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ON HAMEL'S EQUATIONS

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Abstract. This paper reviews recent results on the extension of Hame's formalism to infinite-dimensional mechanical systems and to variational integrators. Of a particular interest are applications to the dynamics and numerical integration of systems with velocity constraints.

1. Introduction

This paper surveys some of the contemporary development of the formalism introduced by Hamel [21] in his habilitation thesis.

Hamel's formalism is an extension of Euler's ideas of using nonmaterial velocity in mechanics. Nonmaterial velocity of a system carries information about system's velocity, but is not the rate of change of system's configuration with respect to time. For a finite degree of freedom system, nonmaterial velocity is a collection of velocity components relative to a set of vector fields that span the fibers of the tangent bundle of the configuration space. In the finite-dimensional setting, this development was carried out by Hamel himself in [21].

One of the reasons for using nonmaterial velocity is that the Euler–Lagrange equations are not always effective for analyzing the dynamics, either analytically or numerically, of a mechanical system of interest. For example, it is difficult to study the motion of the Euler top if the Euler–Lagrange equations are used to represent the dynamics. On the other hand, the use of the angular velocity components relative to a body frame as pioneered by Euler [14] results in a much simpler representation of dynamics. In a similar fashion, Euler [15–17] uses *convective velocity* to represent the dynamics of an ideal incompressible fluid. Euler's approach was further developed by Lagrange [31] for reasonably general Lagrangians on the rotation group and by Poincaré [41] for arbitrary Lie groups (see [35] for details and history).

The nonmaterial velocity used in [31, 41] is associated with a group action. Hamel obtained the equations of motion in terms of nonmaterial velocity that is unrelated to a group action on the configuration space. Hamel's equations include

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both the Euler–Lagrange and Euler–Poincaré equations (for the rigid body for example) as special cases.

As clearly seen from his paper, Hamel was particularly motivated by nonholonomic mechanics. His formalism features the simplicity of analytic representation of constraints and the intrinsic absence of Lagrange multipliers in the equations of motion. It is exceptionally effective for studying (finite-dimensional) constrained systems and understanding their dynamics, both analytically and numerically; see e.g. [2, 7, 25–27, 39, 51] and references therein.

The paper concentrates on the two recent developments: Hamel's formalism for infinite-dimensional mechanical systems and discrete Hamel's mechanics. The former is motivated by the importance of nonmaterial velocity in continuum mechanics as demonstrated by Arnold [1] and Ebin and Marsden [13] as well as by the recent development of infinite-dimensional nonholonomic mechanics (see e.g. [3, 46–48]). Motivation for the latter includes the development of structure-preserving numerical integrators for mechanical systems with velocity constraints, an attempt to restore the concept of ideal constraints in the discrete setting, and related structural stability of variational and nonholonomic integrators. A loss of structural stability has been recently observed in [32, 40].

Being a survey, this paper leaves many technical details out. Interested readers are referred to [2, 44] for such details, applications to systems with symmetry, etc. We concentrate on the formulation of the Hamilton and Lagrange–d'Alembert variational principles for Hamel's equations. In the infinite-dimensional setting, this is done by computing the velocity variation, which requires a construction of an infinite-dimensional version of Hamel's coefficients. Of course, this variation formula is induced by the Lagrangian formulation of continuum mechanics. In the discrete setting, the situation is a bit more delicate. For the constraints to remain ideal after discretization, the velocity variation should be properly defined and in general cannot be derived from discrete Lagrangian mechanics. This is a reflection of an intrinsic dependence of an integrator on the discretization mesh.

The results are illustrated with examples, including an infinite-dimensional generalization of the Chaplygin sleigh and a global energy- and momentum-preserving integrator for the spherical pendulum.

2. Preliminaries

Lagrangian mechanics provides a systematic approach to deriving the equations of motion as well as establishes the equivalence of force balance and variational principles.

2.1. The Euler–Lagrange Equations. A *Lagrangian mechanical system* is specified by a smooth manifold Q called the *configuration space* and a function $L\colon TQ \to \mathbb{R}$ called the *Lagrangian*. In many cases, the Lagrangian is the difference of system's kinetic and potential energies, with the kinetic energy defined by a Riemannian metric and the potential energy being a smooth function on the configuration space Q. If necessary, non-conservative forces can be introduced. For instance, gyroscopic forces are represented by terms in L that are linear in the velocity, but this is not discussed in detail in this paper.

In local coordinates $q = (q^1, \dots, q^n)$ on the configuration space Q we write $L = L(q, \dot{q})$. The dynamics is given by the *Euler–Lagrange equations*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i}, \quad i = 1, \dots, n.$$

These equations were originally derived by Lagrange [31] in 1788 by requiring that simple force balance be *covariant*, i.e. expressible in arbitrary generalized coordinates. A variational derivation of the Euler–Lagrange equations, namely *Hamilton's principle* (see Theorem 2.1 below), came later in the work of Hamilton [22, 23] in 1834/35.

Let $q(t)$, $a \leqslant t \leqslant b$, be a smooth curve in Q. A *variation* of the curve $q(t)$ is a smooth map $\beta\colon [a, b] \times [-\varepsilon, \varepsilon] \to Q$ that satisfies the condition $\beta(t, 0) = q(t)$. This variation defines the vector field

$$\delta q(t) = \left.\frac{\partial \beta(t, \tau)}{\partial \tau}\right|_{\tau=0}$$

along the curve $q(t)$.

THEOREM 2.1. *The following statements are equivalent:*

(i) *The curve $q(t)$, where $a \leqslant t \leqslant b$, is a critical point of the* action functional

$$\int_a^b L(q, \dot{q})\, dt$$

on the space of curves in Q connecting q_a to q_b on the interval $[a, b]$, where we choose variations of the curve $q(t)$ that satisfy $\delta q(a) = \delta q(b) = 0$.

(ii) *The curve $q(t)$ satisfies the Euler–Lagrange equations.*

We point out here that this principle assumes that a variation of the curve $q(t)$ induces the variation $\delta\dot{q}(t)$ of its velocity vector according to the formula

$$\delta\dot{q}(t) := \frac{d}{dt}\delta q(t).$$

For more details and a proof, see e.g. [4, 35].

2.2. The Euler–Poincaré Equations. The classical *Euler equations* for a freely rotating rigid body read

$$J\dot{\Omega} = J\Omega \times \Omega,$$

where Ω is the body angular velocity and J is the inertia tensor. First derived by Euler [14], these equations, as well as the Euler equation for an incompressible inviscid fluid flow,

$$\frac{\partial v}{\partial t} + \nabla_v v = -\nabla p, \quad \operatorname{div} v = 0,$$

were generalized by Poincaré [41, 42] to *any* Lie algebra. These *Euler–Poincaré equations* for a Lagrangian $l(\xi)$ defined on a Lie algebra $\mathfrak{g}$ read

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = \pm \operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi}. \tag{2.1}$$

These equations are variational, with variations satisfying certain constraints, as the following theorem clarifies.

THEOREM 2.2. *Let $\mathfrak{g}$ be a Lie algebra and $l : \mathfrak{g} \to \mathbb{R}$ be a Lagrangian. The following statements are equivalent:*

(i) *The variational principle*

$$\delta \int_a^b l(\xi(t))\, dt = 0$$

holds on $\mathfrak{g}$, using variations of the form

$$\delta\xi = \dot{\eta} \pm \operatorname{ad}_\xi \eta,$$

where η vanishes at the endpoints.

(ii) *The Euler–Poincaré equations (2.1) hold.*

See [6, 9, 24, 34, 35] for details, history, and proofs.

2.3. Hamel's Equations. There are mechanical systems for which the Lagrangian and equations of motion have a simpler structure when the velocity components are measured against a frame that is unrelated to the system's local configuration coordinates. Examples of such a systems include the rigid body and an inviscid fluid flow.

Let $q = (q^1, \dots, q^n)$ be local coordinates on the configuration space Q and $u_i \in TQ$, $i = 1, \dots, n$, be smooth independent *local* vector fields on Q defined in the same coordinate neighborhood. In certain cases, some or all of u_i can be chosen to be *global* vector fields on Q.

Let $\xi = (\xi^1, \dots, \xi^n) \in \mathbb{R}^n$ be the components of the velocity vector $\dot{q} \in TQ$ relative to the frame $u(q) = (u_1(q), \dots, u_n(q))$, i.e.,

$$\dot{q} = \xi^i u_i(q).$$

The Lagrangian of the system written in the local coordinates (q, ξ) on the velocity phase space TQ reads

$$l(q, \xi) := L(q, \xi^i u_i(q)).$$

The coordinates (q, ξ) are a Lagrangian analogue of non-canonical variables of Hamiltonian mechanics.

Define the *Hamel coefficients* or *structure functions* $c_{ij}^k(q)$ by the equations

$$[u_i(q), u_j(q)] = c_{ij}^a(q) u_a(q), \quad a, i, j = 1, \dots, n.$$

The quantities $c_{ij}^a(q)$ vanish if and only if the vector fields $u_i(q)$, $i = 1, \dots, n$, commute.

Viewing u_i as vector fields on TQ whose fiber components equal 0 (that is, taking the vertical lift of the frame vector fields), one defines the directional derivatives $u_i[l]$ for a function $l \colon TQ \to \mathbb{R}$ in a usual way.

The evolution of the variables (q, ξ) is governed by the *Hamel equations*

$$\frac{d}{dt}\frac{\partial l}{\partial \xi^j} = c_{ij}^a \frac{\partial l}{\partial \xi^a}\xi^i + u_j[l], \tag{2.2}$$

$a, i, j = 1, \dots, n$, coupled with the equation $\dot{q} = \xi^i u_i(q)$. These equations were introduced in Hamel [21]. See also [7, 39] for details, history, and contemporary

geometric exposition. If $u_i = \partial/\partial q^i$, the Hamel equations become the Euler–Lagrange equations.

2.4. Ideal Constraints. Assume now that there are *velocity constraints* imposed on the system. We confine our attention to constraints that are linear and homogeneous in the velocity. Accordingly, we consider a configuration space Q and a distribution $\mathcal{D}$ on Q that describes these constraints. Recall that a distribution $\mathcal{D}$ is a collection of linear subspaces of the tangent spaces of Q; we denote these subspaces by $\mathcal{D}_q \subset T_qQ$, one for each $q \in Q$.

A curve $q(t) \in Q$ is said to *satisfy the constraints* if $\dot{q}(t) \in \mathcal{D}_{q(t)}$ for all t. This distribution will, in general, be nonintegrable, i.e., the constraints will be, in general, *nonholonomic*.[1]

As discussed in e.g. [11, 45], it is assumed in classical mechanics that the constraints imposed on the system can be replaced with the *reaction force*. This means that after the force is imposed on the *unconstrained* system, the constraint distribution $\mathcal{D} \subset TQ$ becomes a *conditional invariant manifold* of the *forced unconstrained* Lagrangian system whose dynamics on this invariant manifold is identical to that of the constrained system.

Definition 2.1. Constraints (either holonomic or nonholonomic) are called *ideal* if their reaction force at each $q \in Q$ belong to the null space $\mathcal{D}_q^\circ \subset T_q^*Q$ of $\mathcal{D}_q$.

As shown in [11, 45], the reaction forces of ideal constraints are defined uniquely at each each state $(q, \dot{q}) \in TQ$ satisfying the constraints. For a system subject to ideal constraints, the forced dynamics is equivalent to the Lagrange–d'Alembert principle.

Utilizing Hamel's formalism and assuming the ideal velocity constraints read $\xi^{m+1} = \cdots = \xi^n = 0$, the dynamics of the constrained system is given by (2.2) for $j = 1, \ldots, m$. The remaining $n - m$ equations serve for computing the reaction force, and do not affect the dynamics of the system. For the early development of these equations see Poincaré [41] and Hamel [21].

We refer the readers to [7, 35] for the history and development of the concept of ideal constraints and of variational principles for the Euler–Lagrange, Euler–Poincaré, and Hamel equations.

2.5. The Chaplygin Sleigh. The sleigh is a vertical blade moving on a horizontal plane. There is a single contact point of the blade and the plane. The center of mass of the blade coincides with this contact point. The velocity component of the contact point perpendicular to the blade is set to zero. The sleigh is often thought of as a balanced platform on the top of the blade. See Figure 1 where the platform, the blade, and the contact point are depicted as an oval, a bold segment and a bold dot, respectively.

Let θ be the angular orientation of the sleigh and (x, y) be the coordinates of the contact point as shown in Figure 1. The configuration space for the sleigh is the Euclidean group SE(2), which has the semidirect product structure $\mathrm{SO}(2) \circledS \mathbb{R}^2$.

[1] Constraints are nonholonomic if and only if they cannot be rewritten as *position* constraints.

We parametrize the elements of SE(2) as (θ, x, y). The *body frame* is

$$\frac{\partial}{\partial\theta}, \quad \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}, \quad -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}$$

Using this frame,

$$\dot\theta = \omega, \quad \dot x = v^1\cos\theta - v^2\sin\theta, \quad \dot y = v^1\sin\theta + v^2\cos\theta. \tag{2.3}$$

That is, ω is the angular velocity of the sleigh relative to the vertical line through the contact point, and (v^1, v^2) are the components of the linear velocity of the contact point in the directions along and orthogonal to the blade, respectively. Thus, the constraint reads $v^2 = 0$.

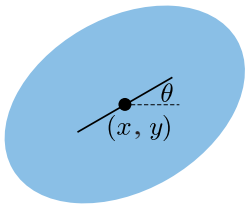


FIGURE 1. The Chaplygin sleigh.

It is convenient to utilize the complex configuration variable $z = x + iy$ on the plane. Similarly, the linear velocity relative to the body frame is written as $v = v^1 + iv^2$, and formulae (2.3) become

$$\dot\theta = \omega, \quad \dot z = e^{i\theta}v, \tag{2.4}$$

whereas the constraint in this complex representation reads

$$v = \bar v.$$

Denote the mass and the moment of inertia of the sleigh by m and J. The Lagrangian is just the kinetic energy of the sleigh, which is the sum of the kinetic energies of the linear and rotational modes of the body:

$$l = \tfrac{1}{2}(J\omega^2 + mv\bar v).$$

The elements of the algebra $\mathfrak{se}(2) = \mathfrak{so}(2) \circledS \mathbb{C}$ in the complex representation are written as $(i\omega, v)$, where $i\omega \in \mathfrak{so}(2)$, $\omega \in \mathbb{R}$, and $v \in \mathbb{C}$. The bracket operation on the algebra $\mathfrak{se}(2)$ reads

$$[(i\omega_1, v_1), (i\omega_2, v_2)] = (0, i\omega_1 v_2 - i\omega_2 v_1). \tag{2.5}$$

Using (2.5), the constrained Hamel equations for the Chaplygin sleigh are computed to be

$$\dot\omega = 0, \quad \dot v = 0. \tag{2.6}$$

Recall that in (2.6) the quantity v is real-valued.

Solving equations (2.4) and (2.6) gives

$$\omega = \text{const}, \quad v = \text{const}, \quad \theta = \theta_0 + \omega t, \quad z = \begin{cases} z_0 + e^{i\theta_0} v t & \text{if } \omega = 0 \\ z_0 - \frac{iv}{\omega} e^{i\theta_0}(e^{i\omega t} - 1) & \text{if } \omega \neq 0 \end{cases}$$

and so generically the sleigh moves along a circle at a uniform rate.

3. Infinite-Dimensional Systems

Here and in Section 4, system's number of degrees of freedom is not assumed finite. As the use of frames and bases in the infinite-dimensional setting is unnatural and not always possible, we introduce a coordinate-free approach to Hamel's formalism. Thus, instead of frames, we use linear velocity substitutions. These substitutions, however, are not induced by a (local) configuration coordinate change.

Below, the functional-analytic details are mostly omitted. These details and can be found in [44]. It is safe to assume that all infinite-dimensional configuration spaces are Banach manifolds, however, the results remain correct for much more general settings, such as convenient spaces.

3.1. Lagrangian Mechanics. Let M be an infinite-dimensional smooth manifold modeled on a vector space W and let TM be its kinematic tangent bundle with the projection $\pi_M \colon TM \to M$. Consider the *initial inclusion map* $i \colon Q \to M$ and the pullback vector bundle $P = i^*TM$. The map i satisfies the following property: A map $f \colon N \to Q$ is smooth if and only if $i \circ f \colon N \to M$ is smooth. Note that Q is usually not a submanifold of M, see [30] for details.

A *Lagrangian* is a smooth function $L \colon P \to \mathbb{R}$. The dynamics for this Lagrangian is defined in a usual way by *Hamilton's principle*: The curve $\gamma \colon [a, b] \to Q$ is a trajectory if

$$\delta \int_a^b L \, dt = 0$$

along γ.

3.2. Hamel's Formalism and Hamilton's Principle. Let U be an open subset of M containing $q \in Q$ and let

$$U \times W \ni (q, \xi) \mapsto (q, \Psi_q \xi) \in \pi_M^{-1}(U) \subset TM \tag{3.1}$$

be a fiber-preserving diffeomorphism that is linear in the second input. Hence, for each $q \in U$, both $\Psi_q \colon W \to T_qM$ and $\Psi_q^{-1} \colon T_qM \to W$ are invertible bounded linear operators smoothly dependent on q in an open subset $i^{-1}(U) \subset Q$.

As the Lagrangian fails to be defined, in general, on TM, it is necessary to consider various forms of equations of motion, such as weak and strong forms. Further, the objects involved may not even be defined on the entire TM but only on its dense subset. See [35] for details and references.

For each $\xi \in W$, the operator $\Psi_q \colon W \to T_qM$ introduced in (3.1) outputs the vector $\Psi_q \xi \in T_qM$ for each $q \in U$. Thus, each $\xi \in W$ defines the vector field

$$\Psi \xi(q) := \Psi_q \xi$$

on U.

Given two vectors $\xi, \eta \in W$, define an antisymmetric bilinear operation $[\,\cdot\,,\,\cdot\,]_q : W \times W \to W$ by

$$\Psi_q[\xi,\eta]_q := [\Psi\,\xi, \Psi\,\eta](q), \tag{3.2}$$

where $[\,\cdot\,,\,\cdot\,]$ is the Jacobi–Lie bracket on the manifold M. Next, for arbitrary ξ, η, $\zeta \in W$, we have

$$\begin{aligned}\Psi_q([[\xi,\eta]_q,\zeta]_q + [[\eta,\zeta]_q,\xi]_q + [[\zeta,\xi]_q,\eta]_q)\\ = [[\Psi\,\xi, \Psi\,\eta], \Psi\,\zeta](q) + [[\Psi\,\eta, \Psi\,\zeta], \Psi\,\xi](q) + [[\Psi\,\zeta, \Psi\,\xi], \Psi\,\eta](q) = 0,\end{aligned}$$

implying, in view of invertibility of Ψ_q, the Jacobi identity for the bracket $[\,\cdot\,,\,\cdot\,]_q$. Therefore, for each $q \in U$, the space W with the operation $[\,\cdot\,,\,\cdot\,]_q$ is a Lie algebra, denoted hereafter W_q.

The *dual* of $[\,\cdot\,,\,\cdot\,]_q$ is, by definition, the operation $[\,\cdot\,,\,\cdot\,]_q^*$: $W_q \times W_q^* \to W_q^*$ given by

$$\big\langle [\xi,\alpha]_q^*, \eta\big\rangle_W := \big\langle \alpha, [\xi,\eta]_q\big\rangle_W, \quad \xi,\eta \in W, \quad \alpha \in W^*.$$

As in the finite-dimensional setting, let $\dot q$ and δq denote the velocity and the virtual displacement at $q \in Q$. From now on, the inverse images of $\dot q$ and δq are written as $\xi, \eta \in W$, that is, $\dot q = \Psi_q\,\xi$ and $\delta q = \Psi_q\,\eta$.

Interpreting ξ as an independent variable that replaces $\dot q$ (locally) defines the Lagrangian as a smooth function of (q,ξ) on $i^{-1}U \times W$:

$$l(q,\xi) := L(q, \Psi_q\,\xi). \tag{3.3}$$

The equations of motion written when (q,ξ) are selected as (local) coordinates on the velocity phase space are called the *Hamel equations*.

Recall that, given a smooth curve $q(t) \in Q$, $t \in [a,b]$, its *variation* is a smooth one-parameter family of curves

$$[a,b] \times [-\varepsilon,\varepsilon] \ni (t,\tau) \mapsto \beta(t,\tau) \in Q \quad \text{such that} \quad \beta(t,0) = q(t).$$

An *infinitesimal variation* δq is defined by

$$\delta q(t,\tau) := \frac{\partial}{\partial\tau}\beta(t,\tau).$$

When this field is evaluated along the curve $q(t)$, we write $\delta q(t)$, i.e.,

$$\delta q(t) := \delta q(t,0) = \frac{\partial}{\partial\tau}\Big|_{\tau=0}\beta(t,\tau). \tag{3.4}$$

Thus, a variation of a smooth curve $q(t) \in Q$ defines a curve $\eta(t) \in W$:

$$\delta q(t) = \Psi_{q(t)}\,\eta(t).$$

THEOREM 3.1 (Hamilton's Principe for Hamel's Equations). *Let $L\colon P \to \mathbb{R}$ be a Lagrangian and l be its representation in local coordinates (q,ξ). Then, the following statements are equivalent:*

(i) *The curve $q(t)$, where $a \leqslant t \leqslant b$, is a critical point of the action functional*

$$\int_a^b L(q,\dot q)\,dt$$

on the space of curves in Q connecting q_a to q_b on the interval $[a,b]$, where we choose variations of the curve $q(t)$ that satisfy $\delta q(a) = \delta q(b) = 0$.

(ii) *The curve $q(t)$ satisfies the weak form of the Euler–Lagrange equations*

$$\int_a^b \left\langle \frac{\delta L}{\delta q} - \frac{d}{dt}\frac{\delta L}{\delta \dot q}, \delta q \right\rangle dt = 0. \tag{3.5}$$

If, additionally, $\delta L/\delta q \in T_q^*M$ and $i_* T_q Q$ is dense in T_qM for every $q \in Q$, the curve $q(t)$ satisfies the strong form of the Euler–Lagrange equations,

$$\frac{d}{dt}\frac{\delta L}{\delta \dot q} - \frac{\delta L}{\delta q} = 0. \tag{3.6}$$

(iii) *The curve $(q(t), \xi(t))$ is a critical point of the functional*

$$\int_a^b l(q,\xi)\, dt \tag{3.7}$$

with respect to variations $\delta\xi$, induced by the variations

$$\delta q = \Psi_q\, \eta, \tag{3.8}$$

and given by

$$\delta\xi = \dot\eta + [\xi,\eta]_q. \tag{3.9}$$

(iv) *The curve $(q(t), \xi(t))$ satisfies the* weak form of the Hamel equations

$$\int_a^b \left\langle \Psi_q^* \frac{\delta l}{\delta q} + \left[\xi, \frac{\delta l}{\delta \xi}\right]_q^* - \frac{d}{dt}\frac{\delta l}{\delta \xi}, \eta \right\rangle dt = 0, \quad \eta \in \Psi_q^{-1}(T_qQ) \tag{3.10}$$

coupled with the equations $\dot q = \Psi_q\, \xi$. If $\delta l/\delta q \in T_q^*M$ and $i_* T_q Q$ is dense in T_qM for every $q \in Q$, the curve $(q(t), \xi(t))$ satisfies the strong form of the Hamel equations

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = \left[\xi, \frac{\delta l}{\delta \xi}\right]_q^* + \Psi_q^* \frac{\delta l}{\delta q} \tag{3.11}$$

coupled with the equation $\dot q = \Psi_q\, \xi$.

For the early development of these equations in the finite-dimensional setting see [21, 41].

Proof. The equivalence of (i) and the weak form of the Euler–Lagrange equations (3.5) is proved by integration by parts:

$$\delta \int_a^b L(q,\dot q)\, dt = \int_a^b \left(\left\langle \frac{\delta L}{\delta q}, \delta q \right\rangle + \left\langle \frac{\delta L}{\delta \dot q}, \delta \dot q \right\rangle \right) dt = \int_a^b \left\langle \frac{\delta L}{\delta q} - \frac{d}{dt}\frac{\delta L}{\delta \dot q}, \delta q \right\rangle dt.$$

The strong form of the Euler–Lagrange equations (3.6) follows from a standard contradiction argument: Skipping technical details,

$$\frac{\delta L}{\delta q} - \frac{d}{dt}\frac{\delta L}{\delta \dot q} \not\equiv 0$$

implies

$$\delta \int_a^b L(q,\dot q)\,dt \neq 0$$

for a certain variation of $q(t)$, which finishes the proof of the equivalence of (i) and (ii).

To prove the equivalence of (i) and (iii), we first compute the quantities $\delta\dot q$ and $d(\delta q)/dt$. Recall that

$$\delta q(t) = \frac{\partial}{\partial\tau}\bigg|_{\tau=0} \beta(t,\tau) = \Psi_{q(t)}\,\eta(t), \quad \text{where} \quad \eta(t)\in W.$$

Using the definition (3.4) of the field δq,

$$\delta\Psi_{q(t)} = \frac{\partial}{\partial\tau}\bigg|_{\tau=0} \Psi_{\beta(t,\tau)} = \delta q(t)[\Psi_{q(t)}] = (\Psi_{q(t)}\,\eta(t))[\Psi_{q(t)}]. \tag{3.12}$$

Hereafter, $v[f]$ denotes the derivative of the function f along the vector filed v; in particular, in (3.12) an operator-valued function is differentiated.

Similarly,

$$\tfrac{d}{dt}\Psi_{q(t)} = \dot q(t)[\Psi_{q(t)}] = (\Psi_{q(t)}\,\xi(t))[\Psi_{q(t)}],$$

and therefore

$$\delta\dot q = (\Psi_q\,\eta)[\Psi_q]\xi + \Psi_q\,\delta\xi, \quad \tfrac{d}{dt}\delta q = (\Psi_q\,\xi)[\Psi_q]\eta + \Psi_q\,\dot\eta.$$

From $\delta\dot q = \frac{d}{dt}\delta q$, we obtain

$$\Psi_q(\delta\xi - \dot\eta) = (\Psi\,\xi)[\Psi\,\eta](q) - (\Psi\,\eta)[\Psi\,\xi](q) = [\Psi\,\xi,\Psi\,\eta](q) = \Psi_q[\xi,\eta]_q,$$

which implies formula (3.9).

To prove the equivalence of (iii) and the weak form of Hamel's equations (3.10), we use the above formula and compute the variation of the action (3.7):

$$\begin{aligned}
\delta\int_a^b l(q,\xi)\,dt &= \int_a^b \left(\left\langle \frac{\delta l}{\delta q}, \delta q\right\rangle + \left\langle \frac{\delta l}{\delta\xi}, \delta\xi\right\rangle\right)dt \\
&= \int_a^b \left(\left\langle \frac{\delta l}{\delta q}, \Psi_q\,\eta\right\rangle + \left\langle \frac{\delta l}{\delta\xi}, \dot\eta + [\xi,\eta]_q\right\rangle\right)dt \\
&= \int_a^b \left\langle \Psi_q^*\frac{\delta l}{\delta q} + \left[\xi, \frac{\delta l}{\delta\xi}\right]_q^* - \frac{d}{dt}\frac{\delta l}{\delta\xi}, \eta\right\rangle dt.
\end{aligned}$$

If $\delta L/\delta q \in T_q^*M$ and i_*T_qQ is dense in T_qM for every $q\in Q$, then for each t the subspace $\Psi_{q(t)}^{-1}(i_*T_{q(t)}Q)$ is dense in W, and the variational derivative vanishes if and only if the strong form of the Hamel equations (3.11) is satisfied. □

Example 3.1. For an incompressible fluid flow in a compact domain $D\subset\mathbb{R}^3$ with a smooth boundary the configuration space is the group $\mathrm{Diff}(D)$ of volume-preserving diffeomorphism of D. Let $q(t)$ be a curve in this group, one may think of $q(t)$ as a particular fluid flow. Following Euler [15–17], one typically uses the *spatial velocity* $\xi = \dot q\circ q^{-1} \in T_e\,\mathrm{Diff}(D)$. Selecting $W = T_e\,\mathrm{Diff}(D) = \mathcal{X}(D)$ (the space of smooth vector fields on D tangent to the boundary) and $\Psi_q = TR_q$ gives

$\dot q = \Psi_q\,\xi$. Therefore, the use of spatial velocity in fluid dynamics is an instance of infinite-dimensional Hamel's formalism. The variation formula (3.9) becomes

$$\delta\xi = \dot\eta - \operatorname{ad}_\xi \eta \equiv \dot\eta + [\xi,\eta],$$

where $[\,\cdot\,,\,\cdot\,]$ is the Jacobi–Lie bracket on D. The dynamics, in the form of Hamel's equations, reads

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} + \operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi} = 0,$$

which are the Euler–Poincaré equations, as established by [1, 41].

REMARK 3.1. The *convective representation* of fluid dynamics is straightforward to obtain by setting $\Psi_q = TL_q$. See [20] for details on the convective representation in continuum mechanics.

EXAMPLE 3.2. For an inextensible string moving in the plane (see Figure 2) the configuration manifold is the space of smooth embeddings $\operatorname{Emb}([0,1],\mathbb{R}^2)$. We will view $\mathbb{R}^2$ as a complex plane. Given $z \in \operatorname{Emb}([0,1],\mathbb{C})$, the inextensibility condition

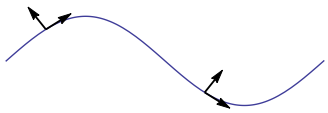


FIGURE 2. An inextensible planar string.

reads $\|z_s\| = 1$, $0 \leqslant s \leqslant 1$. For simplicity, we assume no resistance to bending. Therefore, the Lagrangian reads

$$L(z) = \int_0^1 \frac{1}{2}\big(\|\dot z\|^2 - \lambda(\|z_s\|^2 - 1)\big)\,ds,$$

where $\lambda\colon [0,1] \to \mathbb{R}$ is the Lagrange multiplier (tension). The boundary conditions for the Lagrange multiplier are a part of the requirement $\delta L = 0$. For a free motion of a string, these conditions read

$$\lambda|_{s=0} = \lambda|_{s=1} = 0. \tag{3.13}$$

Let

$$\dot z = \Psi_z\,\xi := z_s\xi, \tag{3.14}$$

so the velocity components to be used to construct Hamel's equations are represented by a complex-valued function $\xi = \xi(s,t)$. Geometrically, the real and imaginary parts of ξ are the tangent and normal velocity components of the points of the string.

The Lagrangian becomes

$$l = \int_0^1 \frac{1}{2}\big(\bar z_s z_s \bar\xi\xi - \lambda(\bar z_s z_s - 1)\big)\,ds,$$

in which the density should be understood as a function of $(z_s, \bar z_s, \xi, \bar\xi)$ and the Lagrange multiplier λ.

Next, the formula (3.2) for the string becomes

$$[\Psi\,\xi,\Psi\,\eta](z)\frac{d}{d\tau}\Big|_{\tau=0}\big((z+\tau z_s\xi)_s\eta-(z+\tau z_s\eta)_s\xi\big)=z_s(\xi_s\eta-\eta_s\xi)=\Psi_z[\xi,\eta]_z.$$

That is,

$$[\xi,\eta]_z=\xi_s\eta-\xi\eta_s. \tag{3.15}$$

Instead of establishing the formulae for the dual bracket and dual operator Ψ^*, it is more efficient in this example to directly work with the variational principle. We have:

$$\frac{\delta l}{\delta z}\delta z+\frac{\delta l}{\delta\xi}\delta\xi+\frac{\delta l}{\delta\bar z}\delta\bar z+\frac{\delta l}{\delta\bar\xi}\delta\bar\xi, \tag{3.16}$$

and since l is real-valued, the two last terms are obtained from the first two by conjugation. Thus, it is sufficient to evaluate the last two terms in (3.16):

$$\begin{aligned}\frac{\delta l}{\delta\bar z}\delta\bar z+\frac{\delta l}{\delta\bar\xi}\delta\bar\xi&=\frac{\delta l}{\delta\bar z}\delta\bar z+\frac{\delta l}{\delta\bar\xi}\big(\bar\xi_s\bar\eta-\bar\xi\bar\eta_s\big)-\frac{d}{dt}\frac{\delta l}{\delta\bar\xi}\bar\eta\\&=\int_0^1\frac12\Big(\big(\lambda z_s-z_s\bar\xi\xi\big)_s\delta\bar z+\bar z_sz_s\xi\big(\bar\xi_s\bar\eta-\bar\xi\bar\eta_s\big)-\frac{d}{dt}\big(\bar z_sz_s\xi\big)\bar\eta\Big)\,ds\\&\quad-\tfrac12\bar z_sz_s\big(\lambda-\bar\xi\xi\big)\bar\eta\big|_{s=0}^{s=1}\\&=\int_0^1\frac12\Big(z_s\bar z_{ss}\xi\bar\xi+z_s\bar z_s\xi\bar\xi_s+\lambda_s\bar z_sz_s+\lambda z_{ss}\bar z_s-\frac{d}{dt}\big(\bar z_sz_s\xi\big)\Big)\bar\eta\,ds-\frac12\bar z_sz_s\lambda\bar\eta\big|_{s=0}^{s=1},\end{aligned}$$

which, after imposing the constraint $\bar z_sz_s=1$, implies

$$\dot\xi=\xi\bar\xi_s+\lambda_s+i\varkappa\big(\lambda-\bar\xi\xi\big) \tag{3.17}$$

as well as the tension conditions (3.13). Here $\varkappa$ is the (signed) curvature of the curve $[0,1]\ni s\mapsto z(s)\in\mathbb{C}$. The right-hand side of (3.17) gives an explicit representation of the terms on the right-hand side of Hamel's equations (3.11) for the string.

Remark 3.2. Alternatively, one defines the operator Ψ by

$$\Psi_z\,\xi:=\frac{z_s}{|z_s|}\xi.$$

Because of the constraint $|z_s|=1$, the resulting Hamel equations are (3.17). However, the bracket $[\xi,\eta]_z$ is now given not by (3.15) but by a slightly different formula. This latter bracket is in fact induced by the (standard) bracket of the Lie algebra of the infinite-dimensional group

$$G=\big\{[0,1]\ni s\mapsto g(s)\in\mathrm{SE}(2)\big\}.$$

To see that, the string dynamics should be interpreted as a motion on G specified by the degenerate Lagrangian

$$l=\int_0^1\frac12\big(\bar\xi\xi-\lambda(\bar z_sz_s-1)\big)\,ds$$

subject to the constraint

$$\frac{z_s}{|z_s|} = e^{i\theta}, \tag{3.18}$$

where (θ, z) are the (standard) coordinates on $\mathrm{SE}(2) = \mathrm{SO}(2) \circledS \mathbb{C}$ and ξ is the $\mathbb{C}$-component of the body velocity $g^{-1}\dot{g}$. One then composes the $\mathbb{C}$-component of the Euler–Poincaré equations on G and imposes constraint (3.18). This results in the string equations (3.17). It is worth noticing that the derivation of these equations using formula (3.14) is simper and more efficient.

4. Systems with Constraints

Here we discuss infinite-dimensional dynamics with velocity constraints. To simplify the exposition, we assume in the rest of the section that $\delta L/\delta q \in T_q^* M$ and $i_* T_q Q$ is dense in $T_q M$ for every $q \in Q$, and thus all results will be stated for strong equations of motion. Similar statements for weak equations are straightforward to obtain.

4.1. The Lagrange–d'Alembert Principle. Recall that in this paper the constraints imposed on the system are assumed linear and homogeneous in the velocity. Such constraints are specified by a vector subbundle $\mathcal{D}$ of the bundle $P = i^* TM$. The base of this subbundle is the manifold Q. This subbundle will, in general, be nonintegrable.

The condition for a curve to satisfy the constraints, $\dot{q}(t) \in \mathcal{D}_{q(t)}$ for all t, is, by itself, insufficient for the development of constrained mechanics. One needs a mechanism for constructing the vector field that captures the dynamics of the constrained system. For the ideal constraints in the finite-dimensional setting, this is accomplished by a projection. Thus, constraints define a submanifold of the velocity phase space and a projection onto this submanifold.

For a projection to be meaningful in the infinite-dimensional case, $\mathcal{D}$ is required to be a locally *splitting subbundle* of P. That is, for each $q \in Q$ there exists a chart (U, h) of M such that

$$Th(\pi_Q^{-1}(U) \cap \mathcal{D}) = h(U) \times W^{\mathcal{D}},$$

where the closed subspace $W^{\mathcal{D}}$ of the model space W is *splitting, or complemented*, i.e., there is a closed subspace $W^{\mathcal{U}}$ of W such that $W^{\mathcal{D}} \oplus W^{\mathcal{U}} = W$ and the projection $\pi^{\mathcal{D}}$ uniquely determined by setting $(\operatorname{Ker} \pi^{\mathcal{D}}, \operatorname{Im} \pi^{\mathcal{D}}) = (W^{\mathcal{U}}, W^{\mathcal{D}})$ is *continuous*.

The following Lagrange–d'Alembert principle is known to be equivalent to the dynamics of systems with ideal constraints:

Definition 4.1. The *Lagrange–d'Alembert equations of motion* for the system are those determined by

$$\delta \int_a^b L(q, \dot{q})\, dt = 0,$$

where we choose variations $\delta q(t)$ of the curve $q(t)$ that satisfy $\delta q(a) = \delta q(b) = 0$ and $\delta q(t) \in \mathcal{D}_{q(t)}$ for each t where $a \leqslant t \leqslant b$.

This principle is supplemented by the condition that the curve $q(t)$ itself satisfies the constraints. Note that we take the variation *before* imposing the constraints; that is, we do not impose the constraints on the family of curves defining the variation. This is well known to be important to obtain the correct mechanical equations (see [5] for a discussion and references).

The Lagrange–d'Alembert principle is equivalent to the equations

$$\frac{d}{dt}\frac{\delta L}{\delta \dot q} - \frac{\delta L}{\delta q} \in \mathcal{D}_q^\circ, \quad \dot q \in \mathcal{D}_q.$$

Here,

$$\mathcal{D}_q^\circ = \{a \in T_q^* M \mid \langle a, v\rangle = 0, v \in \mathcal{D}_q\}.$$

4.2. The Constrained Hamel Equations. Given a system with velocity constraints, that is, a Lagrangian $L\colon P \to \mathbb{R}$ and constraint distribution $\mathcal{D}$, select the operators $\Psi_q\colon W \to T_qQ$ on $U \subset Q$ such that there exist closed subspaces $W^{\mathcal{D}}$, $W^{\mathcal{U}} \subset W$, $W = W^{\mathcal{D}} \oplus W^{\mathcal{U}}$, and $\Psi_q = \Psi_q^{\mathcal{D}} \oplus \Psi_q^{\mathcal{U}}$, where $\Psi_q^{\mathcal{D}}\colon W^{\mathcal{D}} \to \mathcal{D}_q$ and $\Psi_q^{\mathcal{U}}\colon W^{\mathcal{U}} \to \mathcal{U}_q$ and their inverses are bounded linear operators smoothly dependent on $q \in U$. One way to choose the operators Ψ_q is to use the above subbundle chart. In general, $U \neq Q$, as numerous finite-dimensional examples demonstrate.

Each $\dot q \in TM$ is then uniquely decomposed as

$$\dot q = \Psi_q\, \xi^{\mathcal{D}} + \Psi_q\, \xi^{\mathcal{U}}, \quad \text{where} \quad \Psi_q\, \xi^{\mathcal{D}} \in \mathcal{D}_q, \tag{4.1}$$

i.e., $\Psi_q\, \xi^{\mathcal{D}}$ is the component of $\dot q$ along $\mathcal{D}_q$. Similarly, each $\alpha \in W^*$ uniquely decomposes as

$$\alpha = \alpha_{\mathcal{D}} + \alpha_{\mathcal{U}},$$

where $\alpha_{\mathcal{D}}$ and $\alpha_{\mathcal{U}}$ denote the components of α along the duals of $W^{\mathcal{D}}$ and $W^{\mathcal{U}}$, respectively. We have

$$\alpha_{\mathcal{D}} = (\pi^{\mathcal{D}})^* \circ \alpha|_{W^{\mathcal{D}}} \quad \text{and} \quad \alpha_{\mathcal{U}} = \big(\mathrm{id} - (\pi^{\mathcal{D}})^*\big) \circ \alpha|_{W^{\mathcal{U}}},$$

where $(\pi^{\mathcal{D}})^*$ is the adjoint of $\pi^{\mathcal{D}}$. Using (4.1), the constraints read

$$\xi = \xi^{\mathcal{D}} \quad \text{or} \quad \xi^{\mathcal{U}} = 0, \tag{4.2}$$

which implies

$$\delta\xi = \delta\xi^{\mathcal{D}} \quad \text{or} \quad \delta\xi^{\mathcal{U}} = 0. \tag{4.3}$$

The Lagrange–d'Alembert principle then implies the following theorem:

THEOREM 4.1. *The dynamics of a system with velocity constraints is represented by the* strong form of constrained Hamel equations

$$\left(\frac{d}{dt}\frac{\delta l}{\delta \xi} - \left[\xi^{\mathcal{D}}, \frac{\delta l}{\delta \xi}\right]_q^* - \Psi_q^* \frac{\delta l}{\delta q}\right)_{\mathcal{D}} = 0, \quad \xi^{\mathcal{U}} = 0, \quad \dot q = \Psi_q\, \xi^{\mathcal{D}}. \tag{4.4}$$

EXAMPLE 4.1. Consider an inextensible string moving in the plane subject to the vanishing normal velocity constraint, see Figure 2. This is a nonholonomic constraint. One may think of a motion of a sharp string on the horizontal ice.

Using the notations introduced in Example 3.2, the constraint reads $\xi = \bar{\xi}$, i.e., $\xi \in \mathbb{R}$. Equations (4.4) for the constrained string thus become

$$\dot{\xi} = \xi\xi_s + \lambda_s, \tag{4.5}$$

$$\dot{z} = z_s\xi, \quad \xi = \bar{\xi}. \tag{4.6}$$

along with the inextensibility condition.

It is geometrically evident (or can be confirmed with a simple calculation) that the inextensibility condition in the presence of constraint $\xi = \bar{\xi}$ implies $\xi_s = 0$. That is, all points of the string have the same speed, and (4.5) becomes

$$\dot{\xi} = \lambda_s.$$

Using the boundary conditions (3.13), we conclude that

$$\dot{\xi} = 0,$$

i.e., $\xi = \text{const}$ throughout the motion. This is in agreement with the motion of Chaplygin sleigh for which the velocity of the contact point relative to the body frame is constant. Unlike the sleigh, the constrained string motion is not completely determined by the initial state: Any solution of (4.6) is of the form

$$z = \phi(s + \xi t),$$

where ϕ is an arbitrary twice-differentiable complex-valued function. The initial conditions define ϕ on the segment $[0, 1]$. Outside this segment, the function ϕ is unknown, unless, for example, the motion of the front end of the string is prescribed. The motion of the constrained string is therefore *purely kinematic*: The string follows its front end, which moves at a constant speed.

This behavior is similar to that of the degenerate Chaplygin sleigh specified by the Lagrangian $l = \frac{1}{2}\xi\bar{\xi}$ and constraint $\xi = \bar{\xi}$, where $\xi = e^{-i\theta}\dot{z}$. For the degenerate sleigh, the dynamics reads

$$\dot{\xi} = 0, \quad \dot{z} = e^{i\theta}\xi,$$

where $\theta(t)$ is an arbitrary function. Thus, the motions are not identified by the initial conditions.

EXAMPLE 4.2. Consider the Chaplygin sleigh with an inextensible string attached at and allowed to rotate around the contact point of the sleigh and the plane, as shown in Figure 3. Assume that the string is constrained as in Example 4.1.

This system is SE(2)-invariant. The string position z is measured relative to the sleigh. Below, ω and v denote the angular and linear velocity components of the sleigh.

The *absolute* velocity of the string, η, is computed to be

$$\eta = z_s^{-1}(\dot{z} + v + i\omega z),$$

This effectively defines the operator Ψ. The Lagrangian, which is system's kinetic energy, reads

$$l = \frac{1}{2}(J\omega^2 + m\bar{v}v) + \int_0^1 \frac{1}{2}(\bar{\eta}\eta - \lambda(\bar{z}_s z_s - 1))\, ds.$$

The constraint is given by $\bar{\eta} = \eta$.

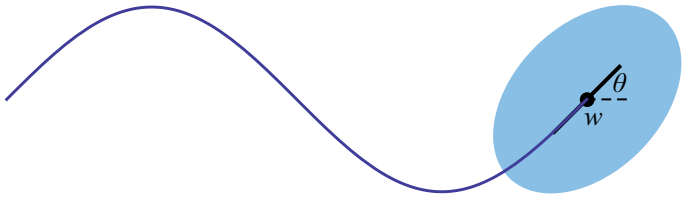


FIGURE 3. The Chaplygin sleigh coupled to a constrained string.

Hamel's equations for this system become

$$J\dot{\omega} = 0, \tag{4.7}$$

$$m\dot{v} = \lambda|_{s=0}, \tag{4.8}$$

$$\dot{\eta} = \eta\eta_s + \lambda|_s, \tag{4.9}$$

where ξ and η are real-valued. These equations should be amended with the coupling conditions

$$z|_{s=0} = 0, \quad z_s|_{s=0} = 1, \quad \eta|_{s=0} = v. \tag{4.10}$$

These simply state that the string is attached to the blade at the contact point of the blade and the velocity of the attached string end equals the velocity of the blade.

Arguing as in Example 4.1, one concludes that η is independent of s. Thus, equation (4.9) becomes

$$\dot{\eta} = \lambda_s. \tag{4.11}$$

Equation (4.7) implies $\omega = \text{const}$.

The tension λ is obtained by integrating (4.11) with respect to s and, since $\lambda|_{s=1} = 0$, we conclude that

$$\lambda = (s-1)\dot{\eta}. \tag{4.12}$$

Therefore, $\lambda|_{s=0} = -\dot{\eta}$, which in combination with (4.8) and (4.10) yields $v = \text{const}$. The velocity coupling condition then implies that the blade moves at the constant speed $\eta = v$. Using (4.12), we conclude that $\lambda = 0$.

Summarizing, the Chaplygin sleigh with the constrained string attached generically undergoes uniform circular motion. Nongeneric trajectories are straight lines. The string (possibly after some period of time) follows the trajectory of the contact point of the sleigh.

It is interesting to point out that in this example the shape dynamics (string's motion) is modulated by the group dynamics (skate's motion). This is the opposite of typical reconstruction in finite-dimensional constrained systems discussed in [5].

We note also that the qualitative dynamics of this system—uniform circular or straight line motion—is consistent with the behavior of integrable Hamiltonian systems. One may raise the question of whether it is integrable in a more precise

sense—with infinitely-many conserved quantities. We intend toaddress this issue in a forthcoming publication.

5. Discrete Mechanics

Discrete mechanics is obtained by discretizing variational principles. See Marsden and West [36] and references therein for a more detailed discussion of discrete mechanics.

5.1. Discrete Lagrangian Mechanics. A discrete analogue of Lagrangian mechanics is the result of discretization of Hamilton's principle; this approach underlies the construction of variational integrators.

A key notion is that of the *discrete Lagrangian*, which is a map $L^d\colon Q\times Q\to\mathbb{R}$ that approximates the action integral along an exact solution of the Euler–Lagrange equations joining the configurations $q_k, q_{k+1}\in Q$,

$$L^d(q_k,q_{k+1})\approx \operatorname*{ext}_{q\in\mathcal{C}([0,h],Q)}\int_0^h L(q,\dot q)\,dt,$$

where $\mathcal{C}([0,h],Q)$ is the space of curves $q:[0,h]\to Q$ with $q(0)=q_k$, $q(h)=q_{k+1}$, and ext denotes extremum.

In the discrete setting, the action integral of Lagrangian mechanics is replaced by an *action sum*

$$S^d(q_0,q_1,\dots,q_N)=\sum_{k=0}^{N-1}L^d(q_k,q_{k+1}),$$

where $q_k\in Q$, $k=0,1,\dots,N$, is a finite sequence in the configuration space. The equations are obtained by the *discrete Hamilton principle*, which extremizes the discrete action given fixed endpoints q_0 and q_N. Taking the extremum over $q_1,\dots,q_{N-1}$ gives the *discrete Euler–Lagrange equations*

$$D_1L^d(q_k,q_{k+1})+D_2L^d(q_{k-1},q_k)=0,\quad k=1,\dots,N-1. \tag{5.1}$$

Here and below, D_iF denotes the partial derivative of the function F with respect to its ith input. Equations (5.1) implicitly define the *update map* $\Phi\colon Q\times Q\to Q\times Q$, where $\Phi(q_{k-1},q_k)=(q_k,q_{k+1})$ and $Q\times Q$ replaces the velocity phase space TQ of continuous-time Lagrangian mechanics.

In the case that Q is a vector space, it may be convenient to use $(q_{k+1/2},v_{k,k+1})$, where $q_{k+1/2}=\frac12(q_k+q_{k+1})$ and $v_{k,k+1}=\frac1h(q_{k+1}-q_k)$, as a state of a discrete mechanical system. In such a representation, the discrete Lagrangian becomes a function of $(q_{k+1/2},v_{k,k+1})$, and the discrete Euler–Lagrange equations read

$$\begin{aligned}\tfrac12\big(D_1L^d(q_{k-1/2},v_{k-1,k})+D_1L^d(q_{k+1/2},v_{k,k+1})\big)\\ +\tfrac1h\big(D_2L^d(q_{k-1/2},v_{k-1,k})-D_2L^d(q_{k+1/2},v_{k,k+1})\big)=0.\end{aligned}$$

These equations are equivalent to the variational principle

$$\begin{aligned}\delta S^d=\sum_{k=0}^{N-1}\big(&D_1L^d(q_{k+1/2},v_{k,k+1})\,\delta q_{k+1/2}\\ &+D_2L^d(q_{k+1/2},v_{k,k+1})\,\delta v_{k,k+1}\big)=0,\end{aligned} \tag{5.2}$$

where the variations $\delta q_{k+1/2}$ and $\delta v_{k,k+1}$ are induced by the variations δq_k and are given by the formulae

$$\delta q_{k+1/2} = \tfrac{1}{2}(\delta q_{k+1} + \delta q_k), \quad \delta v_{k,k+1} = \tfrac{1}{h}(\delta q_{k+1} - \delta q_k).$$

The discrete Hamel formalism introduced below may be interpreted as a generalization of the representation (5.2) of discrete mechanics.

5.2. Discrete Hamel's Equations. In the rest of the paper we assume that Q is a finite-dimensional vector space. Start with a sequence of configurations $\{q_k\}_{k=0}^N$. Given a parameter $\tau \in [0,1]$, define the points $q_{k+\tau} := (1-\tau)q_k + \tau q_{k+1}$ for each $0 \leqslant k \leqslant N-1$. Given $\Psi_q : W \to T_qQ$, the velocity components introduced by this operator at $q_{k+\tau}$ are denoted $\xi_{k,k+1} = (\xi^1_{k,k+1}, \dots, \xi^n_{k,k+1})$. Similar to [8] and [29], the phase space for the suggested discretization of Hamel's equation is the tangent bundle TQ. In local coordinates (q,ξ) on TQ, the discrete Lagrangian $l^d : TQ \to \mathbb{R}$ reads $l^d = l^d(q_{k+\tau}, \xi_{k,k+1})$. To discretize a continuous-time system, we suggest the following procedure:

(i) Select operators $\Psi_q : W \to T_qQ$ and identify the continuous-time Lagrangian $l(q,\xi)$, as in (3.3).
(ii) Construct the discrete Lagrangian using the formula

$$l^d(q_{k+\tau}, \xi_{k,k+1}) = hl(q_{k+\tau}, \xi_{k,k+1}).$$

The action sum then is

$$s^d = \sum_{k=0}^{N-1} l^d(q_{k+\tau}, \xi_{k,k+1}), \tag{5.3}$$

which is an approximation of the action integral (3.7) of the continuous-time system.

Given $\tau \in [0,1]$, define $\eta_{k+\tau}$ by the formula

$$\eta_{k+\tau} = (1-\tau)\eta_k + \tau\eta_{k+1}. \tag{5.4}$$

The quantities η_k, η_{k+1}, and $\eta_{k+\tau}$ will be used below to establish the discrete analogues of the variation formulae (3.8) and (3.9).

Define the *discrete conjugate momentum* by

$$\mu_{k,k+1} := D_2 l^d(q_{k+\tau}, \xi_{k,k+1}). \tag{5.5}$$

Below, we use the notations

$$l^d_{k+\tau} := l^d(q_{k+\tau}, \xi_{k,k+1}), \quad \psi_{q+\tau} := \Psi^*_{q+\tau} D_1 l^d(q_{k+\tau}, \xi_{k,k+1}),$$

etc.

THEOREM 5.1. *The sequence* $\left(q_{k+\tau}, \xi_{k,k+1}\right) \in TQ$ *satisfies the* discrete Hamel equations

$$\begin{aligned} &\tfrac{1}{h}(\mu_{k-1,k} - \mu_{k,k+1}) + \tau\psi_{k-1+\tau} + (1-\tau)\psi_{k+\tau} \\ &\qquad + \tau[\xi_{k-1,k}, \mu_{k-1,k}]^*_{q_{k-1+\tau}} + (1-\tau)[\xi_{k,k+1}, \mu_{k,k+1}]^*_{q_{k+\tau}} = 0 \end{aligned} \tag{5.6}$$

if and only if

$$\delta s^d = \delta \sum_{k=0}^{N-1} l^d(q_{k+\tau}, \xi_{k,k+1}) = 0,$$

where

$$\delta q_{k+\tau} = \Psi_{q_{k+\tau}} \eta_{k+\tau}, \tag{5.7}$$

$$\delta \xi_{k,k+1} = \tfrac{1}{h}(\eta_{k+1} - \eta_k) + [\xi_{k,k+1}, \eta_{k+\tau}]_{q_{k+\tau}}. \tag{5.8}$$

Here $\eta_0 = \eta_N = 0$, *and* $\eta_{k+\tau}$ *is defined in* (5.4), $k = 0, \dots, N-1$.

In order to obtain a complete system of equations, one supplements (5.6) with a discrete analogue of the kinematic equation $\dot q = \Psi_q \xi$. There is a certain freedom in doing that. For now, we assume this discrete analogue to be

$$\frac{\Delta q_k}{h} = \Psi_{k+\tau} \xi_{k,k+1}.$$

We will use a different discretization of the kinematic equation to construct an integrator for the spherical pendulum in Section 6.

REMARK 5.1. Unlike the continuous-time case, the formulae for variations (5.7) and (5.8) *cannot* be derived in a manner presented in the proof of Theorem 3.1. The situation here is somewhat similar to the issue encountered and resolved by Chetaev in his work [10] on the equivalence of the Lagrange–d'Alembert and Gauss principles for systems with nonlinear nonholonomic constraints. Recall that Chetaev's approach was to *define* variations in such a way that the two principles become equivalent.

PROOF. Using formulae (5.7) and (5.8) and computing the variation of the action sum (5.3), one obtains

$$\begin{aligned}
\delta s^d &= \sum_{k=0}^{N-1} D_1 l^d(q_{k+\tau}, \xi_{k,k+1})\, \delta q_{k+\tau} + D_2 l^d(q_{k+\tau}, \xi_{k,k+1})\, \delta \xi_{k,k+1} \\
&= \sum_{k=0}^{N-1} \left\langle D_1 l^d_{k+\tau}, \Psi_{q_{k+\tau}} \eta_{k+\tau} \right\rangle \\
&\quad + \left\langle D_2 l^d_{k+\tau}, (\eta_{k+1} - \eta_k)/h + [\xi_{k,k+1}, \eta_{k+\tau}]_{q_{k+\tau}} \right\rangle \\
&= \sum_{k=1}^{N-1} \left\langle \tfrac{1}{h}(\mu_{k-1,k} - \mu_{k,k+1}), \eta_k \right\rangle \\
&\quad + \left\langle \psi_{k+\tau} + [\mu_{k,k+1}, \xi_{k,k+1}]^*_{q_{k+\tau}}, (1-\tau)\eta_k + \tau \eta_{k+1} \right\rangle \\
&= \sum_{k=1}^{N-1} \left\langle \tfrac{1}{h}(\mu_{k-1,k} - \mu_{k,k+1}), \eta_k \right\rangle + \left\langle \tau \psi_{k-1+\tau} + (1-\tau)\psi_{k+\tau}, \eta_k \right\rangle \\
&\quad + \left\langle \tau [\mu_{k-1,k}, \xi_{k-1,k}]^*_{q_{k-1+\tau}} + (1-\tau)[\mu_{k,k+1}, \xi_{k,k+1}]^*_{q_{k+\tau}}, \eta_k \right\rangle.
\end{aligned}$$

Thus, vanishing of δs^d for arbitrary η_k, $k = 1, \dots, N-1$, is equivalent to discrete Hamel's equations (5.6). □

The formulae for variations (5.7) and (5.8) in the discrete setting are motivated by the following observations. First, recall that in the continuous-time setting the formula (3.9) for $\delta\xi$ follows from the formula

$$\delta(\Psi_q\xi) - \frac{d}{dt}(\Psi_q\eta) = 0. \tag{5.9}$$

A discrete analogue of $\delta(\Psi_q\xi)$ is relatively straightforward to obtain. Indeed, using the formula

$$\delta q_{k+\tau} = \Psi_{q_{k+\tau}}\eta_{k+\tau} \equiv \Psi_{q_{k+\tau}}((1-\tau)\eta_k + \tau\eta_{k+1}))$$

and the interpretation of the operator δ as a directional derivative, just like in formula (3.12), one obtains

$$\delta\Psi_{q_{k+\tau}} = (\Psi\eta_{k+\tau}[\Psi])_{k+\tau},$$

and therefore

$$\begin{aligned}\delta(\Psi_{q_{k+\tau}}\xi_{k+1}) &= \delta\Psi_{q_{k+\tau}}\xi_{k,k+1} + \Psi_{q_{k+\tau}}\delta\xi_{k,k+1}\\ &= \Psi_{q_{k+\tau}}\delta\xi_{k,k+1} + (\Psi\eta_{k+\tau}[\Psi\xi_{k,k+1}])_{k+\tau}.\end{aligned}$$

Hereafter, $u[f]$ denotes the derivative of the function f along the vector field u.

However, a discrete analogue of the formula $\frac{d}{dt}(\Psi_q\eta)$ is not immediately available, as the operation of time differentiation is not intrinsically present in the discrete setting. A workaround that we suggest is to view the transition from q_k to q_{k+1} as a motion along a straight line segment at a uniform rate:

$$q_{k+\tau} = (1-\tau)q_k + \tau q_{k+1}, \quad 0 \leqslant \tau \leqslant 1, \tag{5.10}$$

so that $q_{k+\tau} = q_k$ when $\tau = 0$ and $q_{k+\tau} = q_{k+1}$ when $\tau = 1$. Since the time step is h, the analogue of continuous-time velocity is $\Delta q_k/h$. From (5.10),

$$\frac{\Delta q_k}{h} = \frac{1}{h}\frac{dq_{k+\tau}}{d\tau},$$

leading to an interpretation of the operator

$$\frac{1}{h}\frac{d}{d\tau}$$

as a discrete analogue of time differentiation of continuous-time mechanics.

The discrete analogue of the term $\frac{d}{dt}(\Psi_q\eta)$ thus is

$$\begin{aligned}\frac{1}{h}\frac{d}{d\tau}(\Psi_{q_{k+\tau}}\eta_{k+\tau}) &= \frac{1}{h}\frac{d\Psi_{q_{k+\tau}}}{d\tau}\eta_{k+\tau} + \Psi_{q_{k+\tau}}\frac{1}{h}\frac{d\eta_{k+\tau}}{d\tau}\\ &= \Psi_{q_{k+\tau}}\frac{1}{h}\frac{d\eta_{k+\tau}}{d\tau} + (\Psi\xi_{k,k+1}[\Psi\eta_{k+\tau}])_{k+\tau}\\ &= \Psi_{q_{k+\tau}}\frac{\eta_{k+1}-\eta_k}{h} + (\Psi\xi_{k,k+1}[\Psi\eta_{k+\tau}])_{k+\tau}.\end{aligned}$$

Summarizing, the discrete analogue of (5.9) reads

$$\Psi_{q_{k+\tau}}\delta\xi_{k,k+1} = \Psi_{q_{k+\tau}}\frac{\eta_{k+1}-\eta_k}{h} + [\Psi\xi_{k,k+1}, \Psi\eta_{k+\tau}]_{q_{k+\tau}},$$

which implies formula (5.8) for variation $\delta\xi$.

5.3. Discrete Nonholonomic Systems. Discrete nonholonomic systems (nonholonomic integrators) were introduced by Cortés and Martínez in [12].

Let Q be a configuration space. According to Cortés and Martínez, a *discrete nonholonomic mechanical system* on Q is characterized by:

(i) A discrete Lagrangian $L^d : Q \times Q \to \mathbb{R}$.
(ii) A constraint distribution $\mathcal{D}$ on Q.
(iii) A *discrete constraint manifold* $\mathcal{D}^d \subset Q \times Q$ which has the same dimension as $\mathcal{D}$ and satisfies the condition $(q, q) \in \mathcal{D}^d$ for all $q \in Q$.

The dynamics is given by the following *discrete Lagrange–d'Alembert principle* (see [12]):

$$\sum_{k=1}^{N-1} \big(D_1 L^d(q_k, q_{k+1}) + D_2 L^d(q_{k-1}, q_k)\big)\delta q_k = 0, \quad \delta q_k \in \mathcal{D}_{q_k}, \quad (q_k, q_{k+1}) \in \mathcal{D}^d.$$

As pointed out in [18, 19], the discrete constraint manifold should be carefully selected when a continuous-time nonholonomic system is discretized. For the details on the properties of discrete nonholonomic systems we refer the reader to papers [12,18,19,38]. In a recent paper [29], a somewhat different approach to discretizing nonholonomic systems has been suggested.

Cortés and Martínez also study the dynamics of discrete Chaplygin systems. In particular, given a continuous-time Chaplygin system, they discretize the *Euler–Lagrange equations with constraint reactions*, and conclude that, in general, the resulting discrete system is inconsistent with the outcome of their discrete Lagrange–d'Alembert principle. In other words, *the concept of ideal constraints is not acknowledged by their discretization procedure.*

Lynch and Zenkov [32, 33] proved that the discrete dynamics defined by the Lagrange–d'Alembert principle of Cortés and Martínez may lack *structural stability.* For example, it is possible for the discretization of a continuous-time Chaplygin system to change the dimension and/or stability of manifolds of relative equilibria of the said continuous-time system.

Below, we shall show that a different definition of the discrete Lagrange–d'Alembert principle exists that is free of the aforementioned issues. In particular, the dimension and stability of manifolds of relative equilibria are kept intact if this new version of the Lagrange–d'Alembert principle is utilized.

5.4. Hamel's Formalism for Discrete Nonholonomic Systems. Recall that the Lagrange–d'Alembert principle for continuous-time nonholonomic systems assumes that the variation of action is carried out before imposing the constraints. The outcome is the constrained Hamel equations, as discussed in Section 4.2. In a similar manner, we accept that the dynamics of a discrete nonholonomic system is determined by the *discrete Lagrange–d'Alembert principle*, obtained by *first* taking the variation of the discrete action (5.3) using variations (5.7) and (5.8) subject to the discrete analogue of (4.3), and *then* imposing the discrete constraints. We emphasize that the definition of the discrete Lagrange–d'Alembert principle given

here *is not* the same as the definition of Cortés and Martínez reproduced in Section 5.3.

In the continuous-time setting, the constraints are represented by formula (4.2). We thus suggest that, under the same assumptions on the frame selection as in Section 4.2, the discrete constraints are

$$\xi_{k,k+1} = \xi^{\mathcal{D}}_{k,k+1} \quad \text{or} \quad \xi^{\mathcal{U}}_{k,k+1} = 0.$$

The discrete analogue of (4.3) is

$$\eta_k = \eta^{\mathcal{D}}_k \quad \text{or} \quad \eta^{\mathcal{U}}_k = 0.$$

Arguing like in Section 4.2, one proves the discrete analogue of Theorem 4.1:

THEOREM 5.2. *The dynamics of a discrete system with velocity constraints is given by the* constrained discrete Hamel equations

$$\begin{aligned}(5.11)\qquad &\tfrac{1}{h}(\mu_{k-1,k} - \mu_{k,k+1})_{\mathcal{D}} + (\tau\psi_{k-1+\tau} + (1-\tau)\psi_{k+\tau})_{\mathcal{D}} \\ &\quad + (\tau[\xi^{\mathcal{D}}_{k-1,k}, \mu_{k-1,k}]^*_{q_{k-1+\tau}} + (1-\tau)[\xi^{\mathcal{D}}_{k,k+1}, \mu_{k,k+1}]^*_{q_{k+\tau}})_{\mathcal{D}} = 0,\end{aligned}$$

where $\mu_{k,k+1}$ *is given by formula* (5.5).

Of a special interest is the value $\tau = 1/2$, in which case one verifies that the order of approximation of (5.11) is 2.

5.5. Stability. In this section we link up stability of relative equilibria of Chaplygin systems with structural stability of nonholonomic integrators.

Let H be a commutative Lie group acting freely and properly on the manifold Q. Consider a commutative Chaplygin system characterized by an H-invariant Lagrangian $L\colon Q \to \mathbb{R}$ and constraint distribution $\mathcal{D}$. It is well-known that the dynamics of such systems reduces to its shape dynamics on Q/H. See [5, 37] for details.

Assume additionally that this dynamics on Q/H is invariant with respect to the action of a commutative group G on Q/H. The elements of the group G are denoted g, and we assume that the action of G on Q/H is free and proper, so that Q/H is a principal fiber bundle with the structure group G. Thus, locally, there exist the bundle coordinates (r, g) on Q/H.

Under certain assumptions (see e.g. [28, 50]), the dynamics has a manifold of relative equilibria whose dimension equals $\dim G$. These relative equilibria are the solutions that in the bundle coordinates (r, g) read

$$r = r_e, \quad \dot{g} = \eta_e.$$

As established in Karapetyan [28], some of these relative equilibria may be *partially asymptotically stable*. Karapetyan justifies stability using the center manifold stability analysis, which, for nonholonomic systems under consideration, reduces to verifying that the nonzero spectrum of linearization of the dynamics on Q/H at the relative equilibrium of interest belongs to the left half-plane.[2]

[2]The stability analysis of relative equilibria of nonholonomic systems has a long history, starting form the results of Walker [49] and Routh [43]; see [50] for some of this history and for the energy-momentum method for nonholonomic systems.

Partially asymptotically stable relative equilibria are a part of the ω-limit set of system's dynamics. Similarly, relative equilibria that become partially asymptotically stable after the time reversal are a part of system's α-limit set.

It is important for a long-term numerical integrator to preserve the manifold of relative equilibria and their stability types. Indeed, if the limit sets of an integrator are different from the limit sets of the continuous-time dynamics, this integrator will not adequately simulate the continuous-time dynamics over long time intervals.

As shown in [32,33], it is possible for the discrete Lagrange–d'Alembert principle of Cortés and Martínez to produce discretizations that fail to preserve the manifold of relative equilibria. For instance, it may change the dimension of this manifold, thus changing the structure of the limit sets. Informally, the origin of this effect can be explained as follows: The discrete Lagrange–d'Alembert principle of Cortés and Martínez is capable of introducing reactions that correspond to non-ideal constraints. A typical example would be a reaction force with a dissipative component, whose discrete counterpart causes the aforementioned changes of relative equilibria.

Introduce a discrete Chaplygin system by constructing discrete constrained Hamel's equations (5.11) for the given continuous Chaplygin system. A relative equilibrium of this discrete Chaplygin system with commutative symmetry is a solution

$$r_k = \text{const}, \quad \Delta g_k = \text{const}.$$

Assume now that $\tau = 1/2$ in equations (5.11). Let $h > 0$ be the time step.

THEOREM 5.3 (Lynch and Zenkov [32]). *The introduced discretization preserves the manifold of relative equilibria of the continuous-time Chaplygin system; that is, $r_k = r_e$, $\Delta g_k = h\eta_e$ is a relative equilibrium of the discretization if and only if $r = r_e$, $\dot{g} = \eta_e$ is a relative equilibrium of the continuous-time system. The conditions for partial asymptotic stability of the equilibria of the continuous-time system and of its discretization are the same.*

Summarizing, the discrete Lagrange–d'Alembert principle proposed in this paper ensures the necessary conditions for structural stability of the associated nonholonomic integrator.

6. The Spherical Pendulum

Here we outline the results of Zenkov, Leok, and Bloch [51] on the applications of the discrete Hamel formalism to the energy-momentum-preserving integrator for the spherical pendulum.

6.1. The Spherical Pendulum as a Degenerate Rigid Body. Consider a spherical pendulum whose length is r and mass is m. We view the pendulum as a point mass moving on the sphere of radius r centered at the origin of $\mathbb{R}^3$. The development here is based on the representation

$$\dot{\boldsymbol{\mu}} = \boldsymbol{\mu} \times \boldsymbol{\xi} + mg\boldsymbol{\gamma} \times \boldsymbol{a}, \tag{6.1}$$

$$\dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\xi} \tag{6.2}$$

of pendulum's dynamics; that is, the pendulum is viewed as a rigid body rotating about a fixed point. This rigid body is of course *degenerate*, with the inertia tensor $\mathcal{I} = \operatorname{diag}\{mr^2, mr^2, 0\}$. Here $\boldsymbol{\xi}$ is the angular velocity of the pendulum, $\boldsymbol{\mu}$ is its angular momentum, $\boldsymbol{\gamma}$ is the unit vertical vector (and thus the constraint $\|\boldsymbol{\gamma}\| = 1$ is imposed), and $\boldsymbol{a}$ is the vector from the origin to the center of mass, which for the pendulum is its bob, all written relative to the body frame. Throughout the rest of the paper, the boldface characters represent three-dimensional vectors. The kinetic and potential energies of the pendulum are

$$K = \tfrac{1}{2}\langle \boldsymbol{\mu}, \boldsymbol{\xi}\rangle \equiv \tfrac{1}{2}\langle \mathcal{I}\,\boldsymbol{\xi}, \boldsymbol{\xi}\rangle, \quad U = mg\langle \boldsymbol{\gamma}, \boldsymbol{a}\rangle \equiv mgr\gamma^3,$$

and the Lagrangian reads

$$l(\boldsymbol{\xi}, \boldsymbol{\gamma}) = \tfrac{1}{2}\langle \mathcal{I}\,\boldsymbol{\xi}, \boldsymbol{\xi}\rangle - mg\langle \boldsymbol{\gamma}, \boldsymbol{a}\rangle. \tag{6.3}$$

This Lagrangian is invariant with respect to rotations about $\boldsymbol{\gamma}$, and therefore the vertical component of the *spatial angular momentum* is conserved.

There are two independent components in the vector equation (6.1). We emphasize that the representation (6.1) and (6.2) of the dynamics of the pendulum, though redundant, eliminates the use of local coordinates on the sphere, such as spherical coordinates. Spherical coordinates, while being a nice theoretical tool, introduce artificial singularities at the north and south poles. That is, the equations of motion written in spherical coordinates have denominators vanishing at the poles, but this has nothing to do with the physics of the problem and is solely caused by the geometry of the spherical coordinates. Thus, the use of spherical coordinates in calculations is not advisable.

Another important remark is that the length of the vector $\boldsymbol{\gamma}$ is a *conservation law* of equations (6.1) and (6.2), and thus adding the constraint $\|\boldsymbol{\gamma}\| = 1$ *does not* result in a system of differential-algebraic equations. The latter are known to be a nontrivial object for numerical integration.

Equations (6.1) and (6.2) may be interpreted in a number of ways. In the above, we viewed them as the dynamics of a degenerate rigid body. Since the moment of inertia relative to the direction of the vector $\boldsymbol{a}$ is zero, the third component of the body angular momentum vanishes,

$$\mu_3 = \frac{\partial l}{\partial \xi^3} = 0,$$

and thus there are only two nontrivial equations in (6.1). Thus, one needs five equations to capture the pendulum dynamics. This reflects the fact that rotations about the direction of the pendulum have no influence on the pendulum's motion.

The dynamics then can be simplified by setting

$$\xi^3 = 0, \tag{6.4}$$

which leads to an interpretation of equations (6.1) and (6.2) as the dynamics of the *heavy Suslov top*[3] with a rotationally-invariant inertia tensor and constraint (6.4).

[3]See [4, 39, 45] for the Suslov top.

Summarizing, the dynamics becomes

$$\dot{\boldsymbol{\mu}} = mg\boldsymbol{\gamma} \times \boldsymbol{a}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\xi}, \quad \langle \boldsymbol{\xi}, \boldsymbol{a} \rangle = 0. \tag{6.5}$$

These equations are in fact the constrained Hamel equations, the reconstruction equation, and the constraint, written in the redundant configuration coordinates $\boldsymbol{\gamma} = (\gamma^1, \gamma^2, \gamma^3)$; see [51] for details. Recall that the length of $\boldsymbol{\gamma}$ is the conservation law, so that the constraint $\|\boldsymbol{\gamma}\| = 1$ does not need to be imposed, but the appropriate level set of the conservation law has to be selected.

Our discretization is based on this point of view, i.e., the discrete dynamics will be written in the form of discrete Hamel's equations. The discrete dynamics will posses the discrete version of the conservation law $\|\boldsymbol{\gamma}\| = \text{const}$, so that the algorithm should be capable, in theory, of preserving the length of $\boldsymbol{\gamma}$ up to machine precision.

6.2. Variational Discretization for the Spherical Pendulum. The integrator for the spherical pendulum is constructed by discretizing equations (6.5).

Let the positive real constant h be the time step. Applying the mid-point rule to (6.3), the discrete Lagrangian is computed to be

$$l^d(\boldsymbol{\xi}_{k,k+1}, \boldsymbol{\gamma}_{k+1/2}) = \tfrac{h}{2}\langle \mathcal{I}\,\boldsymbol{\xi}_{k,k+1}, \boldsymbol{\xi}_{k,k+1} \rangle - hU(\boldsymbol{\gamma}_{k+1/2}).$$

Here $\boldsymbol{\xi}_{k,k+1} = (\xi^1_{k,k+1}, \xi^2_{k,k+1}, 0)$ is the discrete analogue of the angular velocity $\boldsymbol{\xi} = (\xi^1, \xi^2, 0)$ and $\boldsymbol{\gamma}_{k+1/2} = \frac{1}{2}(\boldsymbol{\gamma}_{k+1} + \boldsymbol{\gamma}_k)$. The discrete dynamics then reads

$$\tfrac{1}{h}\mathcal{I}(\boldsymbol{\xi}_{k,k+1} - \boldsymbol{\xi}_{k-1,k}) = mg(\boldsymbol{\gamma}_{k+1/2} + \boldsymbol{\gamma}_{k-1/2}) \times \boldsymbol{a}, \tag{6.6}$$

$$\tfrac{1}{h}(\boldsymbol{\gamma}_{k+1/2} - \boldsymbol{\gamma}_{k-1/2}) = \tfrac{1}{2}(\boldsymbol{\gamma}_{k+1/2} + \boldsymbol{\gamma}_{k-1/2}) \times \tfrac{1}{2}(\boldsymbol{\xi}_{k,k+1} + \boldsymbol{\xi}_{k-1,k}). \tag{6.7}$$

We reiterate that there is a certain flexibility in setting up the discrete analogue (6.7) of the continuous-time reconstruction equation (6.2). Our choice may be justified in a number of ways, one of them being energy conservation by the discrete dynamics.

The structure-preserving properties of the proposed integrator for the spherical pendulum are summarized in the following theorem.

THEOREM 6.1 (Zenkov, Leok, and Bloch [51]). *The discrete spherical pendulum dynamics* (6.6) *and* (6.7) *preserves the energy, the vertical component of the spatial angular momentum, and the length of* $\boldsymbol{\gamma}$.

We refer the readers to [51] for the proof and details.

6.3. Simulations. Here we present simulations of the dynamics of the spherical pendulum using the integrator constructed in Section 6.2. For simulations, we select the parameters of the system and the time step to be

$$m = 1 \text{ kg}, \quad r = 9.8 \text{ m}, \quad h = .2 \text{ s}.$$

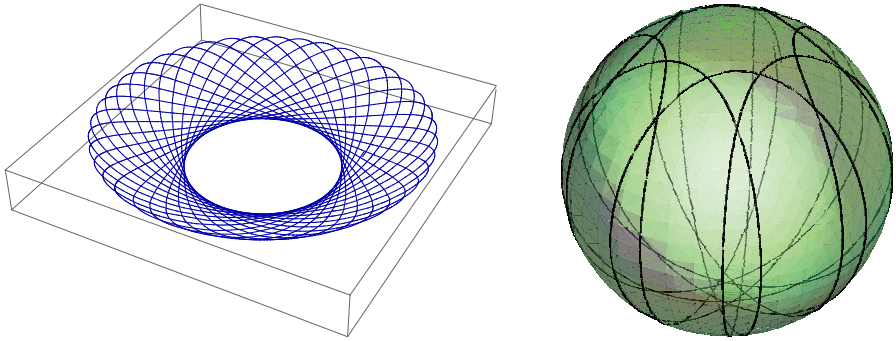
The trajectory of the bob of the pendulum with the initial conditions

$$\xi^1_0 = .6 \text{ rad/s}, \qquad \xi^2_0 = 0 \text{ rad/s},$$

$$\gamma^1_0 = .3 \text{ m}, \qquad \gamma^2_0 = .2 \text{ m}, \qquad \gamma^3_0 = -\sqrt{1 - \left(\gamma^1_0\right)^2 - \left(\gamma^2_0\right)^2} \text{ m}$$

is shown in Figure 4a. As expected, it reveals the quasiperiodic nature of pendulum's dynamics.

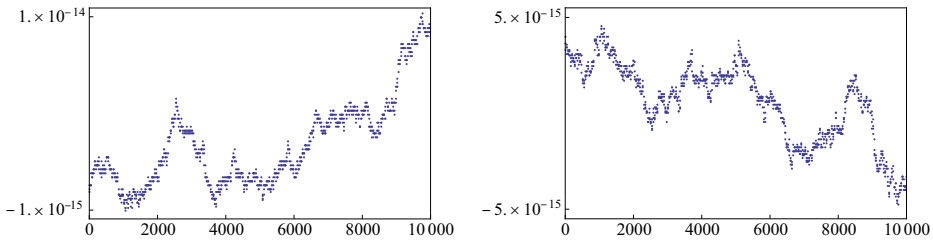
Figure 4b shows pendulum's trajectory that crosses the equator. This simulation demonstrates the global nature of the algorithm, and also seems to do a good job of hinting at the geometric conservation properties of the method.



(A) Small oscillations (B) Large oscillations

FIGURE 4. Trajectories of the pendulum calculated with the Hamel integrator.

Theoretically, if one solves the nonlinear equations exactly, and in the absence of numerical roundoff error, the Hamel variational integrator should exactly preserve the length constraint and the energy. In practice, Figure 5a demonstrates that $\|\boldsymbol{\gamma}\|$ stays to within unit length to about 10^{-14} after 10,000 iterations. Figure 5b demonstrates numerical energy conservation, and the energy error is to about $5 \cdot 10^{-15}$ after 10,000 iterations. Indeed, one notices that the energy error tracks the length error of the simulation, which is presumably due to the relationship between the length of the pendulum and the potential energy of the pendulum. The drift in both appear to be due to accumulation of numerical roundoff error, and could possibly be reduced through the use of compensated summation techniques.



(A) Preservation of the length of $\boldsymbol{\gamma}$ (B) Conservation of energy

FIGURE 5. Numerical properties of the integrator for the pendulum.

For the comparison of the Hamel integrator with simulations using the generalized Störmer–Verlet method and the RATTLE method see [51]. We point out here that the energy error for the Hamel integrator is smaller than those of the Störmer–Verlet and RATTLE methods.

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References

1. V. I. Arnold, *Sur la géometrie differentialle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluids parfaits*, Ann. Inst. Fourier **16** (1966), 319–361.
2. K. R. Ball, D. V. Zenkov, *Hamel's formalism and variational integrators*, Geometry, Mechanics, and Dynamics: The Legacy of Jerry Marsden (D. E. Chang, D. D. Holm, G. Patrick, T. Ratiu, eds.), Fields Inst. Commun. **73**, Springer, NY, 2015, pp. 477–506.
3. E. Binz, M. de León, D. Martín de Diego, D. Socolescu, *Nonholonomic constraints in classical field theories*, Rep. Math. Phys. **49**(2/3) (2002), 151–166.
4. A. M. Bloch, *Nonholonomic mechanics and control*, Interdiscip. Appl. Math. **24**, Springer-Verlag, New York, 2003.
5. A. M. Bloch, P. S. Krishnaprasad, J. E. Marsden, R. Murray, *Nonholonomic mechanical systems with symmetry*, Arch. Ration. Mech. Anal. **136** (1996), 21–99.
6. A. M. Bloch, P. S. Krishnaprasad, J. E. Marsden, T. S. Ratiu, *The Euler–Poincaré equations and double bracket dissipation*, Commun. Math. Phys. **175** (1996), 1–42.
7. A. M. Bloch, J. E. Marsden, D. V. Zenkov, *Quasivelocities and symmetries in nonholonomic systems*, Dyn. Syst. **24**(2) (2009), 187–222.
8. N. Bou-Rabee, J. E. Marsden, *Hamilton–Pontryagin integrators on Lie groups part I: Introduction and structure-preserving properties*, Found. Comput. Math. **9**(2) (2009), 197–219.
9. H. Cendra, D. D. Holm, J. E. Marsden, T. S. Ratiu, *Lagrangian reduction, the Euler-Poincaré equations, and semidirect products*, Am. Math. Soc. Transl. **186** (1988), 1–25.
10. N. G. Chetaev, *On Gauss' principle*, Izv. Fiz-Mat. Obsc. Kazan. Univ., III Ser. **6** (1932–1933), 68–71, (in Russian).
11. ———, *Theoretical mechanics*, Springer–Verlag, New York, 1989.
12. J. Cortés, S. Martínez, *Nonholonomic integrators*, Nonlinearity **14** (2001), 1365–1392.
13. D. G. Ebin, J. E. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. Math. (2) **92** (1970), 102–163.
14. L. Euler, *Decouverte d'un nouveau principe de Mecanique*, Mémoires de l'académie des sciences de Berlin **6** (1752), 185–217.
15. ———, *Principes généraux de l'état d'équilibre des fluides*, Mémoires de l'académie des sciences de Berlin **11** (1757), 217–273.
16. ———, *Principes généraux du mouvement des fluides*, Mémoires de l'académie des sciences de Berlin **11** (1757), 274–315.
17. ———, *Principia motus fluidorum*, Novi Commentarii Academiae Scientiarum Petropolitanae **6** (1761), 271–311.
18. Yu. N. Fedorov, D. V. Zenkov, *Discrete nonholonomic LL systems on Lie groups*, Nonlinearity **18** (2005), 2211–2241.
19. ———, *Dynamics of the discrete Chaplygin sleigh*, Discrete Contin. Dyn. Syst. **2005** Suppl. (2005), 258–267.
20. F. Gay-Balmaz, J. E. Marsden, T. S. Ratiu, *Reduced variational formulations in free boundary continuum mechanics*, J. Nonlinear Sci. **22**(4) (2012), 463–497.
21. G. Hamel, *Die Lagrange–Eulersche Gleichungen der Mechanik*, Schlömilch Z. **50** (1904), 1–57.

22. W. R. Hamilton, *On a general method in dynamics, part I*, Philos. Trans. R. Soc. Lond. **124** (1834), 247–308.
23. ———, *On a general method in dynamics, part II*, Philos. Trans. R. Soc. Lond. **125** (1835), 95–144.
24. D. D Holm, J. E. Marsden, T. S. Ratiu, *The Euler–Poincaré equations and semidirect products with applications to continuum theories*, Adv. Math. **137** (1998), 1–81.
25. T. R. Kane, D. A. Levinson, *Formulation of equations of motion for complex spacecraft*, J. Guid. Control **3**(2) (1980), 99–112.
26. ———, *The use of Kane's dynamical equations in robotics*, Int. J. Robot. Res. **2**(3) (1983), 3–21.
27. ———, *Dynamics, Theory and Applications*, McGraw Hill, NY, 1985.
28. A. V Karapetyan, *On the problem of steady motions of nonholonomic systems*, J. Appl. Math. Mech. **44** (1980), 418–426.
29. M. Kobilarov, J. E. Marsden, G. S. Sukhatme, *Geometric discretization of nonholonomic systems with symmetries*, Discrete Contin. Dyn. Syst., Ser. S **3**(1) (2010), 61–84.
30. A. Kriegl, P. W. Michor, *The convenient setting of global analysis*, Math. Surv. Monogr. **53**, Am. Math. Soc., Providence, Rhode Island, 1997.
31. J. L. Lagrange, *Mécanique analytique*, Chez la Veuve Desaint, Paris, 1788.
32. C. Lynch, D. Zenkov, *Stability of stationary motions of discrete-time nonholonomic systems*, Problems of Analytical Mechanics and Stability Theory. Collection of Papers Dedicated to the Memory of Academician Valentin V. Rumyantsev (V. V. Kozlov, S. N. Vassilyev, A. V. Karapetyan, N. N. Krasovskiy, V. N. Tkhai, F. L. Chernousko, eds.), Fizmatlit, Moscow, 2009, 259–271 (in Russian).
33. C. Lynch, D. V. Zenkov, *Stability of relative equilibria of discrete nonholonomic systems with semidirect symmetry*, preprint, 2010.
34. J. E. Marsden, *Lectures on mechanics*, Lond. Math. Soc. Lect. Note Ser. **174**, Cambridge University Press, 1992.
35. J. E. Marsden, T. S. Ratiu, *Introduction to mechanics and symmetry*, 2nd ed., Texts Appl. Math. **17**, Springer–Verlag, New York, 1999.
36. J. E. Marsden, M. West, *Discrete mechanics and variational integrators*, Acta Numerica **10** (2001), 357–514.
37. J. M Maruskin, A. M. Bloch, J. E. Marsden, D. V. Zenkov, *A fiber bundle approach to the transpositional relations in nonholonomic mechanics*, J. Nonlinear Sci. **22**(4) (2012), 431–461.
38. R. McLachlan, M. Perlmutter, *Integrators for nonholonomic mechanical systems*, J. Nonlinear Sci. **16** (2006), 283–328.
39. Ju. I. Neimark, N. A. Fufaev, *Dynamics of nonholonomic systems*, Transl. Math. Monogr. **33**, Am. Math. Soc., Providence, Rhode Island, 1972.
40. Y. Peng, S. Huynh, D. V. Zenkov, A. M. Bloch, *Controlled Lagrangians and stabilization of discrete spacecraft with rotor*, Proc. CDC **51** (2012), 1285–1290.
41. H. Poincaré, *Sur une forme nouvelle des équations de la mécanique*, C. R. Acad. Sci., Paris **132** (1901), 369–371.
42. ———, *Sur la precession des corps deformables*, Bull. Astron. **27** (1910), 321–356.
43. E. J Routh, *Treatise on the dynamics of a system of rigid bodies*, MacMillan, London, 1860.
44. D. Shi, Y. Berchenko-Kogan, D. V. Zenkov, A. M. Bloch, *Hamel's formalism for infinite-dimensional mechanical systems*, J. Nonlinear Sci. (2016), to appear. link.springer.com/article/10.1007/s00332-016-9332-7
45. G. K. Suslov, *Theoretical mechanics*, 3rd ed., GITTL, Moscow–Leningrad, 1946.
46. J. Vankerschaver, *The momentum map for nonholonomic field theories with symmetry*, Int. J. Geom. Methods Mod. Phys. **2**(6) (2005), 1029–1041.
47. ———, *A class of nonholonomic kinematic constraints in elasticity*, J. Phys. A: Math. Theor. **40** (2007), 3889–3913.
48. ———, *Continuous and discrete aspects of Lagrangian field theories with nonholonomic constraints*, Ph.D. thesis, Ghent University, 2007.

49. G. T. Walker, *On a dynamical top*, Quart. J. **28** (1896), 175–184.
50. D. V. Zenkov, A. M. Bloch, J. E. Marsden, *The energy-momentum method for the stability of nonholonomic systems*, Dyn. Stab. Syst. **13** (1998), 128–166.
51. D. V. Zenkov, M. Leok, A. M. Bloch, *Hamel's formalism and variational integrators on a sphere*, Proc. CDC **51** (2012), 7504–7510.

О ХАМЕЛОВИМ ЈЕДНАЧИНАМА

Резиме. У раду се даје преглед недавних резултата о проширењу Хамеловог формализма на бесконачно-димензионе механичке системе и варијационе интеграторе. Од посебног значаја су примене у динамици и нумеричкој интеграцији система са везама по брзинама.

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