

## BILLIARDS IN CONFOCAL QUADRICS AS A PLURI-LAGRANGIAN SYSTEM

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ABSTRACT. We illustrate the theory of one-dimensional pluri-Lagrangian systems with the example of commuting billiard maps in confocal quadrics.

### 1. Introduction

The aim of this note is to illustrate some of the issues of the theory of one-dimensional pluri-Lagrangian systems, developed recently in [5], with a well known example of billiards in quadrics [3, 6]. In Section 2 we recall the main positions of the theory of pluri-Lagrangian systems, including a novel explanation of the so called spectrality property introduced in [2]. Then in Section 3 we recall some basic facts about billiards in quadrics. The main new contribution is contained in Section 4, where we use the spectrality property to derive the full set of integrals of motion for commuting billiard maps in confocal quadrics.

### 2. Reminder on discrete 1-dimensional pluri-Lagrangian systems

Suppose we are given a 1-parameter family of pairwise commuting symplectic maps  $F_\lambda : T^*M \rightarrow T^*M$ ,  $F_\lambda(q, p) = (\tilde{q}, \tilde{p})$ , possessing generating (Lagrange) functions  $L(q, \tilde{q}; \lambda)$ , so that

$$(2.1) \quad F_\lambda : p = -\frac{\partial L(q, \tilde{q}; \lambda)}{\partial q}, \quad \tilde{p} = \frac{\partial L(q, \tilde{q}; \lambda)}{\partial \tilde{q}}.$$

When considering a second such map, say  $F_\mu : T^*M \rightarrow T^*M$ , we will denote its action by a hat:  $F_\mu(q, p) = (\hat{q}, \hat{p})$ ,

$$(2.2) \quad F_\mu : p = -\frac{\partial L(q, \hat{q}; \mu)}{\partial q}, \quad \hat{p} = \frac{\partial L(q, \hat{q}; \mu)}{\partial \hat{q}}.$$

The commutativity of these maps allows us to define, for any  $(q_0, p_0) \in T^*M$ , the function  $(q, p) : \mathbb{Z}^2 \rightarrow T^*M$  by setting

$$(q(n+e_1), p(n+e_1)) = F_\lambda(q(n), p(n)), \quad (q(n+e_2), p(n+e_2)) = F_\mu(q(n), p(n)), \quad \forall n \in \mathbb{Z},$$

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see Figure 1(A). This justifies our general short-hand notation for functions on  $\mathbb{Z}^2$ : if  $q$  stands for  $q(n)$ , then  $\tilde{q}$  stands for  $q(n + e_1)$ , while  $\hat{q}$  stands for  $q(n + e_2)$ . We introduce a discrete 1-form  $L$  on  $\mathbb{Z}^2$  by setting (slightly abusing the notations)  $L(n, n + e_1) = L(q, \tilde{q}; \lambda)$ , respectively  $L(n, n + e_2) = L(q, \hat{q}; \mu)$ .

From (2.1), (2.2) we easily see that the following *corner equations* hold true everywhere on  $\mathbb{Z}^2$ :

$$(E) \quad \frac{\partial L(q, \tilde{q}; \lambda)}{\partial q} - \frac{\partial L(q, \hat{q}; \mu)}{\partial q} = 0,$$

$$(E_1) \quad \frac{\partial L(q, \tilde{q}; \lambda)}{\partial \tilde{q}} + \frac{\partial L(\tilde{q}, \hat{q}; \mu)}{\partial \tilde{q}} = 0,$$

$$(E_2) \quad \frac{\partial L(q, \hat{q}; \mu)}{\partial \hat{q}} + \frac{\partial L(\hat{q}, \tilde{q}; \lambda)}{\partial \hat{q}} = 0,$$

$$(E_{12}) \quad \frac{\partial L(\hat{q}, \tilde{q}; \lambda)}{\partial \hat{q}} - \frac{\partial L(\tilde{q}, \hat{q}; \mu)}{\partial \hat{q}} = 0.$$

These four corner equations (E)–(E<sub>12</sub>) correspond to the four vertices of an elementary square of the lattice  $\mathbb{Z}^2$ , as on Figure 1(B).

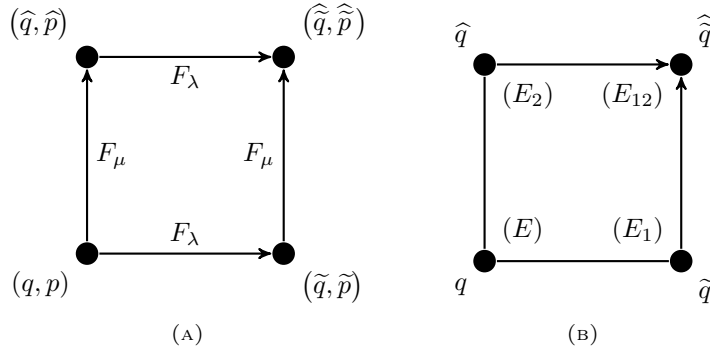


FIGURE 1. (A) Defining  $(q, p) : \mathbb{Z}^2 \rightarrow T^*M$  for two commuting maps  $F_\lambda$  and  $F_\mu$ . (B) Four corner equations. Their consistency means the following. If we start with the data  $q, \tilde{q}, \hat{q}$  related by the corner equation (E), and solve the corner equations (E<sub>1</sub>) and (E<sub>2</sub>) for  $\hat{q}$ , then the two values of  $\hat{q}$  coincide identically and satisfy the corner equation (E<sub>12</sub>).

The corner equations tell us that any solution  $q : \mathbb{Z}^2 \rightarrow M$  delivers a critical point to the action functional

$$S_\Gamma = \sum_{\sigma \in \Gamma} L(\sigma)$$

for any directed path  $\Gamma$  in  $\mathbb{Z}^2$  (under variations that fix the fields at the endpoints of the path  $\Gamma$ ). In other words, the field  $q : \mathbb{Z}^2 \rightarrow M$  solves the *pluri-Lagrangian problem* for the Lagrangian 1-form  $L$  [5].

THEOREM 2.1. *The value  $dL(\sigma)$  for all elementary squares  $\sigma = (n, n + e_1, n + e_1 + e_2, n + e_2)$  is constant on solutions of the system of corner equations (E)–(E<sub>12</sub>):*

$$(2.3) \quad dL(\sigma) := L(q, \tilde{q}; \lambda) + L(\tilde{q}, \hat{\tilde{q}}; \mu) - L(\hat{q}, \hat{\tilde{q}}; \lambda) - L(q, \hat{q}; \mu) = c(\lambda, \mu).$$

PROOF. The expression on the left-hand side of equation (2.3) is a function on the manifold of solutions of the system of corner equations. The manifold of solutions is of dimension  $2 \dim M$ , as it can be parametrized by  $(q, p)$  or by  $(q, \tilde{q})$ . It is enough to prove that  $\partial(dL(\sigma))/\partial q = 0$  and  $\partial(dL(\sigma))/\partial \tilde{q} = 0$ . We prove a stronger statement: if one considers  $dL(\sigma)$  as a function on the manifold of dimension  $4 \dim M$ , parametrized by  $q, \tilde{q}, \hat{q}$ , and  $\hat{\tilde{q}}$ , then the gradient of this function vanishes on the submanifold of solutions of corner equations, of dimension  $2 \dim M$ . But this is obvious, since vanishing of the partial derivatives of  $dL(\sigma)$  with respect to its 4 arguments is nothing but the corresponding corner equations.  $\square$

THEOREM 2.2. *For a family  $F_\lambda$  of commuting symplectic maps, the discrete pluri-Lagrangian 1-form  $L$  is closed on solutions,  $dL = c(\lambda, \mu) = 0$ , if and only if  $\partial L(q, \tilde{q}; \lambda)/\partial \lambda$  is a common integral of motion for all  $F_\mu$ .*

PROOF. Clearly, the possible dependence of the constant  $c(\lambda, \mu)$  on the parameters  $\lambda, \mu$  is skew-symmetric:  $c(\lambda, \mu) = -c(\mu, \lambda)$ . Therefore,  $c(\lambda, \mu) = 0$  is equivalent to  $\partial c(\lambda, \mu)/\partial \lambda = 0$ , that is, to

$$(2.4) \quad \frac{\partial L(q, \tilde{q}; \lambda)}{\partial \lambda} - \frac{\partial L(\hat{q}, \hat{\tilde{q}}; \lambda)}{\partial \lambda} = 0$$

(see equation (2.3); terms involving  $\partial \hat{q}/\partial \lambda$  and  $\partial \hat{\tilde{q}}/\partial \lambda$  vanish due to the corresponding corner equations). The latter equation is equivalent to saying that  $\partial L(q, \tilde{q}; \lambda)/\partial \lambda$  is an integral of motion for  $F_\mu$ .  $\square$

The latter property is a re-formulation of the mysterious “spectrality property” discovered by Kuznetsov and Sklyanin for Bäcklund transformations [2].

### 3. Billiard in a quadric

We consider the billiard in an ellipsoid

$$(3.1) \quad \mathcal{Q} = \left\{ x \in \mathbb{R}^n : \langle x, A^{-1}x \rangle = \sum_{i=1}^n \frac{x_i^2}{a_i^2} = 1 \right\}.$$

Let  $\{x_k\}_{k \in \mathbb{Z}}$ ,  $x_k \in \mathcal{Q}$ , be an orbit of this billiard. Denote by

$$v_k = \frac{x_{k+1} - x_k}{|x_{k+1} - x_k|} \in S^{n-1}$$

the unit vector along the line  $(x_k x_{k+1})$ . Then the following equations define the billiard map:

$$(3.2) \quad B : \begin{cases} x_{k+1} - x_k = \mu_k v_k, \\ v_k - v_{k-1} = \nu_k A^{-1} x_k. \end{cases}$$

Here the numbers  $\mu_k, \nu_k$  can be determined from the conditions  $v_k \in S^{n-1}, x_k \in \mathcal{Q}$ , so that

$$(3.3) \quad \mu_k = |x_{k+1} - x_k|, \quad \nu_k = \langle v_k - v_{k-1}, A(v_k - v_{k-1}) \rangle^{1/2}.$$

One can obtain alternative expressions for  $\mu_k, \nu_k$  by the following arguments. Suppose that  $x_k \in \mathcal{Q}$ , and determine  $\mu_k$  from the condition that  $x_{k+1} = x_k + \mu_k v_k \in \mathcal{Q}$ . This gives:

$$\langle x_k + \mu_k v_k, A^{-1}(x_k + \mu_k v_k) \rangle = 1 \quad \Leftrightarrow \quad 2\mu_k \langle x_k, A^{-1}v_k \rangle + \mu_k^2 \langle v_k, A^{-1}v_k \rangle = 0,$$

so that

$$(3.4) \quad \mu_k = -\frac{2\langle x_k, A^{-1}v_k \rangle}{\langle v_k, A^{-1}v_k \rangle} = \frac{2\langle x_{k+1}, A^{-1}v_k \rangle}{\langle v_k, A^{-1}v_k \rangle}.$$

(The second expression follows in the same way by assuming that  $x_{k+1} \in \mathcal{Q}$  and requiring that  $x_k = x_{k+1} - \mu_k v_k \in \mathcal{Q}$ .)

Similarly, suppose that  $v_{k-1} \in S^{n-1}$  and require that  $v_k = v_{k-1} + \nu_k A^{-1}x_k \in S^{n-1}$ . This gives:

$$\langle v_{k-1} + \nu_k A^{-1}x_k, v_{k-1} + \nu_k A^{-1}x_k \rangle = 1 \Leftrightarrow 2\nu_k \langle v_{k-1}, A^{-1}x_k \rangle + \nu_k^2 \langle A^{-1}x_k, A^{-1}x_k \rangle = 0,$$

so that

$$(3.5) \quad \nu_k = -\frac{2\langle v_{k-1}, A^{-1}x_k \rangle}{\langle A^{-1}x_k, A^{-1}x_k \rangle} = \frac{2\langle v_k, A^{-1}x_k \rangle}{\langle A^{-1}x_k, A^{-1}x_k \rangle}.$$

By the way, these alternative expressions for  $\mu_k, \nu_k$  immediately imply the following result.

**PROPOSITION 3.1.** *The quantity  $I = \langle x, A^{-1}v \rangle$  is an integral of motion of the billiard map.*

**PROOF.** Comparing the both expressions in (3.4), we find:

$$(3.6) \quad \langle x_{k+1}, A^{-1}v_k \rangle = -\langle x_k, A^{-1}v_k \rangle.$$

Similarly, comparing the both expressions in from (3.5), we find:

$$(3.7) \quad \langle x_k, A^{-1}v_k \rangle = -\langle x_k, A^{-1}v_{k-1} \rangle.$$

Combining (3.6) with (the shifted version of) (3.7), we show the desired result.  $\square$

We are now in a position to give a Lagrangian formulation of the billiard map. Actually, there are two different such formulations. One of them is pretty well known. I do not know a reference for the second (“dual” one), however, it is based on the so called skew hodograph transformation introduced by Veselov [7, 4].

**First (traditional) Lagrangian formulation.** Eliminate variables  $v_k$  from equation (3.2):

$$\frac{x_{k+1} - x_k}{\mu_k} - \frac{x_k - x_{k-1}}{\mu_{k-1}} = \nu_k A^{-1}x_k,$$

or, according to the first expression in (3.3),

$$\frac{x_{k+1} - x_k}{|x_{k+1} - x_k|} - \frac{x_k - x_{k-1}}{|x_k - x_{k-1}|} = \nu_k A^{-1}x_k.$$

This can be considered as the Euler-Lagrange equation for the discrete Lagrange function

$$L : \mathcal{Q} \times \mathcal{Q} \rightarrow \mathbb{R}, \quad L(x_k, x_{k+1}) = |x_{k+1} - x_k|.$$

Here, one can interpret  $\nu_k$  as the Lagrange multiplier, which should be chosen so as to assure that  $x_{k+1} \in \mathcal{Q}$ , provided  $x_{k-1} \in \mathcal{Q}$  and  $x_k \in \mathcal{Q}$ .

**Second (“dual”) Lagrangian formulation.** Eliminate variables  $x_k$  from equation (3.2):

$$\frac{A(v_{k+1} - v_k)}{\nu_{k+1}} - \frac{A(v_k - v_{k-1})}{\nu_k} = \mu_k v_k,$$

or, according to the second expression in (3.3),

$$\frac{A(v_{k+1} - v_k)}{\langle v_{k+1} - v_k, A(v_{k+1} - v_k) \rangle^{1/2}} - \frac{A(v_k - v_{k-1})}{\langle v_k - v_{k-1}, A(v_k - v_{k-1}) \rangle^{1/2}} = \mu_k v_k.$$

This can be considered as the Euler-Lagrange equation for the discrete Lagrange function

$$L : S^{n-1} \times S^{n-1} \rightarrow \mathbb{R}, \quad L(v_k, v_{k+1}) = \langle v_{k+1} - v_k, A(v_{k+1} - v_k) \rangle^{1/2}.$$

Here, one can interpret  $\mu_k$  as the Lagrange multiplier, which should be chosen so as to assure that  $v_{k+1} \in S^{n-1}$ , provided  $v_{k-1} \in S^{n-1}$  and  $v_k \in S^{n-1}$ .

One can consider the billiard map as the map on the space  $\mathcal{L}$  of oriented lines in  $\mathbb{R}^n$ . This space can be parametrized as follows:

$$\mathcal{L} \ni \ell = \{x + tv : t \in \mathbb{R}\} \quad \leftrightarrow \quad (v, x) \in S^{n-1} \times \mathbb{R}^n.$$

Of course, in this representation one is allowed to replace  $x \in \ell$  by any other  $x' = x + t_0 v \in \ell$ . A canonical choice of the representative  $x'$  is the point on  $\ell$  nearest to the origin 0, that is  $x' = x - \langle x, v \rangle v$ . Clearly, this representative can be considered as  $x' \in T_v S^{n-1} \simeq T_v^* S^{n-1}$ . Thus, one can identify  $\mathcal{L}$  with  $T^* S^{n-1}$ , and the billiard map can be considered as a map  $B : T^* S^{n-1} \rightarrow T^* S^{n-1}$ , with the generating (Lagrange) function  $L : S^{n-1} \times S^{n-1} \rightarrow \mathbb{R}$ . As a consequence, the map  $B : T^* S^{n-1} \rightarrow T^* S^{n-1}$  preserves the canonical 2-form on  $T^* S^{n-1}$ .

#### 4. Commuting billiard maps

We use the following classical result (see [6])<sup>1</sup>.

**THEOREM 4.1.** *For any two quadrics  $\mathcal{Q}_\lambda$  and  $\mathcal{Q}_\mu$  from the confocal family*

$$\mathcal{Q}_\lambda = \{x \in \mathbb{R}^n : Q_\lambda(x) = 1\},$$

where

$$Q_\lambda(x) := \langle x, (A + \lambda I)^{-1} x \rangle = \sum_{i=1}^n \frac{x_i^2}{a_i^2 + \lambda},$$

the corresponding maps  $B_\lambda : T^* S^{n-1} \rightarrow T^* S^{n-1}$  and  $B_\mu : T^* S^{n-1} \rightarrow T^* S^{n-1}$  commute.

This places the billiards in confocal quadrics into the context of the theory of one-dimensional pluri-Lagrangian systems. With the help of this theory, we are going to prove the following statement [3].

<sup>1</sup>A conjecture by Tabachnikov [6] that the commutation of billiard maps characterizes confocal quadrics has been settled, under certain assumptions, in [1].

THEOREM 4.2. *The maps  $B_\mu : T^*S^{n-1} \rightarrow T^*S^{n-1}$  have a set of common integrals of motion given by*

$$F_i(v, x) = v_i^2 + \sum_{j \neq i} \frac{(x_i v_j - x_j v_i)^2}{a_i^2 - a_j^2}, \quad 1 \leq i \leq n.$$

Only  $n - 1$  of them are functionally independent, due to  $\sum_{i=1}^n F_i = \langle v, v \rangle = 1$ .

PROOF. According to Theorem 2.1, the expression

$$dL(\lambda, \mu) := L(v, \tilde{v}; \lambda) + L(\tilde{v}, \hat{v}; \mu) - L(\hat{v}, \tilde{v}; \lambda) - L(v, \hat{v}; \mu)$$

is a constant (depending maybe on  $\lambda, \mu$ ). The value of this constant is easily determined on a concrete billiard trajectory aligned along the big axis of either of the ellipsoids  $\mathcal{Q}_\lambda, \mathcal{Q}_\mu$ . For such a trajectory,  $v = (1, 0, \dots, 0)$ ,  $\tilde{v} = \hat{v} = -v$ , and  $\hat{\tilde{v}} = v$ . Recall that

$$L(v, \tilde{v}; \lambda) = \langle \tilde{v} - v, (A + \lambda I)(\tilde{v} - v) \rangle^{1/2}.$$

There follows immediately that  $dL(\lambda, \mu) = 0$ . Now Theorem 2.2 implies that the quantity  $\partial L(v, \tilde{v}; \lambda) / \partial \lambda$  is a common integral of motion for all  $B_\mu$ . A direct computation gives:

$$\begin{aligned} \frac{\partial L(v, \tilde{v}; \lambda)}{\partial \lambda} &= \frac{\langle v - \tilde{v}, v - \tilde{v} \rangle}{\langle v - \tilde{v}, (A + \lambda I)(v - \tilde{v}) \rangle^{1/2}} \\ &= \frac{\nu^2 \langle (A + \lambda I)^{-1} x, (A + \lambda I)^{-1} x \rangle}{\nu} \quad (\text{used eqs. (3.2), (3.3)}) \\ &= \nu \langle (A + \lambda I)^{-1} x, (A + \lambda I)^{-1} x \rangle \\ &= 2 \langle x, (A + \lambda I)^{-1} v \rangle \quad (\text{used eq. (3.5)}). \end{aligned}$$

Thus, the quantity

$$Q_\lambda(x, v) := \langle x, (A + \lambda I)^{-1} v \rangle = \sum_{i=1}^n \frac{x_i v_i}{\lambda + a_i^2}$$

with  $v \in S^{n-1}$ ,  $x \in \mathcal{Q}_\lambda$ , is an integral of motion of all maps  $B_\mu : \mathcal{L} \rightarrow \mathcal{L}$  (compare with Proposition 3.1). However, for the map  $B_\mu$ , the parametrization of the line  $\ell = \{x + tv : t \in \mathbb{R}\} \in \mathcal{L}$  by means of a point  $x \in \ell \cap \mathcal{Q}_\lambda$  is unnatural and rather inconvenient. Actually, it would be preferable to take, for any  $B_\mu$ , a representative from  $\ell \cap \mathcal{Q}_\mu$ , but a still better option would be an expression not depending on the representative at all. This is easily achieved. Observe that, as soon as  $Q_\lambda(x) = 1$ , we have

$$Q_\lambda^2(x, v) = Q_\lambda(v) - Q_\lambda(v)Q_\lambda(x) + Q_\lambda^2(x, v),$$

and the combination of the last two terms on the right-hand side is invariant under the change of the representative  $x \mapsto x + tv$ :

$$\begin{aligned} Q_\lambda^2(x, v) - Q_\lambda(v)Q_\lambda(x) &= \sum_{i,j=1}^n \frac{x_i v_i x_j v_j - x_i^2 v_j^2}{(\lambda + a_i^2)(\lambda + a_j^2)} = \sum_{1 \leq i \neq j \leq n} \frac{x_i v_i x_j v_j - x_i^2 v_j^2}{(\lambda + a_i^2)(\lambda + a_j^2)} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{1 \leq i \neq j \leq n} \frac{1}{a_j^2 - a_i^2} \left( \frac{1}{\lambda + a_i^2} - \frac{1}{\lambda + a_j^2} \right) (x_i v_i x_j v_j - x_i^2 v_j^2) \\
 &= \sum_{1 \leq i \neq j \leq n} \frac{1}{a_j^2 - a_i^2} \cdot \frac{1}{\lambda + a_i^2} \cdot (x_i v_i x_j v_j - x_i^2 v_j^2 + x_j v_j x_i v_i - x_j^2 v_i^2) \\
 &= \sum_{i=1}^n \frac{1}{\lambda + a_i^2} \sum_{j \neq i} \frac{(x_i v_j - x_j v_i)^2}{a_i^2 - a_j^2}.
 \end{aligned}$$

As a result, we see that the maps  $B_\mu : \mathcal{L} \rightarrow \mathcal{L}$  have the following integral of motion:

$$\sum_{i=1}^n \frac{1}{\lambda + a_i^2} \left( v_i^2 + \sum_{j \neq i} \frac{(x_i v_j - x_j v_i)^2}{a_i^2 - a_j^2} \right) = \sum_{i=1}^n \frac{F_i}{\lambda + a_i^2}.$$

Of course, this holds true also for each  $F_i$  individually. □

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**БИЛИЈАРИ У КОНФОКАЛНИМ КВАДРИКАМА  
КАО ПЛУРИ-ЛАГРАНЖЕВ СИСТЕМ**

РЕЗИМЕ. Теорију једнодимензионих плури-Лагранжевих система илуструјемо примером комутирајућих билијарских пресликавања у конфокалним квадрикама.

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