \{(z_1^{k_1} : \cdots : z_n^{k_n})\}$ *is a compact holomorphic manifold* [17,18] *with singularities with respect to this action.*

PROOF. Let $\pi_0, \pi$ be canonical projections of

$$\pi_0 \colon \mathbb{C}^n \to P^{n-1}, \pi \colon \mathbb{C}^n \to P_k^{n-1}.$$

Consider the mapping of the complex projective space $P^{n-1}$, which is

$$f \colon P^{n-1} \to P_k^{n-1},$$

induced by a function $f_0$

$$f^0 \colon (z_1, \ldots, z_n) \to (z_1^{k_1}, \ldots, z_n^{k_n}), f = \pi \circ f^0 \circ \pi_0^{-1}.$$

It is clear, that this mapping is defined correctly and it has the singular points $z_i = 0$. The mapping $f$ is open therefore any atlas of the manifold $P^{n-1}$ induces an atlas of the space $P_k^{n-1}$. At the points $z_i = 0$ the manifold $P_k^{n-1}$ is holomorphic, but at the points $z_i = z_j = 0$ singularities, similar to an apex of the cone (see [18]), are possible. A compactness of the manifold $P_k^{n-1}$ is a result of the compactness of $P^{n-1}$. □

PROPOSITION 3.2. *The projection*

$$\pi \colon \mathbb{C}^{3n} + \mathbb{C}^{3n} \to \mathbf{P}_*^{6n-1},$$

*where* $* = (\overbrace{1, .., 1}^{n}, \overbrace{2, .., 2}^{n})$, *is determined by the next formula:*

$$\pi \colon (p_1, \ldots, p_n, q_1, \ldots, q_n) \to (p_1 : \cdots : p_n : \frac{q_1}{|q_1|^2} : \cdots : \frac{q_n}{|q_n|^2}),$$

*and induces the structure of the holomorphic $\mathbb{C}$-one-dimensional foliation* [19] *with the singularities $\mathcal{F}$ of the compact holomorphic manifolds $\mathbf{P}_*^{6n-1}$, having singularities by-turn.*

PROOF. According to the definition of $\pi$ the vector

$$(\alpha p_1, \ldots, \alpha p_n, \alpha^{-2} q_1, \ldots, \alpha^{-2} q_n)$$

is projected onto

$$(\alpha p_1 : \cdots : \alpha p_n : \alpha^2 \frac{q_1}{|q_1|^2} : \cdots : \alpha^2 \frac{q_n}{|q_n|^2}). \qquad \square$$

REMARK 3.2. The projection $\pi$ can be defined more naturally:

$$\pi \colon (p_1, \ldots, p_n, q_1, \ldots, q_n) \to (p_1 : \cdots : p_n : q_1 : \cdots : q_n),$$

but in this case an image of this mapping is a noncompact manifold.

REMARK 3.3. We can use the mapping $\pi^{-1}$ which has the following presentation:

$$\pi^{-1} \colon (p_1 : \cdots : p_n : q_1, \ldots, q_n) \to (p_1 : \cdots : p_n : \frac{q_1}{|q_1|^2} : \cdots : \frac{q_n}{|q_n|^2}),$$

if it is necessary.

REMARK 3.4. The foliation $\mathcal{F}$ is integrable as there is the invariant mapping $J \colon \mathbf{P}_*^{6n-1} \smallsetminus \mathbf{X} \to \mathbf{P}^6$,

$$J \colon z \to (\mathcal{H}(w) : \mathcal{I}_1^2(w) : \mathcal{I}_2^2(w) : \mathcal{I}_3^2(w) : \mathcal{M}_1^2(w) : \mathcal{M}_2^2(w) : \mathcal{M}_3^2(w)),$$

where $w = \pi^{-1}(z)$,

$$\mathbf{X} = \{ z \in \mathbf{P}_*^{6n-1} : \mathcal{H}(\pi^{-1}(z)) = 0, \mathcal{I}(\pi^{-1}(z)) = 0, \mathcal{M}(\pi^{-1}(z)) = 0 \}.$$

Moreover a surface $\mathbf{X}$ is a fibre invariant for the foliation $\mathcal{F}$ too.

PROPOSITION 3.3. *All the singular points of the foliation $\mathcal{F}$ are the following: $\pi$-projections of the solutions $(\tilde{p}^0, \tilde{q}^0)$ of the characteristic system (3.1), $\pi$-projections of the singular points of the system (2.2), i.e. the points*

$$\{ \pi(p_1, \ldots, p_n, q_1, \ldots, q_n) : \exists i, j \quad |q_i - q_j| = 0 \}$$

*and*

$$\{ (p_1 : \cdots : p_n : q_1 : \cdots : q_n) \in \mathbf{P}_*^{6n-1} : \exists i \quad |q_i| = 0 \}.$$

PROOF. Evidently singular points of the equation (2.2)

$$\{ \pi(p_1, \ldots, p_n, q_1, \ldots, q_n) : \exists i, j \quad |q_i - q_j| = 0 \}$$

are projected onto singular points of the foliation $\mathcal{F}$.

Moreover if the vector $(\dot{p}, \dot{q})$ touches the $\pi$-pre-image of $\pi(p, q)$ the point $\pi(p, q)$ will be a singular point of the foliation too. Such points satisfy the system

(3.1)
$$\begin{cases} -2\tilde{q}_i + 3\frac{\tilde{p}_i}{m_i} = 0, \\ \tilde{p}_i - 3Gm_i \sum_j m_j \frac{\tilde{q}_i - \tilde{q}_j}{|\tilde{q}_i - \tilde{q}_j|^3} = 0. \end{cases}$$

We call this system characteristic. At last the points of the manifold $\mathbf{P}^{6n-1}_*$, which have not got a pre-image, i.e. the points $(p_1 : \cdots : p_n : q_1 : \cdots : q_n) : |q_i| = 0$, may be singular points. □

REMARK 3.5. A solution of the characteristic system (3.1) determines a central configuration [6] specifying some partial solutions of the three-body problem that were known to Euler and Lagrange.

Further we make some calculations for the 3-body problem.

## 4. The change of variables for the 3-body problem

In this case the problem has dimension 18. Using the first integrals, this dimension can be lowered to 8 [20]. However the differential equations derived in such a way are rather inconvenient for any further investigations, and besides, the dimension of the reduced problem remains rather high. Therefore we change the variables without lowering the problem's dimension trying to make the equations, defining this problem, as simple as possible. At first we make a change of variables, switching to relative coordinates:

$$(4.1) \qquad \begin{cases} x_1 = (q_2 - q_3) \cdot G^{-1/3}, \sigma \\ y_1 = \left(\frac{p_2}{m_2} - \frac{p_3}{m_3}\right) \cdot G^{-1/3}, \sigma \end{cases}$$

here and lower $\sigma$ denotes a circle permutation of indexes $(1, 2, 3)$. The system (4.1) takes the next form

$$(4.2) \qquad \begin{cases} \dot{x}_1 = y_1, \sigma \\ \dot{y}_1 = \left(m_1\left(\frac{x_2}{|x_2|^3} + \frac{x_3}{|x_3|^3}\right) - (m_2 + m_3)\frac{x_1}{|x_1|^3}\right), \sigma \end{cases}$$

If the system (4.2) is solved it will be simple to find $p_i, q_i$. In this case we express the variables $p_i$ through $y_i$ and the value of the first integral $\mathcal{I} = p_1 + p_2 + p_3$. Then we find $q_1 = \int p_1 dt$, $\sigma$, and may consider the initial values of $q_i$ to be known.

Now suppose that

$$\dot{u} = \sum_\sigma \frac{x_1}{|x_1|^3}, \quad m = \sum_\sigma m_1, \quad mz_1 = y_1 - m_1 u, \sigma,$$

then

$$(4.3) \qquad \begin{cases} \dot{z}_1 = -\frac{x_1}{|x_1|^3}, \sigma \\ \dot{x}_1 = mz_1 + m_1 u, \sigma \\ \dot{u} = \sum_\sigma \frac{x_1}{|x_1|^3}. \end{cases}$$

Since

$$u + \sum_\sigma z_1 = \frac{1}{m}\sum_\sigma y_1 = \frac{1}{m}\sum_\sigma \left(\frac{p_2}{m_2} - \frac{p_3}{m_3}\right)G^{-1/3} \equiv 0,$$

then $u = -\sum_\sigma z_1$ and finally we get the following system:

$$(4.4) \qquad \begin{cases} \dot{x}_1 = mz_1 - m_1 \sum_\sigma z_1, \sigma \\ \dot{z}_1 = -\frac{x_1}{|x_1|^3}, \sigma \end{cases}$$

REMARK 4.1. The system (4.4) has an essentially simpler presentation than the classical system (4.2), that gives us evident technical advantages in our further calculations. At the same time, as it was mentioned in the remark 2.1, these systems are connected not only by a simple change of variables, but also by the operation of integration. Nevertheless, the main result of this paper concerning a description of entire and finite valued solutions, proved for (4.4) is also true for the classical system (4.2).

THEOREM 4.1. *The system of differential equations* (4.4) *is equivalent*[2] *to the three-body problem* (2.2) *and is a canonical Hamiltonian system with coordinates* $(x_i/m_i, z_i)$. *Its Hamiltonian has the form:*

$$\mathcal{H} = \sum_\sigma \frac{mz_1^2}{2m_1} - \frac{1}{2}\Big(\sum_\sigma z_1\Big)^2 - \sum_\sigma \frac{1}{m_1|x_1|}.$$

*The system of differential equations* (4.4) *has the following first integrals:*

$$\mathcal{I} = \sum_\sigma x_1, \quad \mathcal{M} = \sum_\sigma \frac{1}{m_1} z_1 \times x_1.$$

PROOF.

$$\mathcal{I}_1 = \sum_\sigma (q_1 - q_2)G^{-1/3} \equiv 0.$$

$$\dot{\mathcal{I}}_2 = \sum_\sigma \frac{1}{m_1}(\dot{z}_1 \times x_1 + z_1 \times \dot{x}_1)$$

$$= \sum_\sigma \Big(\frac{1}{m_1}\Big(x_1 \times \frac{x_1}{|x_1|^3} + z_1 \times \Big(mz_1 - m_1 \sum_\sigma z_1\Big)\Big)\Big)$$

$$= \sum_\sigma \Big(z_1 \times \sum_\sigma z_1\Big) = \sum_\sigma z_1 \times \sum_\sigma z_1 = 0. \qquad \square$$

The fact that the system (4.4) is canonical can be proved directly.

REMARK 4.2. The integral $\mathcal{I}$ of a kinetic momentum of the system (2.2) is absent for the system (4.4). Values of the integrals of an angular momentum $\mathcal{M}$ of these systems are equal modulo a constant.

Now we make one more change of variables [8], allowing to investigate an asymptotics of singular points of solutions of the three-body problem.

Let $t_* \in \mathbb{C}$ be a singular point of the solution $(x_i(t), z_i(t))$ (i. e. $t_*$ is a singular point of one of the coordinate functions of $(x_i(t), z_i(t))$ Get rid of branching in $t_*$, if any, by representing $x_i(t) = \hat{x}_i(\operatorname{Ln}(t - t_*)^\alpha)$, $z_i(t) = \hat{z}_i(\operatorname{Ln}(t - t_*)^\alpha)$, where $\hat{x}_i(\tau), \hat{z}_i(\tau)$ are single-valued functions for $\operatorname{Re}\tau \to -\infty$.

REMARK 4.3. The adduced reasoning allows us to pass to a new system (4.5) of differential equations, using which we can indicate a simple way of finding an asymptotic behaviour of solutions in the neighbourhood of singular points of the initial system (4.4). At the same time we get not only the asymptotics of finite

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[2]see remark 2.1

valued solutions at singular points, but also the asymptotics of infinite valued solutions at their singular points. We can not state, that the suggested way allows us to find all the singular points of solutions of the considered problem.

The system (4.4) is transformed into

$$\begin{cases} \dot{\hat{x}}_1 = \frac{1}{\alpha}e^{\tau/\alpha}\big(mz_1 - m_1\sum_\sigma z_1\big), \sigma \\ \dot{\hat{z}}_1 = -\frac{1}{\alpha}e^{\tau/\alpha}\frac{\hat{x}_1}{|x_1|^3}, \sigma; \end{cases}$$

where a derivative is taken with respect to $\tau$.

In order to make the right-hand side of the equations independent of $\tau$, we make a replacement of the variable in the following form: $\tilde{x}_i(\tau) = e^{\beta\tau}\,\hat{x}_i(\tau)$, $\tilde{z}_i(\tau) = e^{\gamma\tau}\hat{z}_i(\tau)$. Then we have

$$\begin{cases} \dot{\tilde{x}}_1 = \beta\tilde{x}_1 + \frac{1}{\alpha}e^{\tau\beta+\tau/\alpha}\big(m\hat{z}_1 - m_1\sum_\sigma \hat{z}_1\big), \sigma \\ \dot{\tilde{z}}_1 = \gamma\tilde{z}_1 - \frac{1}{\alpha}e^{\tau\gamma+\tau/\alpha}\frac{\hat{x}_1}{|x_1|^3}, \sigma. \end{cases}$$

We see that the right-hand side is independent of $\tau$, if $\beta+\frac{1}{\alpha} = \gamma$, $\gamma+\frac{1}{\alpha} = -2\beta$, i.e. if $\gamma = \frac{1}{3\alpha}, \beta = -\frac{2}{3\alpha}$.

Taking $\alpha = \frac{1}{3}$, we obtain the following system:

(4.5) $$\begin{cases} \dot{\tilde{x}}_1 = -2\tilde{x}_1 + 3\big(m\tilde{z}_1 - m_1\sum_\sigma \tilde{z}_1\big), \sigma \\ \dot{\tilde{z}}_1 = \tilde{z}_1 - 3\frac{\tilde{x}_1}{|x_1|^3}, \sigma. \end{cases}$$

A dependence between the differential systems (4.4) and (4.5) is expressed by the next relations:

(4.6) $$z_i(t) = (t-t_*)^{-\frac{1}{3}}\tilde{z}_i(\text{Ln}(t-t_*)^{\frac{1}{3}}), \quad x_i(t) = (t-t_*)^{\frac{2}{3}}\tilde{x}_i(\text{Ln}(t-t_*)^{\frac{1}{3}})$$

PROPOSITION 4.1. *A solution of the system* (4.4) *does not have a singularity at the point $t_*$ if and only if the corresponding solutions of* (4.5) *by* (4.6) *have an asymptotic behaviour $\tilde{x}_i \sim \tilde{x}_{i0}e^{-2\tau}, \tilde{z}_i \sim \tilde{z}_{i0}e^\tau$ at $\text{Re}\,\tau \to -\infty$.*

PROOF. It is enough to substitute the asymptotics $\tilde{x}_i \sim \tilde{x}_{i0}e^{-2\tau}$, $\tilde{z}_i \sim \tilde{z}_{i0}e^\tau$ to (4.6). □

REMARK 4.4. Having found an asymptotic behaviour of the solutions $\tilde{x}_i, \tilde{z}_i$ in the neighbourhood of the singular points, we can obtain an asymptotic behaviour of the singular points of the solutions (4.4) having the representation (4.6).

The projection $\pi$ from the proposition 3.1 for the system (4.4) has the following form:

$$\pi\colon (z_1, z_2, z_3, x_1, x_2, x_3) \to \Big(z_1 : z_2 : z_3 : \frac{x_1}{|x_1|^2} : \frac{x_2}{|x_2|^2} : \frac{x_3}{|x_3|^2}\Big)$$
$$= (z_1 : z_2 : z_3 : w_1 : w_2 : w_3).$$

A correspondent system of differential equations for the fibres of $\mathcal{F}$ on the manifold $\mathbf{P}_*^{17}$ obtains the following form:

(4.7) $$\begin{cases} \dot{w}_1 = -2w_1\big(w_1, mz_1 - m_1\sum_\sigma z_1\big) + |w_1|^2\big(mz_1 - m_1\sum_\sigma z_1\big), \sigma \\ \dot{z}_1 = -w_1|w_1|, \sigma \end{cases}$$

A convenience of the system (4.7) consists in the fact that it is defined on a compact manifold and its right-hand side is always determined. At the same time the system (4.4) is more convenient for calculations, as essentially more simple. Denote also, that both the system (4.4) and the system (4.7) define a right hand side of $\dot{z}_i$ up to a sign, but for the foliation $\mathbf{P}^{17}_*$ the solutions $\alpha p_i(\alpha^3 t)$, $\alpha^{-2} q_i(\alpha^3 t)$ are indistinguishable: assuming $\alpha = -1$, we obtain that, the solutions $(w_i(t), z_i(t))$ and $(w_i(t), -z_i(t))$ are projected onto the same fibre under the projection $\pi$.

PROPOSITION 4.2. *The singular points of the foliation $\mathcal{F}$ on the compact holomorphic manifold $\mathbf{P}^{17}_*$ are the projections $\pi(\tilde{x}^0_i, \tilde{z}^0_i)$ of the solutions of a characteristic system*

$$(4.8) \qquad \begin{cases} -2\tilde{x}_1 + 3\big(m\tilde{z}_1 - m_1 \sum_\sigma \tilde{z}_1\big) = 0, \sigma \\ \tilde{z}_1 - 3\frac{\tilde{x}_1}{|x_1|^3} = 0, \sigma. \end{cases}$$

*In addition to this, the points*

$$\{(z_1 : z_2 : z_3 : w_1 : w_2 : w_3) \in \mathbf{P}^{17}_* : \exists i \quad |w_i| = 0\}$$

*form a singular invariant surface in $\mathbf{P}^{17}_*$, which is traversed by trajectories, forming a set of trajectories of a general form.*

PROOF. This proposition repeats the proposition 3.3 for the three-body problem (2.2).

As for the singular points of the form $\{(z_1 : z_2 : z_3 : w_1 : w_2 : w_3) \in \mathbf{P}^{17}_* : \exists i |w_i| = 0\}$, which pretend to be singular points thanks to a singularity of the functions $|w_i|$, these points form an invariant surface in $\mathbf{P}^{17}_*$. We can make sure that it is true by finding the derivative of the function $|w_i|^2$ along the vector field (4.7). This derivative is identically equal to zero on the surface $|w_i|^2 = 0$. $\qquad \square$

The roots of the characteristic system (4.8) were already known to Euler and Lagrange. We present their finding for completing the paper taking into account that this finding is simple. Substitute $\tilde{z}_1$ into the first equation of (4.8). Omitting the sign $\sim$ for simplicity, we get

$$\frac{x_1}{m_1} = \frac{9}{2}\Big(\frac{m}{m_1}\frac{x_1}{|x_1|^3} - \sum_\sigma \frac{x_1}{|x_1|^3}\Big), \sigma.$$

Then we subtract these equations one from another and get:

$$\frac{x_1}{m_1} - \frac{x_2}{m_2} = \frac{9m}{2}\Big(\frac{x_1}{|x_1|^3 m_1} - \frac{x_2}{|x_2|^3 m_2}\Big), \sigma.$$

or

$$(4.9) \qquad \frac{x_1}{m_1}\Big(1 - \frac{9m}{2|x_1|^3}\Big) = \frac{x_2}{m_2}\Big(1 - \frac{9m}{2|x_2|^3}\Big) = \frac{x_3}{m_3}\Big(1 - \frac{9m}{2|x_3|^3}\Big).$$

Let all the vectors $x_i$ be collinear. Let also $x_1, x_2 > 0, x_3 < 0$. Denote $x_2 = \rho x_1, x_3 = -(1+\rho)x_1$ and substitute $x_2, x_3$ into (4.9). Then we get following quintic polynomial:

$$(m_2 + m_3)\rho^5 + (3m_2 + 2m_3)\rho^4 + (3m_2 + m_3)\rho^3$$

$$- (3m_1 + m_3)\rho^2 - (3m_1 + 2m_3)\rho - (m_1 + m_3) = 0.$$

Similarly it is possibly to consider all the rest cases when vectors $x_i$ are collinear.
Now let vectors $x_i$ be non-collinear then

$$|x_1|^3 = |x_2|^3 = |x_3|^3 = \frac{2}{9m};$$

and besides $x_1 + x_2 + x_3 = 0$.

The singular points of the foliation $\mathcal{F}$ are interesting on the way to finding asymptotic behaviours of singular points of the solutions of the three body problem, first of all.

DEFINITION 4.1. The singular points $\pi(z)$ of the foliation $\mathcal{F}$, where $z$ is a root of the characteristic system (4.8) and the vectors $x_i$ are non-collinear, will be called $\alpha$-singular points. The singular points $\pi(z)$ of the foliation $\mathcal{F}$, where $z$ is a root of the characteristic system and the vectors $x_i$ are collinear, will be called $\beta$-singular points.

Let $(\tilde{x}_i^0, \tilde{z}_i^0), i = 1, 2, 3$ be a solution of the characteristic system (4.8). At the same time it is a singular point of the system of differential equations (4.5). The linearised system (4.5) has the following form in the neighbourhood of such a point:

$$(4.10) \qquad \begin{cases} \dot{\tilde{x}}_1 = -2\tilde{x}_1 + 3\big(m\tilde{z}_1 - m_1 \sum_\sigma \tilde{z}_1\big), \sigma \\ \dot{\tilde{z}}_1 = \tilde{z}_1 - 3\frac{\tilde{x}_1}{|\tilde{x}_1^0|^3} + 9\frac{\tilde{x}_1^0}{|\tilde{x}_1^0|^5}(\tilde{x}_1^0, \tilde{x}_1), \sigma. \end{cases}$$

## 5. An asymptotic behaviour of the singular points of the foliation $\mathcal{F}$

First of all consider an asymptotic behaviour of the $\alpha$-singular points of the foliation $\mathcal{F}$.

Suppose that $|\tilde{x}_1^0|^3 = |\tilde{x}_2^0|^3 = |\tilde{x}_1^0|^3 = \frac{9m}{2} = a^3$, then a linear system for finding eigenvectors of the right-hand side of (4.10) has the following form:

$$(5.1) \qquad \begin{cases} (m_2 + m_3)z_1 - m_1(z_2 + z_3) = \frac{\lambda+2}{3}x_1\sigma \\ -\frac{3}{a^3}x_1 + \frac{9}{a^5}(\tilde{x}_1^0, x_1)\tilde{x}_1^0 = (\lambda - 1)z_1, \sigma. \end{cases}$$

THEOREM 5.1. *Eigenvalues and corresponding eigenvectors of the linear system* (4.10) *are the following:* $\lambda = 1$.

*The eigenvector* $u_1$ *has the form:*

$$x_1 = 0, \quad z_1 = m_1 r, \sigma, \quad r \in \mathbb{C}^3.$$

*A dimension of the eigenspace is equal to 3:* $\lambda(\lambda + 1) = 0$.
*The eigenvectors* $u_0, u_{-1}$ *satisfy the conditions:*

$$x_1 \perp \tilde{x}_1^0, \quad z_1 = \frac{3x_1}{a^3(1 - \lambda)}, \sigma.$$

*A dimension of the eigenspaces for* $\lambda = 0$ *and* $\lambda = -1$ *is equal to 3:* $(\lambda+3)(\lambda-2) = 0$.
*The eigenvectors* $u_{-3}, u_2$ *have the form:* $x_1 = \nu\tilde{x}_1^0, z_1 = \mu\tilde{x}_1^0, \sigma$,
*A dimension of the eigenspace for* $\lambda = -3$ *and* $\lambda = 2$ *is equal to 1.*

*The rest of the eigenvalues $\lambda_k, k = 1, \ldots, 4$ are the roots of the equation*

$$\lambda(\lambda + 1)(\lambda + 3)(\lambda - 2) + \frac{27}{m^2} \sum_\sigma m_1 m_2 = 0.$$

*A dimension of the eigenspaces for these $\lambda$ is equal to $1$.*
*It is possible to select a proper basis from all the eigenvectors, mentioned above.*

PROOF. $\lambda = 1$. We get the next presentation from the second equation of the system (5.1):

$$x_1 = \frac{3}{a^2}(\tilde{x}_1^0, x_1)\tilde{x}_1^0 = \mu\tilde{x}_1^0 = \frac{3\mu}{a^2}(\tilde{x}_1^0, \tilde{x}_1^0) = \tilde{x}_1^0 = \mu\tilde{x}_1^0 = 3\mu\tilde{x}_1^0, \sigma.$$

We see that $\mu = 0$ hence $x_1 = 0, \sigma$. The values of $z_i$ can be found from the first equation of the system (5.1).

$\lambda \neq 1$. Substitute the presentation of $z_i$ from the second equation of the system (5.1) to the first equation, taking into account that $\sum_\sigma \tilde{x}_1^0 = \sum_\sigma x_1 = 0$.

As a result we obtain the following system:

(5.2)
$$\begin{cases} m(\tilde{x}_1^0, x_1)\tilde{x}_1^0 + m_1((\tilde{x}_2^0, x_1 + x_2) + (\tilde{x}_1^0, x_2))\tilde{x}_1^0 \\ +m_1((\tilde{x}_1^0, x_1 + x_2) + (\tilde{x}_2^0, x_1))\tilde{x}_2^0 = \frac{1}{27}a^5\lambda(\lambda + 1)x_1 \\ m(\tilde{x}_2^0, x_2)\tilde{x}_2^0 + m_2((\tilde{x}_1^0, x_1 + x_2) + (\tilde{x}_2^0, x_1))\tilde{x}_2^0 \\ +m_2((\tilde{x}_2^0, x_1 + x_2) + (\tilde{x}_1^0, x_2))\tilde{x}_1^0 = \frac{1}{27}a^5\lambda(\lambda + 1)x_2 \end{cases}$$

$\lambda = -1$ or $\lambda = 0$. Multiply the first and the second equations of the system (5.1) by $m_2$ and $m_1$ correspondingly and then subtract the second equation from the first one. We get:

(5.3)
$$m_2\tilde{x}_1^0(\tilde{x}_1^0, x_1) - m_1\tilde{x}_2^0(\tilde{x}_2^0, x_2) = 0 \Rightarrow (\tilde{x}_1^0 \perp x_1), \sigma.$$

Now we can state with assurance that the system (5.1) is correct if $\lambda(\lambda+1) = 0$ and the orthogonal condition (5.3) is held true. The vectors $x_i$ have two components: the first component lies on the plane $\{\tilde{x}_1^0, \sigma\}$, and the second component lies on the plane that is orthogonal to the first plane. There is only one free parameter for the components lying in the first plane taking into account that $\sum_\sigma x_1 = 0$ and there are two free parameters for the components, lying on the second plane on the same condition.

$\lambda = -3$ or $\lambda = 2$. Now let us find the eigenvectors having the following presentation: $x_1 = \nu\tilde{x}_1^0, z_1 = \mu\tilde{z}_1^0, \sigma$. Using the characteristic system (4.8) we obtain the next relations:

$$\begin{cases} 2\mu = (\lambda + 2)\nu \\ 2\nu = (\lambda - 1)\mu \end{cases}$$

from where we get $(\lambda + 3)(\lambda - 2) = 0$.

$\lambda \neq -1, 0, 1$. In this case (as it follows from (5.2)) vectors $x_i$ lie on the plane $\{\tilde{x}_1^0, \sigma\}$. Denote the expression $a^3\lambda(\lambda + 1)/27$ by $\mu$. and get a problem of the 4th degree of finding eigenvectors. Taking into account the fact that we already know the two vectors $\lambda = -3, 2$, a degree of the problem can be reduced to 2. To achieve

this, consider a basis of a 4-dimensional space of the operator (5.2), having the following form:

$$e_1 = \begin{pmatrix} \tilde{x}_2^0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ \tilde{x}_1^0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} \tilde{x}_1^0 \\ \tilde{x}_2^0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} \tilde{x}_1^0 + 2\tilde{x}_2^0 \\ -2\tilde{x}_1^0 - \tilde{x}_2^0 \end{pmatrix}$$

In this basis the vectors $e_3, e_4$ are eigenvectors. The basal vector $e_3$ is defined by the eigenvector of the operator (5.1), corresponding to the values $\lambda = -3, 2(x_1 = \tilde{x}_1^0), \sigma$. The basal vector $e_4$ corresponds to the eigenvalues $\lambda = -1, 0$, being defined by the condition $x_1 \perp \tilde{x}_1^0$ or $x_1 = \tilde{x}_1^0 + 2\tilde{x}_2^0, \sigma$. In the basis $(\tilde{x}_1^0, \tilde{x}_2^0)$ $x_2 = \tilde{x}_2^0 + 2\tilde{x}_3^0 = -\tilde{x}_2^0 - 2\tilde{x}_1^0$.

In the basis $e_1, \ldots, e_4$ a matrix of the operator (5.2) has the following form:

$$\begin{pmatrix} \frac{1}{2}(-m_1 + m_2 + m) & \frac{1}{2}(2m_1 + m_2 - m) & 0 & 0 \\ \frac{1}{2}(m_1 + 2m_2 - m) & \frac{1}{2}(m_1 - m_2 + m) & 0 & 0 \\ \frac{1}{4}(2m_1 + m_2 - m) & \frac{1}{4}(m_1 + 2m_2 - m) & 6 & 0 \\ \frac{1}{2}(2m_1 - m_2 - m) & \frac{1}{2}(m_1 - 2m_2 + m) & 0 & 0 \end{pmatrix},$$

from where a characteristic polynomial can be easily found.

A total number of eigenvectors is equal to 15, that is equal to a dimension of the considered subspace $\mathbb{C}^{18}$, which is defined by the condition $x_1 + x_2 + x_3 = 0$. □

Now consider an asymptotic behaviour of the $\beta$-singular points of the foliation $\mathcal{F}$. Thus, we try to find a solution of the following system:

$$(5.4) \qquad \begin{cases} (m_2 + m_3)z_1 - m_1(z_2 + z_3) = \frac{\lambda + 2}{3}x_1, \sigma \\ -\frac{3}{|\tilde{x}_1^0|^3}x_1 + \frac{9}{|\tilde{x}_1^0|^5}(\tilde{x}_1^0, x_1)\tilde{x}_1^0 = (\lambda - 1)z_1, \sigma. \end{cases}$$

Without any restrictions of generality we can suppose that the vectors $\tilde{x}_i^0, \tilde{z}_i^0$ have the forms $(a_i, 0, 0), (b_i, 0, 0)$ correspondingly. In this case the operator (5.4)) has three following eigenspaces: $x_i, z_i \in V_1 = \{(*, 0, 0)\}$, $x_i, z_i \in V_2 = \{(0, *, 0)\}$, and $x_i, z_i \in V_3 = \{(0, 0, *)\}$. Considering the problem (5.4)) in the every proper subspace we get the following theorem.

THEOREM 5.2. *The eigenvalues and the eigenvectors, that are corresponding to these eigenvalues, of the linear system* (5.4) *are the following:*

$\lambda = 1$. *The eigenvector $u_1$ has the form:*

$$x_1 = 0, \ z_1 = m_1 r, \sigma, \quad r \in \mathbb{C}^3.$$

*A dimension of the eigenspace is equal to* 3.

*In the space $V_1$ $(\lambda + 3)(\lambda - 2) = 0$. The eigenvectors $u_{-3}, u_2$ have the following form: $x_1 = \nu \tilde{x}_1^0, z_1 = \mu \tilde{z}_1^0, \sigma$, $2\mu = (\lambda + 2)\nu$. A dimension of the eigenspaces for $\lambda = -3$ and $\lambda = 2$ is equal to* 1.

$\lambda \neq -3, 1, 2$. *The eigenvalues $\lambda_1, \lambda_2$ are found as the roots of the equation*

$$\lambda^2 + \lambda + 2 - 18 \sum_\sigma \frac{m_1 + m_2}{a_3^3} = 0.$$

*A dimension of the eigenspaces for every root is equal to* 1.

*In the space $V_2$ $\lambda(\lambda + 1) = 0$. The eigenvectors $u_0, u_{-1}$ have the following form: $x_1 = \nu(0, a_1, 0)$, $z_1 = \mu(0, b_1, 0), \sigma$, $2\mu = (\lambda - 1)\nu$. A dimension of the eigenspaces for $\lambda = 0$ and $\lambda = 1$ is equal to* 1.

$\lambda \neq 0, 1, -1$. *The eigenvalues* $\lambda_3, \lambda_4$ *are found as the roots of the equation*

$$\lambda^2 + \lambda - 4 + 9 \sum_\sigma \frac{m_1 + m_2}{a_3^3} = 0.$$

*A dimension of the eigenspace for every root is equal to* 1.

   *In the space* $V_3$ *the eigenvalues are the same as in the space* $V_2$. *The eigenvectors are found similarly. It is possible to select a proper basis from all the eigenvectors, mentioned above.*

   PROOF. First of all, consider the case $\lambda = 1$. Then we'll be able to substitute $z_i$ from the second equation (5.4) to the first one. The second equation of the system (5.4) for the space $V_1$ and the spaces $V_2, V_3$ has the form

$$\frac{6}{|\tilde{x}_1^0|^3} x_1 = (\lambda - 1)z_1, \sigma, \qquad \frac{-3}{|\tilde{x}_1^0|^3} x_1 = (\lambda - 1)z_1, \sigma,$$

correspondingly. In either of these cases $x_i = 0$ and as well as for the $\alpha$-points we get the eigenvector. Consider the space $V_1$.

   Repeating the proof of the Theorem 5.1 let us find the eigenvectors in such a form: $x_1 = \nu \tilde{x}_1^0$, $z_1 = \mu \tilde{z}_1^0, \sigma$, In this case we get the equation $(\lambda + 3)(\lambda - 2) = 0$.

   In order to find the rest of the roots $\lambda$, substitute $z_i$ from second equation (5.4) to the first one. At the same time, denote the product $(\lambda - 1)(\lambda + 2)$ by $\mu$ and replace $-x_1 - x_2$ by $x_3$. We shall get a quadratic polynomial by $\mu$. Finally, taking into account that the derived polynomial is exactly divided by $(\lambda+3)(\lambda-2) = \mu+4$, we get the first power polynomial

$$\mu + 4 - 18 \sum_\sigma \frac{m_1 + m_2}{a_3^3} = 0.$$

Now consider the space $V_2$ and let us pay our attention to the fact that the system (5.4) for the spaces $V_1$ and $V_2$ doesn't differ much from the same one for the space $V_1$. That is why the eigenvectors for $V_2$ can be found in the form $x_i = \nu(0, a_i, 0)$, $z_i = \mu(0, b_i, 0)$ and similarly for $V_3$. So we get the following system:

$$\begin{cases} 2\mu = (\lambda + 2)\nu \\ -\nu = (\lambda - 1)\mu \end{cases}$$

from where it follows that $\lambda^2 + \lambda = 0$.

   Repeating the same calculations as for the space $V_1$ we get the polynomial

$$\mu - 2 + 9 \sum_\sigma \frac{m_1 + m_2}{a_3^3} = 0,$$

where $\mu = (\lambda - 1)(\lambda + 2)$.

   The calculations for the space $V_3$ exactly coincide with the case of $V_2$.

   A total number of eigenvectors is equal to 15, that is equal to a dimension of the considered subspace $\mathbb{C}^{18}$, which is defined by the condition $x_1 + x_2 + x_3 = 0$.  $\square$

   Undoubtedly, we can't but pay our attention to the surprising coincidence of properties of the eigenvectors of the $\alpha$ and $\beta$-points, the reason of which will be explained further.

THEOREM 5.3. *The operators* $(5.1)$, $(5.4)$, *which linearize the differential equations* $(4.10)$ *in the neighbourhoods of the $\alpha$ and $\beta$-points, have the following entire eigenvalues for any masses $m_i$ and the following eigenvectors, corresponding to these eigenvalues:*

$\lambda = 1$. *The eigenvector $u_1$ has the following form:*

$$x_1 = 0, \quad z_1 = m_1 r, \sigma, \quad r \in \mathbb{C}^3.$$

$\lambda(\lambda + 1) = 0$. *The eigenvectors $u_0, u_{-1}$ satisfy the following conditions:*

$$x_1 \perp \tilde{x}_1^0, \quad z_1 = \frac{3x_1}{a^3(1 - \lambda)}, \sigma.$$

$(\lambda + 3)(\lambda - 2) = 0$. *The eigenvectors $u_{-3}, u_2$ has the following form:*

$$x_1 = \nu \tilde{x}_1^0, \quad z_1 = \mu \tilde{z}_1^0, \sigma, \quad 2\mu = (\lambda + 2)\nu.$$

PROOF. The correctness of the theorem follows from theorems $5.1$ and $5.2$.  □

Suppose that the functions $x_1(t), x_2(t)$, from the system $(4.4)$ are known. Then $x_3(t) = -x_1(t) - x_2(t)$, $z_1(t) = \int \frac{x_1(t)}{|x_1(t)|^3} dt, \sigma$.

If we know the functions $w_1(t), w_2(t)$, we can find $x_1(t), x_2(t)$, as $x_1(t) = \frac{w_1(t)}{|w_1(t)|^2}, \sigma$.

At last if we know the functions $z_1(t), z_2(t)$, then $x_1(t) = \frac{\dot{z}_1(t)}{|\dot{z}_1(t)|^{3/2}}, \sigma$, according to $(4.4)$.

Taking into account the reasoning adduced above we call any collection $x(t) = ((x_1(t), x_2(t)), w(t) = (w_1(t), w_2(t)), z(t) = (z_1(t), z_2(t)), \sigma$, a solution of the three-body problem. The properties of the foliation $\mathcal{F}$ are important for profound studying the three-body problem but we'll try to investigate the singular points of the solutions $x(t), w(t), z(t)$.

PROPOSITION 5.1. *If $t_*$ is a singular point of the solution of the three-body problem* $(4.7)$ *then $\max_i\{|w_i|\} \to \infty$, $t \to t_*$. At the same time $\|w(t)\| \to \infty$.*

PROOF. Let the functions $|w_i(t)|$ be bounded in the neighbourhood of the singular point $t_*$. It means that the right hand side $\dot{z}$ of $(4.7)$ is also bounded in the neighbourhood of $t_*$, thus the solution $(w(t), z(t))$ of the differential equation $(4.7)$ is holomorphic in this neighbourhood. We get the contradiction, hence $\max_i\{|w_i|\} \to \infty$, $t \to t_*$ and $\|w(t)\| \to \infty$.  □

THEOREM 5.4. *All the solutions*

$$w_1(t) = \frac{x_1(t)}{|x_1(t)|^2}, \sigma,$$

*corresponding to the trajectories, entering the singular points of the problem $(4.5)$, have an asymptotic behaviour of the next form:*

$$(5.5) \qquad x(t) = \tilde{x}^0 t^{2/3} + \kappa_1 u_1 t + \kappa_2 u_2 t^{4/3} + \sum_k \mu_k v_k t^{(2+\lambda_k)/3} + \dots,$$

*where $\tilde{x}^0$ is a solution of the characteristic system $(4.8)$, $\kappa_1$, $\kappa_2$, $\mu_k$ are free parameters, $u_1$, $u_2$ are eigenvectors of the operators $(5.1)$, $(5.4)$, $v_k$ are eigenvectors of these operators for eigenvalues $\lambda_k > 0$ (see theorems $5.1$, $5.2$).*

PROOF. Using the results of the theorems 5.1 and 5.2, get the first approximation of an asymptotic behaviour of solutions of (4.5) and then apply the Picard operator. $\square$

REMARK 5.1. As it was noted before, a coincidence of entire eigenvalues of the operators, linearizing the $\alpha$ and $\beta-$singular points is not accidental and is explained by availability of invariant surfaces of the three body problem (2.3), particularly, of the surfaces, defined by the first integrals. This fact can be verified by substituting the obtained asymtotics to the first integrals and also by differentiating these integrals along the vector field (4.5).

For the $\alpha$-singular points the eigenvalue $\lambda = 1$ corresponds to the degree of the first integral $\mathcal{I}$ of the problem (2.2) with values in $\mathbb{C}^3$; for the system (4.4) this integral disappears due to the substitutions (4.2), (4.3), but an invariant surface together with an eigenvalue equal to 1 hold out. The eigenvalue $\lambda = 0$ appears because of an invariance of the problem (4.4) under effect of the group $\mathrm{So}(3)$, having a dimension, equal to 3, and is described by the coefficient $\tilde{x}^0$ in the asymptotics (5.5). The eigenvalue $\lambda = 2$ corresponds to a degree of the Hamiltonian $\mathcal{H}$. The eigenvalue $\lambda = -3$ characterizes an invariant trajectory, along which a projection on a compact manifold during a factorization of the flow occurs. In the asymptotics this trajectory is presented by a summand $\tilde{x}^0 t^{2/3}$. The integral $\mathcal{M}$ with values in $\mathbb{C}^3$ has a degree, equal to $-1$, but is not described by the asymptotics (5.5), if $\mathcal{M} \neq 0$, as the corresponding trajectory of the system (4.5) does not enter into the singular point.

For the $\beta$-singular points a set of positions of the directing vector $\tilde{x}^0$ is a two-dimensional sphere, thus its dimension is equal to 2. The rest of properties are similar.

Finally, let us note, that no less than two eigenvalues among computed ones $\lambda_i$ for the $\alpha$-singular points, and no less than three eigenvalues for the $\beta$-singular points are negative or not real.

REMARK 5.2. If $\lambda < 0$, $\tau \to -\infty$, because (4.5), the trajectories, outgoing from the singular points specify the expansion of $x(t)$ in powers $t^{-1/3}$, due to (4.6), i.e. the asymptotic behaviour when $t \to \infty$.

REMARK 5.3. An asymptotics of the motion (5.5) before and after the collision does not correspond to neither a elastic, nor an inelastic collision. That is why hereinafter we consider only the motions without collisions in the real part of the physical space $\mathbb{R}^3$.

Upon this occasion, note, that the singular points (5.5) don't have an asymptotic behaviour of a general position. Undoubtedly, this fact is quite natural, for the reason that these singular points characterize triple collisions or movements of the bodies, converging to some stable configurations.

All the other singular points of the foliation $\mathcal{F}$ can be obtained when the right hand side of (4.4) is not defined: these are binary collisions when $\|x_i\| = 0$, or $|x_i(t)| = 0$, $Im(t) \neq 0$.

DEFINITION 5.1. Let us call the singular points of the solutions of the system (4.4), that are specified by the condition $|x_i(t)| = 0$, $Im(t) \neq 0$, the points of quasi-collisions.

Certainly, all the types of collisions and solutions, defining them, are interesting for us, even if the measure of such solutions is equal to zero. Nevertheless, taking into account the problem of the solar system stability, we consider the solutions with quasi-collisions only, especially as trajectories, intersecting the surface $|x_i| = 0$, form a set of a full measure.

## 6. An asymptotic behaviour of quasi-collisions

Let $|x_1| = 0$, $Im(t) \neq 0$ in the system (4.4). Then the variables $z_2$, $z_3$ are small in comparison with $z_1$, hence the system (4.4) is approximately described by the system

$$(6.1) \qquad \begin{cases} \dot{x}_1 = mz_1, \\ \dot{z}_1 = -\frac{x_1}{|x_1|^3}. \end{cases}$$

We can also get this system, if the mass of one of these three bodies is equal to zero. This is natural because it is clear that we speak about the situation when the influence of one of the bodies on the two others is minimal. The problem (6.1) is plane; denote the coordinates of the vector $x_1 = (\chi_1, \chi_2)$, $z_1 = (\psi_1, \psi_2)$, and $|x_1| = |\chi|$, and make sure, an integral of moment is available

$$\dot{\mathcal{M}} = (\dot{\chi}_1 \chi_2 - \chi_1 \dot{\chi}_2)\dot{} = \ddot{\chi}_1 \chi_2 - \chi_1 \ddot{\chi}_2 = m\left(\frac{\chi_1}{|\chi|^3}\chi_2 - \chi_1 \frac{\chi_2}{|\chi|^3}\right) = 0.$$

It means that in polar coordinates

$$\dot{\chi}_1 \chi_2 - \chi_1 \dot{\chi}_2 = (r\cos(\varphi))\dot{}\, r\sin(\varphi) - r\cos(\varphi)(r\sin(\varphi))\dot{} = -r^2 \dot{\varphi} = \mathcal{M}.$$

The integral from the Theorem 4.1 takes the following form:

$$(6.2) \qquad \mathcal{H} = \frac{m}{2}z_1^2 - \frac{1}{|\chi|}$$

and then we have:

$$\dot{\chi}_1 = \dot{r}\cos(\varphi) - r\dot{\varphi}\sin(\varphi),$$
$$\dot{\chi}_2 = \dot{r}\sin(\varphi) + r\dot{\varphi}\cos(\varphi),$$
$$\mathcal{H} = \frac{1}{2m}(\dot{r}^2 + r^2\dot{\varphi}^2) - r^{-1},$$

$$(6.3) \qquad \begin{cases} \dot{\varphi} = -\mathcal{M}r^{-2}, \\ \dot{r} = \sqrt{-\mathcal{M}^2 r^{-2} + 2m(\mathcal{H} + r^{-1})}. \end{cases}$$

We can get rid of time and then we obtain

$$\frac{dr}{d\varphi} = -\frac{r\sqrt{-M^2 + 2m(\mathcal{H}r^2 + r)}}{\mathcal{M}^2}.$$

By means of the classic Binet replacing [21] $\omega = 1/r - m/\mathcal{M}^2$, we make the equation more simple:

$$\frac{dr}{d\omega} = \sqrt{A^2 - \omega^2}, \quad A^2 = \frac{2\mathcal{H}m}{\mathcal{M}^2} + \frac{m^2}{\mathcal{M}^4}.$$

However, such a change of variables is not doing much good, if considering the three body problem. Moreover, polar coordinates are also not convenient, because

circular functions have infinite-valued inverse functions. That is why we continue investigating the asymptotics of singular points using the previous coordinates $(z, x)$.

We're interested in considering the easiest case, when one of the points moves on a parabola relative to the other one. Thus we consider the motion of the following form:

$$\chi_2 = k\chi_1^2 - a. \tag{6.4}$$

Further representations are obtained, using the relations (6.1)–(6.4), being quite simple. That is the reason for which we do not comment them at length.

$$(6.4),(6.1) \Rightarrow \dot{\chi}_2 = 2k\chi_1\dot{\chi}_1 = 2mk\chi_1\psi_1 = m\psi_2 \Rightarrow$$

$$\psi_2 = 2k\chi_1\psi_1, \tag{6.5}$$

$$(6.5),(6.1),(6.4) \Rightarrow \dot{\psi}_2 = 2km\psi_1^2 - \frac{2k\chi_1^2}{|\chi|^3} = \frac{a - k\chi_1^2}{|\chi|^3} \Rightarrow$$

$$2km\psi_1^2 = \frac{a + k\chi_1^2}{|\chi|^3}. \tag{6.6}$$

$$(6.4),(6.5) \Rightarrow \mathcal{M} = 2k\chi_1^2\psi_1 - (k\chi_1^2 - a)\psi_1 \Rightarrow$$

$$\psi_1 = \frac{\mathcal{M}}{a + k\chi_1^2}. \tag{6.7}$$

$$(6.6),(6.7) \Rightarrow \frac{2km\mathcal{M}}{(a + k\chi_1^2)^2} = \frac{a + k\chi_1^2}{|\chi|^3} = \frac{a + k\chi_1^2}{(\chi_1^2 + (a - k\chi_1^2)^2)^{3/2}} \Rightarrow$$

$$|\chi| = (\chi_1^2 + (a - k\chi_1^2)^2)^{1/2} = a + k\chi_1^2 \Rightarrow \tag{6.8}$$

$$4ka = 1. \tag{6.9}$$

$$(6.6),(6.7),(6.8),(6.9) \Rightarrow 2\mathcal{M}^2 km = 1 \Rightarrow$$

$$a = \frac{\mathcal{M}^2 m}{2}, \quad k = \frac{1}{2\mathcal{M}^2 m}$$

Thus, we have a solution in the following form:

$$\psi_1 = \frac{2\mathcal{M}^3 m}{\chi_1^2 + \mathcal{M}^4 m^2}, \quad \psi_2 = \frac{4\mathcal{M}^3 km\chi_1}{\chi_1^2 + \mathcal{M}^4 m^2}, \quad \chi_2 = \frac{\chi_1^2}{2\mathcal{M}^2 m} - \frac{\mathcal{M}^2 m}{2},$$

$$\dot{\chi}_1 = \frac{2\mathcal{M}^3 m^2}{\chi_1^2 + \mathcal{M}^4 m^2} \quad \Leftrightarrow \quad \frac{\chi_1^3}{3\mathcal{M}^4 m^2} + \chi_1 = \frac{2t}{\mathcal{M}},$$

where the last but one relation follows from (6.1), (6.7). On the assumption of the formulae, we have just obtained, it is easy to get the time dependence $\chi_2(t)$ and $|\chi(t)|$:

$$(\chi_2 + a)\left(\frac{\chi_2 + a}{3\mathcal{M}^4 m^2 k} + 1\right)^2 = \frac{4t^2 k^2}{\mathcal{M}^2}, \quad |\chi(t)| = a + k\chi_1(t)^2 = 2a + \chi_2(t),$$

and an asymptotic behaviour of the solution $\chi_1(t)$.

$$\chi_1 = -\mathcal{M}^2 mi + \sqrt{2i\mathcal{M}mt} + \frac{t}{3\mathcal{M}} + \frac{5t^{3/2}}{18\mathcal{M}^2\sqrt{2i\mathcal{M}m}} + \frac{4it^2}{27\mathcal{M}^4 m} + \cdots,$$

$$\chi_2 = -\mathcal{M}^2 m - i\sqrt{2i\mathcal{M}mt} + \frac{2it}{3\mathcal{M}} + \frac{7it^{3/2}}{18\mathcal{M}^2 m\sqrt{2i\mathcal{M}m}} + \frac{5t^2}{27\mathcal{M}^4 m} + \ldots,$$

$$\psi_1 = \frac{\dot{\chi}_1}{m} = \frac{\sqrt{\mathcal{M}i}}{\sqrt{2mt}} + \frac{1}{3\mathcal{M}m} + \frac{5t^{1/2}}{12\mathcal{M}^2\sqrt{2i\mathcal{M}m}} + \frac{8it}{27\mathcal{M}^4 m^2} + \ldots,$$

$$(6.10) \quad \psi_2 = \frac{\dot{\chi}_2}{m} = -i\frac{\sqrt{\mathcal{M}i}}{\sqrt{2mt}} + \frac{2i}{3\mathcal{M}m} + \frac{7it^{1/2}}{12m\mathcal{M}^2\sqrt{2i\mathcal{M}m}} + \frac{10t^2}{27\mathcal{M}^4 m^2} + \ldots$$

Now let us consider the hyperbolic and the elliptic motions.

$$\sqrt{(\chi_1 + a)^2 + \chi_2^2} \pm \sqrt{\chi_1^2 + \chi_2^2} = b \Rightarrow k^2\chi_2^2 - \chi_1(\chi_1 + a) = \frac{c^2}{4},$$

$$a^2 - b^2 = c^2, \quad k = \frac{b}{c}.$$

Having differentiated it twice, we get:

$$2k^2\chi_2\psi_2 - 2\chi_1\psi_1 - a\psi_1 = 0,$$

$$2k^2 m\psi_2^2 - \frac{2k^2\chi_2^2}{|\chi|^3} - 2m\psi_1^2 + \frac{2\chi_1^2}{|\chi|^3} + \frac{a\chi_1}{|\chi|^3} = 0.$$

After having expressed $\psi_2$ by $\mathcal{H}$, we get:

$$(6.11) \qquad (ml^2\dot{\chi}_1^2 - 2hk^2)^2 = \frac{4(8l^2\chi_1^2 + 6a\chi_1 + c^2)^2 k^6}{(4l^2\chi_1^2 + 4a\chi_1 + c^2)^3}.$$

$$\frac{\sqrt{2\mathcal{H}m|\chi|^2 + 2m|\chi| - \mathcal{M}^2}}{2\mathcal{H}m} - \frac{\text{Ln}(m(2\mathcal{H}|\chi| + 1) + \sqrt{2\mathcal{H}m|\chi|^2 + 2m|\chi| - \mathcal{M}^2})}{(2\mathcal{H})^{3/2}\sqrt{m}} = t,$$

$$|\chi|^2 = 4l^2\chi_1^2 + 4a\chi_1 + c^2, \quad \chi_2^2 = |\chi|^2 - \chi_1^2.$$

It is not easy to integrate this differential equation in an explicit form. That is why we'll find its solution by integration (6.3).

A solution of the two body problem (6.1) has the following form in the general case:

$$\frac{\sqrt{2\mathcal{H}m|\chi|^2 + 2m|\chi| - \mathcal{M}^2}}{2\mathcal{H}m} - \frac{\text{Ln}(m(2\mathcal{H}|\chi| + 1) + \sqrt{2\mathcal{H}m|\chi|^2 + 2m|\chi| - \mathcal{M}^2})}{(2\mathcal{H})^{3/2}\sqrt{m}} = t,$$

$$|\chi|^2 = 4l^2\chi_1^2 + 4a\chi_1 + c^2, \quad \chi_2^2 = |\chi|^2 - \chi_1^2.$$

As we see, the solution is not polynomial in contrast to the parabolic case. Moreover, it is infinite-valued, and this fact explains the reason of a complexity of finding partial integrable cases with hidden symmetries. Note also, that all the known solutions in the problem of a heavy rigid body's movement (except the Hess case) are expressed in either elliptic or hyper-elliptic functions, and that is why, to find them is much easier than in the three body problem.

An asymptotics of quasi-collisions for the general case can be found from (6.11); as well as in the parabolic case it has quite a simple form:

$$(6.12) \qquad l\chi_1 = -\frac{c}{2} + \left(-\frac{ck}{m}\right)^{1/4}\sqrt{t} + \cdots \Rightarrow \chi_2 = i\chi_1 + \ldots,$$

that is, as well as in the parabolic case, we get the asymptotics $\chi_i = \chi_{i0} + \chi_{i1}\sqrt{t} + \ldots$ in the point of a quasi-collision.

Thus, the asymptotic of a singular point of a quasi-collision of a solution of the two body problem is obtained. It is quite obvious, that the asymptotics (6.12) will change little and one will be able to obtain it by the Picard iterations, if the third body participates in a quasi-collision.

To finish the classification of singular points of the three body problem, we also need to consider a possibility of existence of such a singular point, that the projection $\pi(x(t), z(t))$ of the solution $(x(t), z(t))$, containing this point, would not enter the singular point $(\tilde{x}_i^0, \tilde{z}_i^0)$, $i = 1, 2, 3$, of the foliation $\mathcal{F}$, but would wind round some limit set.

To prove an absence of such singular points in the solutions of the three body problem, first let us prove the next statement.

PROPOSITION 6.1. *Let a limit set of the projection* $\pi(w(t), z(t)) = (\tilde{w}(t), \tilde{z}(t))$ *of a not constant solution* $(w(t), z(t))$ *of the three body problem* (4.4) *not contain any singular points of the foliation* $\mathcal{F}$*. Then the solution has a singular point* $t_* \in \mathbb{C}$ *and at the same time* $\|(\tilde{w}(t)\|$, $\|\tilde{z}(t)\| \to \infty$.

PROOF. Let $(w(t), z(t))$ be a solution of the problem (4.4) and $Y = \pi(w(t), z(t))$ is a fibre of the foliation $\mathcal{F}$, corresponding to the solution. Suppose also, that the solution $(w(t), z(t))$ does not have a singular point. Then the fibre $Y$ also does not have a singular point, as all possible singular points of the foliation have corresponding singular points of the solution.

Let $y \in Y$ be some point of the fibre and $\pi(w(t_0), z(t_0)) = y$, $\|w(t_0), z(t_0)\| = \max_i\{\|w_i(t_0), z_i(t_0)\|\} = 1$. As the solution is not a constant, a trajectory $\gamma \subset \mathbb{C}$, along which the condition $\|w(t_1), z(t_1)\| > 2$ will be met from the point $t_0$ to the point $t_1$, exists. Moreover, one can find a neighbourhood $U$ in $P_*^{17}$ where for all $\acute{y} \in U$, if $\pi(w(t_0), z(t_0)) = \acute{y}$, $\|w(t_0), z(t_0)\| = 1$, then along the curve $\gamma$ the condition $\|w(t_1), z(t_1)\| > 2$ will be also met. Then we choose a finite sub cover $U_i$ from the open cover $Y$, consisting of such neighbourhoods, as a bundle manifold is compact. Suppose $|t_{0i} - t_{1i}| < T$ for all $i$. We can build the trajectory $\Gamma$, containing such points $t_0, t_1, \ldots, t_n, \ldots$, that from the point $t_{k-1}$ to the point $t_k$ the module $\|w(t_0), z(t_0)\|$ increases more than twice. Then the singular point $t_* \in \mathbb{C}$ will be situated not farther than $T + \frac{T}{2} + \frac{T}{2^2} + \cdots = 2T$ and $\|(\tilde{w}(t)\|, \|\tilde{z}(t)\| \to \infty$ from the initial point $t_0$. $\qquad\square$

THEOREM 6.1. *All the singular points of the solutions* $(x(t), z(t))$ *of the three body problem* (2.3) *are reduced to the* $\alpha$*-singular, the* $\beta$*-singular points and the singular points of quasi-collisions of the bodies.*

PROOF. If we suppose that there exist some other singular points except those, that are listed in the formulation of the theorem, then the projection $\pi(x(t), z(t))$ of the solution $(x(t), z(t))$, containing at least one such a point, should not enter the singular point $(\tilde{x}_i^0, \tilde{z}_i^0)$, $i = 1, 2, 3$, of the foliation $\mathcal{F}$, but wind round some limit set. But then according to the proposition 6.1 $\|(\tilde{w}(t)\| \to \infty$, that is equivalent to $\|(\tilde{x}(t)\| \to 0$, but this represents a collision of the bodies. $\qquad\square$

For the reason that all the types of singular points have been considered, the next theorem is correct.

THEOREM 6.2. *For almost all the initial conditions all the singular points of solutions of the three body problem* $x(t) = (x_1(t), x_2(t), x_3(t)) \in \mathbb{C}^9$

$$\begin{cases} \dot{x}_1 = mz_1 - m_1 \sum_\sigma z_1, \sigma \\ \dot{z}_1 = -\frac{x_1}{|x_1|^3}, \sigma \end{cases}$$

*have an asymptotics of the next form:*

$$x(t) = a + b\sqrt{t} + \dots, a, b \in \mathbb{C}^3.$$

*If such a solution* $x(t)$ *has a finite number of singular points, it is defined by a finite valued function.*

Reasoning from the full classification of singular points of the three body problem (2.3), all the solutions of this problem can be divided into classes, depending on the availability and nature of their singular points:

1) entire solutions;
2) finite valued solutions;
3) infinite valued solutions.

The entire solutions can be found similarly to the way, it was done for the problem of a heavy rigid body's movement in [8].

A possibility of existence of finite valued solutions follows from the fact that asymptotic behaviours of quasi-collisions are two-valued (see the theorem 6.2) and also from the existence of asymptotic behaviours of double and triple collisions with rational values of $\lambda$ (see the theorems 5.1, 5.2). Such solutions will define one-dimensional trajectories in the phase space, and from the proved fact it follows that these trajectories may be found among the solutions, defining the bodies scattering with speeds, converging to zero so as in the parabolic case of the two body problem.

The existence of infinite valued solutions does not raise any doubts, however finding even exact periodic solutions of this class is quite a difficult task. Although, perhaps it is possible to describe some quasi-periodic solutions, that exist according to the results of the KAM-theory (see, for example [6]).

It is worth telling, that a lot of solutions have been already found with the help of computer calculations. These solutions seem to be periodic or quasi periodic (see, for example [1]), which makes any efforts to find the exact solutions of the three body problem quite relevant.

## 7. Entire and finite valued solutions of the three body problem

The fact of presence or absence of entire solutions of the three body problem does not seem to be evident, for example, from a physical point of view: it is quite easy to imagine possible motions of the three bodies in $\mathbb{R}^3$, but it is much more difficult to imagine such "motions" in $\mathbb{C}^3$.

Entire functions do not have a convenient representation, that could be substituted to the differential equations of the three body problem. At the same time a

solution of the two body problem, defined by entire functions, is well-known: it is a rotation of a satellite in a circular orbit. For the system (6.1) we can assign, for example, $x_1 = (m\sin(\omega t), m\cos(\omega t))$, $z_1 = (\omega\cos(\omega t), -\omega\sin(\omega t))$, $m\omega^2 = 1$.

It is already known, that there are similar solutions for the three body problem among the solutions of Euler and Lagrange, but we need to find a full list of entire solutions.

The entire solutions $|x_i(t)|$ do not have any points of quasi-collisions, as these solutions do not have singularities at all. That is why the solution of the system (4.7)

$$\begin{cases} \dot{w}_1 = -2w_1\big(w_1, mz_1 - m_1\sum_\sigma z_1\big) + |w_1|^2\big(mz_1 - m_1\sum_\sigma z_1\big), \sigma \\ \dot{z}_1 = -w_1|w_1|, \sigma \end{cases}$$

where $w_1(t) = x_1(t)/|x_1(t)|^2, \sigma$, will also be entire.

As it follows from the proposition 4.2, if an entire solution $(w(t),\, z(t))$ with inconstant functions $|x_i(t)|$ exists, then a limit set of its $\pi-$projection in view of $\|w_1\| \to \infty, \sigma$, contains at least one point of the integral surface $|w_1| = 0, \sigma$, of the foliation $\mathcal{F}$.

It means that for such a solution $\|x_1\| \to 0$, $|x_1| \to \infty, \sigma$, as $|w_1(t)| = 1/|x_1(t)|, \sigma$, that is impossible. Consequently, $|x_1(t)| = const, \sigma$.

The theorem 2.1 is proved.

Provided $|x_1(t)| = const., \sigma$, the three body problem becomes linear, and solving it reduces to an algebraic problem of finding the suitable initial conditions.

However, we can act simpler. For the reason that the distances between the points remain the same, and the equations have such a representation

$$\ddot{x}_1 = -m\mu_1 x_1 + m_1\sum_\sigma x_1\mu_1, \sigma,$$

where $\mu_1 = |x_1|^{-3} = const, \sigma$, the motion along the exponent contradicts the proved restraints, and the uniform rotation is possible only when

$$\kappa_1 x_1 = x_2\mu_2 + x_3\mu_1, \sigma.$$

Taking into consideration that $\sum_\sigma x_1 = 0$, we get the system

$$\begin{cases} (\kappa_1 + \mu_3)x_1 + (\mu_3 - \mu_2)x_2 = 0, \\ (\kappa_3 + \mu_1)x_1 + (\kappa_3 + \mu_2)x_2 = 0, \\ (\mu_1 - \mu_3)x_1 + (\kappa_3 + \mu_3)x_2 = 0, \end{cases}$$

that has a solution when the coefficients near $x_1, x_2$ are proportional (this will be the Euler case) or equal to zero (this will be the Lagrange case).

Suppose that a solution of the three body problem is finite valued. As it follows from analytic properties of the two body problem, a finite valued solution is possible only when the relative motion of the bodies is parabolic and the condition $z(t) \to 0$, holds if $x(t) \to \infty$.

It is obvious that in the three body problem if the last condition holds, then a solution will also be finite valued. Otherwise, the point of the condensation of an infinite number of singular points of quasi-collisions (if these ones exist) lies on

the singular surface $|x_1(t)| = 0, \sigma$. At the same time this point is also the point of an infinite branching, but it is reached by the solution when $\|x_i(t)\| \to \infty$ for some $i$. However, in the parabolic case there is no infinite branching at the infinity, thus it does not exist at all, and the problem of finding the finite valued solutions becomes relevant.

In the three body problem, if the bodies fly from the infinity to the infinity with the rates, converging to zero under $t \to \pm\infty$, the solutions have three quasi-collisions, i.e. exactly 6 singular points with the branching of the second degree. In this case it is easy to find the asymptotics of $|x_i(t)|$ at the infinity. If we know all the singular points together with their asymptotic behaviours, we can state that, for example, for a vector, defined by the two bodies, there exist a coordinate, for which

$$(7.1) \qquad \dot{\chi}_1 = \frac{a_1}{\mathcal{P}_1^{[2]}(\chi_1)} + \frac{a_2}{\mathcal{P}_2^{[2]}(\chi_1)} + \frac{a_3}{\mathcal{P}_3^{[2]}(\chi_1)},$$

where $\mathcal{P}_i^{[2]}(\chi_1)$ ia a polynomial of the second degree, from where we get

$$\dot{\chi}_1 \left( \mathcal{P}_4^{[2]}(\chi_1) + \frac{b_{11}\chi_1 + b_{12}}{c_{11}\chi_1^2 + c_{12}\chi_1 + c_{13}} + \frac{b_{21}\chi_1 + b_{22}}{c_{21}\chi_1^2 + c_{22}\chi_1 + c_{23}} \right) = C_1,$$

that results in a finite valued solution only if

$$\dot{\chi}_1 \left( \mathcal{P}_4^{[2]}(\chi_1) + \frac{b}{(\chi_1 + c_1 + ic_2)^2} + \frac{b}{(\chi_1 + c_1 - ic_2)^2} \right) = C_1$$

or

$$\int \mathcal{P}_4^{[2]}(\chi_1) d\chi_1 - \frac{2b(\chi_1 + c_1)}{\chi_1^2 + c_1^2 + c_2^2 + 2c_1\chi_1} = C_1 t + C_2,$$

as when integration logarithms should not appear, the singular points $\chi_1 \to const$ under $t \to \infty$ should be complex, but the solution itself should be real. Moving the variables $\chi_1, t$ on a constant, simplify the dependency:

$$\int \mathcal{P}_4^{[2]}(\chi_1) d\chi_1 - \frac{2b\chi_1}{\chi_1^2 + c^2} = Ct,$$

After having given the specified more precise definitions in the starting equation (7.1) we have only one singular point of a quasi-collision: $\chi_1 = \pm ic$.

In this case all the variables have two singular complex-conjugate points $t_*, \bar{t}_*$ with asymptotic behaviours of the form (6.10):

$$\chi_i = \chi_{i0} + \chi_{i1}\sqrt{t - t_*} + \ldots, \quad t \to t_*$$

and with the asymptotics at the infinity

$$\chi_i = \chi_{i\infty}(t - t_*)^{k/3} + \ldots, \quad t \to \infty, \quad k = 1, 2.$$

For our convenience, when describing an exact finite valued solution, denote the coordinates of the vectors between two bodies by the variables $\chi_i(t), \zeta_i(t), \xi_i(t)$, $i = 1, 2, 3$.

Note also, that all these coordinates for a concrete solution are connected by relations, the degree of which is not more than 2, as otherwise an asymptotic behaviour of the sought solution at the infinity wouldn't meet the required conditions.

Rewrite the relation (7.1) in the next form:

$$(7.2) \qquad \tfrac{1}{3}\chi_1^3 + b^2\chi_1 - at + c_1 = 0.$$

We suppose in (7.2) the coefficient near $\chi_1(t)^2$ to be equal to zero, as this can be achieved by adding an appropriate constant to the variable $\chi_1(t)$. The coefficient near $\chi_1(t)$ should be positive, in order for the derivative $\dot{\chi}_1(t)$ to be equal to zero at the complex-conjugate points.

According to (4.3) and (4.4)

$$\ddot{\chi} + m\frac{\chi}{|\chi|^3} - m_1\dot{u},$$

and we get the first coordinate $\dot{u}$, taking the second derivative (7.2), and supposing

$$|\chi| = \kappa_1(\chi_1^2 + b^2),$$

reasoning from the known asymptotic behaviour at the singular point and at the infinity of this function.

$$\dot{u}_1 = \frac{\chi_1(m - 2a^2\kappa_1^3)}{\kappa_1^3 m_1(\chi_1^2 + b^2)^3}$$

The first coordinate of the second body should have the next form:

$$\zeta_1 = c_2\zeta_1^2 + c_1\zeta_1 + c_0$$

and at the same time it should identically satisfy the next equation:

$$(7.3) \qquad \ddot{\zeta}_1 + m\frac{\zeta_1}{|\zeta|^3} - m_1\dot{u}_1, \quad |\zeta| = \kappa_2(\chi_1^2 + b^2).$$

A direct substitution shows that (7.3) can be true only if $|\zeta| = |\chi|$ or $\chi$ and $\zeta$ are proportional. The same condition, naturally, is also necessary for the third body. So all the finite-valued solutions of (2.3) lead to the solutions of Euler or Lagrange of the classical three body problem (2.2).

Finally, let us pay our attention to the fact, that when we pass from the system (4.2) to the system (4.4) and inversely, we have to integrate the function

$$\dot{u} = \sum_{\sigma} \frac{x_1}{|x_1|^3}.$$

With such changes of variables it is evident, that the singular points of solutions of (4.2) pass to the singular points of solutions of (4.4). However, for the finite valued solutions, considered for the system (4.4), according to the theorem 6.1 we have a full classification of singular points of the solutions $x(t)$ and these singularities are such, that finite valued solutions remain finite valued after integration.

The same reasoning is all the more true for entire solutions of the system (4.2), as these solutions do not have singularities at all (by definition).

The theorem 2.2 is proved.

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# О ЦЕЛИМ И КОНАЧНО-ВРЕДНОСНИМ РЕШЕЊИМА ПРОБЛЕМА ТРИ ТЕЛА

Резиме. Дат је комплетан опис решења проблема три тела које су целе и коначно-вредносне функције времена. Показано је да сва коначно-вредносна решења припадају класи решења које су дали Ојлер и Лагранж.

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