THEORETICAL AND APPLIED MECHANICS
Volume 43 (2016) Issue 2, 169–190 DOI: 10.2298/TAM161130013B

DYNAMICS OF A BODY SLIDING ON A ROUGH PLANE AND SUPPORTED AT THREE POINTS

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Abstract. This paper is concerned with the problem of a rigid body (tripod) moving with three points in contact with a horizontal plane under the action of dry friction forces. It is shown that the regime of asymptotic motion (final dynamics) of the tripod can be pure rotation, pure sliding, or sliding and rotation can cease simultaneously, which is determined by the position of the tripod's supports relative to the radius of inertia. In addition, the dependence of the trajectory of the center of mass on the system parameters is investigated. A comparison is made with the well-known theoretical and experimental studies on the motion of bodies with a flat base.

1. Introduction

This paper is concerned with the problem of a rigid body (tripod) moving with three points in contact with a horizontal plane under the action of dry friction forces.

The system under consideration has a model character, since the equations of motion for it can be written without making additional hypotheses and assumptions (requiring the use of elasticity theory) as to how the pressure of the body is distributed over the plane of sliding: the reaction forces at three points of contact are uniquely defined from the conditions for projections of forces and their momenta. From this point of view, the laws found in the system's dynamics can be used to verify and refine the dynamical friction models.

N. E. Zhukovskii [1] was the first to address a simpler problem, that of a bench with two point supports (*bipod*), as an example of application of the conditions for equilibrium of the system with friction. However, the dynamics of the bipod does not differ significantly from that of the tripod as far as the simplification of analysis is concerned, and can be obtained in the limit when two supports approach each other. Afterwards, some particular motions of this system were discussed in [3, 10, 13]. In particular, purely rotational motions were found for which the

2010 *Mathematics Subject Classification:* 70F40, 70E18.
Key words and phrases: dry friction, linear pressure distribution, planar motion, Coulomb's law, asymptotic motion, dynamics, tripod.

body rotates about one of its fixed supports [3], and it was numerically shown [10] that these solutions can be limit solutions as the motion approaches a stop. The trajectories in absolute space are constructed in [3, 13]. It is shown that in the general case the velocity of the center of mass changes direction in the course of motion, so that the trajectory meanders. However, this effect is not observed in the case of “roughening” of trajectories [3]. We also mention the paper [13], in which the author considers the motion of a bipod on an inclined plane and gives an example of translational motion arising when the initial motion is purely rotational.

In 1912, P. Field [9] was the first to present an approximate analysis of the motion of a body with three point supports (*tripod*). Some partial solutions to this system and conditions for their existence and stability were found in [2]. In [4], more general quadratures were obtained, but they are entirely useless for analysis of the dynamics. A modern investigation of the motion of a tripod was carried out by Shegelski and coauthors in [11]. This paper presents numerically constructed trajectories for different tripods. In particular, it is demonstrated that in some cases the trajectories of an equilateral tripod are similar to those of a body with an annular contact area. However, a counterexample of an "undulatory" trajectory of the tripod is also given, which contradicts the well-known laws of the dynamics of bodies with a flat base [6].

Unfortunately, the final dynamics of the tripod has been poorly studied so far. Mention can be made of [5], where conditions for the existence of asymptotic (when the motion approaches a stop) stable translational motions of the tripod have been found. It is appropriate here to present the results of [10, 12], where the regimes of asymptotic motion of bodies with a flat base are investigated and it is shown that the final dynamics is determined by the ratio between the radius of the contact patch and the radius of inertia of the body. When its value is small, asymptotic pure rotation is realized, when it is large, asymptotic translational motion takes place, and when the value is close to unity, the rotation and sliding cease simultaneously.

We give particular attention to comparison of the laws of dynamics of a tripod and a body with a flat round base (disk); for a detailed review, analytical and experimental study of such a body see [6] and [7]. It is well known from theoretical works that the pattern of absolute motion of the disk depends on the choice of the model of the pressure distribution of the body on the plane of sliding. In the case of homogeneous pressure distribution the trajectories of the center of mass of the body are rectilinear [8, 10], which is not confirmed experimentally. Experiments have shown that, in the case of a linear law of pressure distribution, the trajectories of the center of mass deviate in the direction opposite to that of rotation. Both models of pressure distribution predict simultaneous termination of rotation and sliding of the body, which finds a qualitative, but no quantitative, experimental confirmation [7]. We note that the difficulties in comparing theoretical and experimental results on the motion of a disk are mainly due to the uncertainty of distribution of normal reactions. The probabilistic approach to this problem has been proposed recently in [14].

Some results of our research can be verified experimentally, which is very important for identification of various friction models in describing the system dynamics.

We are planning to carry it out by using modern high-precision equipment (see [7]). At the end of the paper we present the results under discussion and formulate unsolved problems.

2. Equations of motion for a dry friction model

2.1. Equations of motion of a balanced body. Consider the problem of a rigid body moving on a horizontal rough plane and supported at three points (tripod). We shall assume that the body is *balanced.* This implies that the distribution of mass m is such that one of the principal axes of inertia of the body coincides with the normal to the surface, the center of mass also lies on this axis at height h. We define two coordinate systems (see Fig. 1):

1. coordinate system $OXYZ$ — the origin lies at some point of the plane, and the axis OZ is perpendicular to the plane;
2. *a moving* coordinate system $Cxyz$ — the origin C coincides with the center of mass of the body, the axis Cz is perpendicular to the plane and coincides with the principal axis of inertia, the axes Cx and Cy are parallel to the plane, and the axis Cx passes above one of the supports of the body at height h.

Let $\boldsymbol{R}_P = (X, Y)$ be the coordinates of the projection of the tripod's center of mass onto the plane of sliding and let α be the rotation angle of the axes of the moving coordinate system $Cxyz$ relative to the fixed coordinate system $OXYZ$. The configuration space of the system of interest is the product $\mathcal{M} = \{(X, Y, \alpha)\} = \mathbb{R}^2 \times SO(2)$, where the first factor describes the position of the projection of the center of mass onto the plane of sliding and the second factor describes the orientation of the tripod.

REMARK 2.1. In [11], the moving coordinate system has been chosen such that one of its axes coincides with the linear velocity vector of the tripod. However, such a choice requires constantly recalculating the projections of the position of the supporting points onto its axes. The advantage of the moving system proposed in this paper is that the coordinate axes are rigidly attached to the supporting points.

Let us write the equations of motion in the moving coordinate system $Cxyz$. Let $\boldsymbol{r}_i (i = 1, 2, 3)$ be the radius vectors of the supporting points

$$\boldsymbol{r}_1 = (x_1, 0, -h), \quad \boldsymbol{r}_2 = (x_2, y_2, -h), \quad \boldsymbol{r}_3 = (x_3, y_3, -h)$$

and let $\boldsymbol{\omega} = (0, 0, \omega)$ and $\boldsymbol{v} = (v_x, v_y, 0)$ be, respectively, the angular velocity and the velocity of the center of mass of the tripod, referred to the moving axes $Cxyz$. The equations of motion in vector form are

$$m\dot{\boldsymbol{v}} + m(\boldsymbol{\omega} \times \boldsymbol{v}) = \boldsymbol{F} + \boldsymbol{N} + m\boldsymbol{g}, \quad (\mathbf{I}\boldsymbol{\omega})^{\cdot} = \boldsymbol{M} + \boldsymbol{T}, \tag{2.1}$$

where $\boldsymbol{F} = (F_x, F_y, 0)$ is the friction force, $\boldsymbol{N} = (0, 0, N)$ is the normal reaction, $\boldsymbol{M} = (M_x, M_y, M_z)$ is the friction torque, $\boldsymbol{T} = (T_x, T_y, 0)$ is the moment of the reaction force (since the reaction force is vertical, its moment is parallel to the plane of sliding), $\boldsymbol{g}$ is the free-fall acceleration, and $\mathbf{I}$ is the inertia tensor, which for a balanced body has the following block-diagonal form:

$$\mathbf{I} = \begin{pmatrix} I_{xx} & I_{xy} & 0 \\ I_{xy} & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}.$$

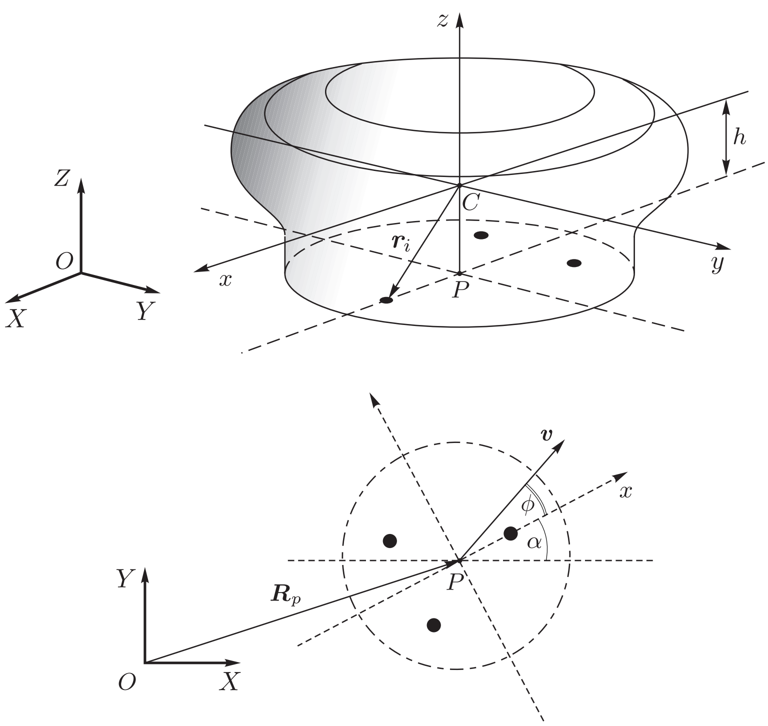

FIGURE 1. Sliding rigid body with three point supports (tripod), fixed ($OXYZ$) and moving ($Cxyz$) coordinate systems.

The equations of motion (2.1) may be written in terms of the moving axes $Cxyz$

$$m(\dot v_x - \omega v_y) = F_x, \qquad m(\dot v_y + \omega v_x) = F_y, \qquad I_{zz}\dot\omega = M_z,$$
$$F_x = \sum_{i=1}^{3} F_{xi}, \qquad F_y = \sum_{i=1}^{3} F_{yi}, \qquad M_z = \sum_{i=1}^{3}(x_i F_{yi} - y_i F_{xi}), \tag{2.2}$$

where F_{xi} and F_{yi} are the projections of the friction forces for the i-th support.

In addition, for a body sliding on a horizontal plane, we need to write the *conditions for motion without loss of contact*, which in this case are the requirements for compensation of the vertical projections of forces and horizontal projections of the moments:

$$
\begin{aligned}
&\sum_{i=1}^{3} N_i - mg = 0, \qquad M_x + T_x = 0, \qquad M_y + T_y = 0,\\
&M_x = \sum_{i=1}^{3} hF_{yi}, \qquad M_y = -\sum_{i=1}^{3} hF_{xi},\\
&T_x = \sum_{i=1}^{3} y_i N_i, \qquad T_y = -\sum_{i=1}^{3} x_i N_i,
\end{aligned}
\tag{2.3}
$$

where N_i is the normal reaction of the i-th support.

Remark 2.2. Moreover, it should be borne in mind that in this case the reaction forces cannot become negative, i. e.,

$$N_i \geqslant 0, \quad i = 1, 2, 3.$$

According to the Amontons–Coulomb dry friction model, the friction force acts at each supporting point:

$$\boldsymbol{F}_i = -fN_i \frac{\boldsymbol{v}_i}{|\boldsymbol{v}_i|}, \quad \boldsymbol{v}_i = \boldsymbol{v} + (\boldsymbol{\omega} \times \boldsymbol{r_i}),$$

where f is the coefficient of friction and $\boldsymbol{v}_i$ is the velocity of the supporting point. In this case, the scalar expressions for projections of the friction force and the friction torque are

$$F_x = -f \sum_{i=1}^{3} N_i V_{xi}, \quad F_y = -f \sum_{i=1}^{3} N_i V_{yi}, \quad M_z = -f \sum_{i=1}^{3} N_i (x_i V_{yi} - y_i V_{xi}), \tag{2.4}$$

$$V_{xi} = \frac{v_x - \omega y_i}{\sqrt{(v_x - \omega y_i)^2 + (v_y + \omega x_i)^2}}, \quad V_{yi} = \frac{v_y + \omega x_i}{\sqrt{(v_x - \omega y_i)^2 + (v_y + \omega x_i)^2}}.$$

Substituting the known expressions for the friction force and friction torque into the conditions for motion without loss of contact (2.3), we obtain equations defining the normal reactions N_i:

$$\sum_{i=1}^{3} N_i (x_i - fhV_{xi}) = 0, \quad \sum_{i=1}^{3} N_i (y_i - fhV_{yi}) = 0 \quad \sum_{i=1}^{3} N_i = mg.$$

The solution of these equations can be conveniently represented using three-dimensional vectors

$$
\begin{aligned}
&\boldsymbol{a} = (a_1, a_2, a_3), \qquad \boldsymbol{b} = (b_1, b_2, b_3), \qquad \boldsymbol{e} = (1, 1, 1), \quad \hat{\boldsymbol{N}} = (N_1, N_2, N_3),\\
&a_i = x_i - fhV_{ix}, \qquad b_i = y_i - fhV_{iy},
\end{aligned}
\tag{2.5}
$$

whose components are not the projections on the axes of a coordinate system, but scalar quantities relating to one of the three supporting points. Hence the expression defining the components $\hat{\boldsymbol{N}}$ can be written in vector form

$$\hat{\boldsymbol{N}} = mg \frac{(\boldsymbol{b} \times \boldsymbol{a})}{(\boldsymbol{b}, \boldsymbol{a} \times \boldsymbol{e})}. \tag{2.6}$$

Substituting the known expressions for projections of the friction force and friction torque (2.4) and normal reaction force (2.6) in (2.2), we obtain *equations of motion for a balanced body with three points in contact with a horizontal plane in the moving coordinate system*

$$
\begin{gathered}
m(\dot{v}_x - \omega v_y) = F_x, \quad m(\dot{v}_y + \omega v_x) = F_y, \quad I_{zz}\dot{\omega} = M_z, \\
F_x = -f\sum_{i=1}^{3} N_i V_{xi}, \quad F_y = -f\sum_{i=1}^{3} N_i V_{yi}, \\
M_z = -f\sum_{i=1}^{3} N_i(x_i V_{yi} - y_i V_{xi}), \\
N_i = mg\frac{\varepsilon_{ijk} b_j a_k}{\varepsilon_{ijk} b_i a_j e_k},
\end{gathered} \tag{2.7}
$$

where ε_{ijk} is the antisymmetric Levi-Civita tensor.

The evolution of the position and orientation of the body relative to the fixed coordinate system $OXYZ$ is governed by the equations

$$
\dot{X} = v_x \cos\alpha - v_y \sin\alpha, \quad \dot{Y} = v_x \sin\alpha + v_y \cos\alpha, \quad \dot{\alpha} = \omega, \tag{2.8}
$$

where V_{xi}, V_{yi} and N_i are given by (2.4), (2.5) and (2.7).

2.2. Energy dissipation. In the general case, the kinetic energy of a rigid body has the form

$$
T = \frac{1}{2}m\boldsymbol{v}^2 + \frac{1}{2}(\boldsymbol{\omega}, I\boldsymbol{\omega}). \tag{2.9}
$$

Differentiating (2.9) with respect to time and substituting the equations of motion (2.7) and (2.8), we obtain an expression for the evolution of the kinetic energy of the tripod depending on the components of the friction force and friction torque, the linear velocity of the center of mass and the angular velocity of the body

$$
\dot{T} = F_x v_x + F_y v_y + M_z \omega. \tag{2.10}
$$

Substituting (2.4) into (2.10), we bring this expression to the form

$$
\dot{T} = -f\sum_{i=1}^{3} N_i\sqrt{(v_x - \omega y_i)^2 + (v_y + \omega x_i)^2},
$$

from which it follows that $\dot{T} < 0$.

The kinetic energy of the tripod decreases, which, generally speaking, leads to a stop of the tripod in finite time. However, this can be rigorously proved only in a few particular cases where the solution is given by explicit expressions. In the general case, it can only be shown that at a sufficiently small initial velocity, due to the reaction (2.6) being different from zero, the body stops in finite time. In the case of an arbitrary initial velocity, the proof requires analysis of the reactions (2.6), which occur in the process of motion and can be obtained only numerically. Some of them can become equal to zero, resulting generally in the loss of contact or in different paradoxes which were first noted by Painlevé. In this paper, we ignore these issues. However, for a real analysis of the dynamics, it is necessary (but also much more difficult) to consider a more complicated model incorporating these effects.

3. Analysis of the dynamics at final stages of motion

Since the motion of the tripod occurs in a finite time interval, we can investigate some problems of the final dynamics and its stability by restricting ourselves to an asymptotic subsystem of smaller dimension. This transition to a reduced system is physically obvious, although it requires some justification. One can avoid proofs by introducing special definitions, for example, τ-stability of translational motion at instants close to a stop [5].

In this section we restrict ourselves to exploring the dynamics of the asymptotic subsystem. To do so, we write the equations of motion (2.7) in a cylindrical coordinate system, use the regularization procedure (rescale time) to exclude a singularity at the instant of stop and discard higher-order terms of smallness due to a decrease in the energy of the system. We note that the reduced system approximates well the complete system at the final stage of motion, i.e., at small energies.

3.1. Dimensionless equations of motion. We introduce the new cylindrical coordinate system $O\mathcal{R}\phi\zeta$ by using the relations

$$v_x = \mathcal{R}\cos\phi\sqrt{1-\zeta^2}, \quad v_y = \mathcal{R}\sin\phi\sqrt{1-\zeta^2}, \quad \rho\omega = \mathcal{R}\zeta,$$

where $\mathcal{R} = \sqrt{v_x^2 + v_y^2 + \rho^2\omega^2}$ is a quantity related to the kinetic energy by $T = \frac{m\mathcal{R}^2}{2}$, ϕ is the angle between the axis Cx and the velocity vector of the center of mass $\boldsymbol{v}$, the variable ζ determines the ratio between the linear and angular velocities

$$\frac{\sqrt{1-\zeta^2}}{\zeta} = \frac{v}{\omega\rho} = k, \tag{3.1}$$

and $\rho = \sqrt{\frac{I_{zz}}{m}}$ is the radius of inertia of the balanced body relative to the axis Oz.

We recall that the value of the parameter k at the instant of stop determines the asymptotic motion of the body. At $k = \mathrm{const}$ the translational and rotational motions of the body cease simultaneously, at $k = 0$, which corresponds to $\zeta = 1, v = 0$, the body asymptotically tends to pure rotation, and at $k \to \infty$, which corresponds to $\zeta = 0, \omega = 0$, the body asymptotically tends to pure sliding.

To facilitate further qualitative analysis, we also pass on to dimensionless coordinates of the position of the tripod's supports, $\bar{x}_i, \bar{y}_i$, and to the coefficient of effective friction, μ:

$$\bar{x}_i = \frac{x_i}{\rho}, \quad \bar{y}_i = \frac{y_i}{\rho}, \quad \mu = \frac{h}{\rho}f.$$

We write the equations of motion (2.7) in the new coordinate system $O\mathcal{R}\phi\zeta$:

$$\frac{\dot{\mathcal{R}}}{gf} = \sqrt{1-\zeta^2}(\bar{F}_y\sin\phi + \bar{F}_x\cos\phi) + \bar{M}_z\zeta,$$

$$\begin{aligned}\frac{\mathcal{R}}{gf}\sqrt{1-\zeta^2}\dot{\phi} &= \bar{F}_y\cos\phi - \bar{F}_x\sin\phi - \frac{\mathcal{R}^2\zeta\sqrt{1-\zeta^2}}{\rho g f},\\ \frac{\mathcal{R}}{gf}\dot{\zeta} &= -\zeta\sqrt{1-\zeta^2}(\bar{F}_y\sin\phi + \bar{F}_x\cos\phi) + \bar{M}_z(1-\zeta^2),\end{aligned} \tag{3.2}$$

where $\bar{F}_x, \bar{F}_y, \bar{M}_z$ are the dimensionless expressions for projections of the friction force and the friction torque

$$\bar{F}_x = -\sum_{i=1}^{3}\bar{N}_i\bar{V}_{xi}, \quad \bar{F}_y = -\sum_{i=1}^{3}\bar{N}_i\bar{V}_{yi}, \quad \bar{M}_z = -\sum_{i=1}^{3}\bar{N}_i(\bar{x}_i\bar{V}_{yi} - \bar{y}_i\bar{V}_{xi}),$$

$$\bar{N}_i = \frac{\varepsilon_{ijk}\bar{a}_j\bar{b}_k}{\varepsilon_{ijk}\bar{b}_i\bar{a}_j e_k}, \quad \bar{a}_i = \bar{x}_i - \mu\bar{V}_{xi}, \quad \bar{b}_i = \bar{y}_i - \mu\bar{V}_{yi}, \quad e = (1,1,1),$$

$$\bar{V}_{xi} = \frac{\sqrt{1-\zeta^2}\cos\phi - \zeta\bar{y}_i}{\sqrt{(\sqrt{1-\zeta^2}\cos\phi - \zeta\bar{y}_i)^2 + (\sqrt{1-\zeta^2}\sin\phi + \zeta\bar{x}_i)^2}},$$

$$\bar{V}_{yi} = \frac{\sqrt{1-\zeta^2}\sin\phi + \zeta\bar{x}_i}{\sqrt{(\sqrt{1-\zeta^2}\cos\phi - \zeta\bar{y}_i)^2 + (\sqrt{1-\zeta^2}\sin\phi + \zeta\bar{x}_i)^2}}.$$

3.2. Regularization and reduction near the state of rest. In the system (3.1), we rescale time, as is done in the three-body problem of celestial mechanics, to regularize the singularity:

$$d\tau = \frac{gf}{\mathcal{R}}dt, \quad \frac{d\mathcal{R}}{d\tau} = \mathcal{R}', \quad \frac{d\phi}{d\tau} = \phi', \quad \frac{d\zeta}{d\tau} = \zeta'.$$

This yields a system of differential equations defining the functions $\mathcal{R}(\tau), \phi(\tau), \xi(\tau)$:

$$\begin{aligned}\mathcal{R}' &= \mathcal{R}\left(\sqrt{1-\zeta^2}(\bar{F}_y\sin\phi + \bar{F}_x\cos\phi) + \bar{M}_z\zeta\right),\\ \sqrt{1-\zeta^2}\phi' &= \bar{F}_y\cos\phi - \bar{F}_x\sin\phi - \frac{\mathcal{R}^2\zeta\sqrt{1-\zeta^2}}{\rho g f},\\ \zeta' &= -\zeta\sqrt{1-\zeta^2}(\bar{F}_y\sin\phi + \bar{F}_x\cos\phi) + \bar{M}_z(1-\zeta^2).\end{aligned} \tag{3.3}$$

In the last two equations, at small values of $\mathcal{R}$ we single out the asymptotic (reduced) system

$$\begin{aligned}\sqrt{1-\zeta^2}\phi' &= \bar{F}_y\cos\phi - \bar{F}_x\sin\phi,\\ \zeta' &= -\zeta\sqrt{1-\zeta^2}(\bar{F}_y\sin\phi + \bar{F}_x\cos\phi) + \bar{M}_z(1-\zeta^2),\end{aligned} \tag{3.4}$$

which describes the dynamics of the tripod in the limiting cases of small energies of motion, for example, at stages close to the instant of stop.

The properties of the system (3.4) reflect the characteristic properties of the initial system when the stop is approached, and so we examine the system (3.4) in more detail.

3.3. Simplifying assumption on the position of the tripod's supports. We note that, despite the reduction of order of the system, the analysis of the asymptotic motion of the tripod is difficult due to a large set of parameters $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{y}_1, \bar{y}_2, \bar{y}_3, \mu)$. To decrease their number, we introduce the following assumption on the position of the tripod's supports to simplify analysis and calculations.

Consider a particular case where *the supports lie at the apices of an isosceles triangle* inscribed into a circle of radius $\bar{x}_1$ with center in the projection of the tripod's center of mass onto the plane of sliding (see Fig. 2). In this case, $\bar{y}_1 = 0, \bar{x}_2 = \bar{x}_3, \bar{y}_2 = -\bar{y}_3 = \sqrt{\bar{x}_1{}^2 - \bar{x}_2{}^2}$, hence, the number of parameters decreases to three $(\bar{x}_1, \bar{x}_2, \mu)$. The domain of definition of parameters $\bar{x}_1$ and $\bar{x}_2$ is $\bar{x}_1 > 0, -\bar{x}_1 < \bar{x}_2 < 0$.

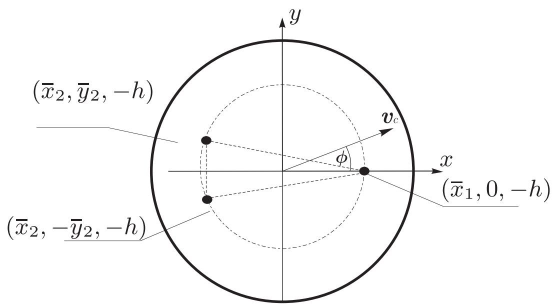

FIGURE 2. Supports of the tripod in the case where the points of contact lie at the apices of an isosceles triangle.

We investigate possible regimes of motion and singular points of the two-dimensional asymptotic system (3.4) *depending on the geometric and inertial properties of the tripod: the coefficient of effective friction* μ *and the relative position of the supports* $\bar{x}_i, \bar{y}_i$.

3.4. Asymptotic stability of translational motion. The complete system (3.1) and hence the asymptotic system (3.4)have *partial solutions*(and, in this case, *degenerate families of singular points* (ϕ, ζ^*)) which correspond to purely translational and purely rotational motions of the tripod. The translational motion is realized at the value $\zeta_1^* = 0$, which according to (3.1) corresponds to $\omega = 0$, and the rotational motion is realized at $\zeta_2^* = \pm 1$, which corresponds to $v = 0$.

We show that *asymptotic stable translational motions are possible in the system* and that the pattern of these motions depends on the geometric and inertial properties of the tripod.

To analyze the stability of a degenerate family of singular points $(\phi, \zeta_1^* = 0)$ with respect to the parameters and the variables (ϕ, ζ), we use the methods of qualitative analysis of dynamical systems near the state of equilibrium (see, e. g., [15]). We linearize the system (3.4) and investigate the vector field near the axis $\zeta = 0$.

The linearized equations of motion of the system (3.4) are

$$\begin{aligned} \phi' &= -\mu\zeta + O(\zeta^2), \quad \mu = \frac{h}{\rho} f \\ \zeta' &= K(\phi, \mu, \bar{x}_i, \bar{y}_i)\zeta + O(\zeta^2), \end{aligned} \tag{3.5}$$

where

$$\begin{gathered} K(\phi, \mu, \bar{x}_i, \bar{y}_i) = 1 - \frac{\sum_{i=1}^{3} A_i B_i + C_i D_i}{2S}, \\ A_i = (\cos(\phi)\bar{x}_i + \sin(\phi)\bar{y}_i)^2, \\ B_i = \mu\cos(\phi)(\bar{y}_j - \bar{y}_k) - \mu\sin(\phi)(\bar{x}_j - \bar{x}_k) + \bar{x}_j\bar{y}_k - \bar{x}_j\bar{y}_k, \\ C_i = \sin(\phi)\bar{x}_i - \cos(\phi)\bar{y}_i, \\ D_i = -\mu^2(\cos(\phi)(\bar{x}_j - \bar{x}_k) + \sin(\phi)(\bar{y}_j - \bar{y}_k)), \\ S = \tfrac{1}{2}((\bar{x}_1 - \bar{x}_3)(\bar{y}_2 - \bar{y}_3) - (\bar{x}_2 - \bar{x}_3)(\bar{y}_2 - \bar{y}_3)), \end{gathered}$$

S being the area of the triangle whose apices are the point supports of the tripod.

When the supports of the tripod lie at the apices of an isosceles triangle, the function $K(\phi, \mu, \bar{x}_i, \bar{y}_i)$ has the form

$$\begin{aligned} K_2(\phi, \mu, \bar{x}_1, \bar{x}_2) = &- 2\mu\cos^3\phi\bar{x}_1 + \cos^2\phi(\bar{x}_1^2 + 2\bar{x}_1\bar{x}_2) \\ &+ \cos\phi(\mu\bar{x}_1 - \mu\bar{x}_2) + \mu^2 - \bar{x}_1^2 - \bar{x}_2\bar{x}_1 + 1, \end{aligned}$$

and in the case of imposition of the additional condition that the tripod's supports lie at the apices of an equilateral triangle the above function has an even simpler form

$$K_3(\phi, \mu, \bar{x}_1) = -\tfrac{1}{2}\bar{x}_1^2 + \mu\bar{x}_1(\tfrac{3}{2}\cos\phi - 2\cos^3\phi) + \mu^2 + 1.$$

Applying elementary considerations of qualitative analysis of dynamical systems, one may assert that, if the equations of motion of the tripod which are linearized near $\zeta = 0$ have the form (3.5), then the stability of the degenerate family of singular points $(\phi, \zeta^* = 0)$ with respect to the variables (ϕ, ζ) depends on the sign of the function $K(\phi, \mu, \bar{x}_i, \bar{y}_i)$ as follows:

(1) when $K(\phi, \mu, \bar{x}_i, \bar{y}_i) < 0$, the motion is *asymptotically stable*,
(2) when $K(\phi, \mu, \bar{x}_i, \bar{y}_i) > 0$, the motion is *unstable*.

Indeed, on the plane (ϕ, ζ) the linearized system (3.5) defines a vector field whose qualitative analysis near degenerate equilibrium states lying on the straight line $(\phi, \zeta^* = 0)$ gives sufficient information on their stability [15]. The trajectories of the system are continuous lines tending to or moving away from some singular point $(\phi^*, \zeta^* = 0)$. The directions of motion on the trajectories are determined by the coefficients on the right-hand sides of the system (3.5). Under the condition that the coefficient $-\mu$ in the equation for ϕ' is always negative, the trajectories will tend to the axis $\zeta = 0$ only in the case $K(\phi, \mu, \bar{x}_i, \bar{y}_i) < 0$, which will lead to asymptotic stability of motion. When $K(\phi, \mu, \bar{x}_i, \bar{y}_i) > 0$, the trajectories will move away from the axis $\zeta = 0$, which will lead to instability of translational motion.

Analysis of the sign definiteness of the functions $K_2(\phi, \mu, \bar{x}_1, \bar{x}_2)$ and $K_3(\phi, \mu, \bar{x}_1)$ depending on the parameters $\bar{x}_1, \bar{x}_2, \mu$ allows us to make the following assertions about the asymptotic stability of the translational motion of the tripod.

1. The translational motion of the tripod whose supports lie at the apices of an isosceles triangle are asymptotically stable in some directions of sliding, ϕ, when $\bar{x}_1 > \sqrt{\mu^2+1}$ (see Fig. 3, region 5).

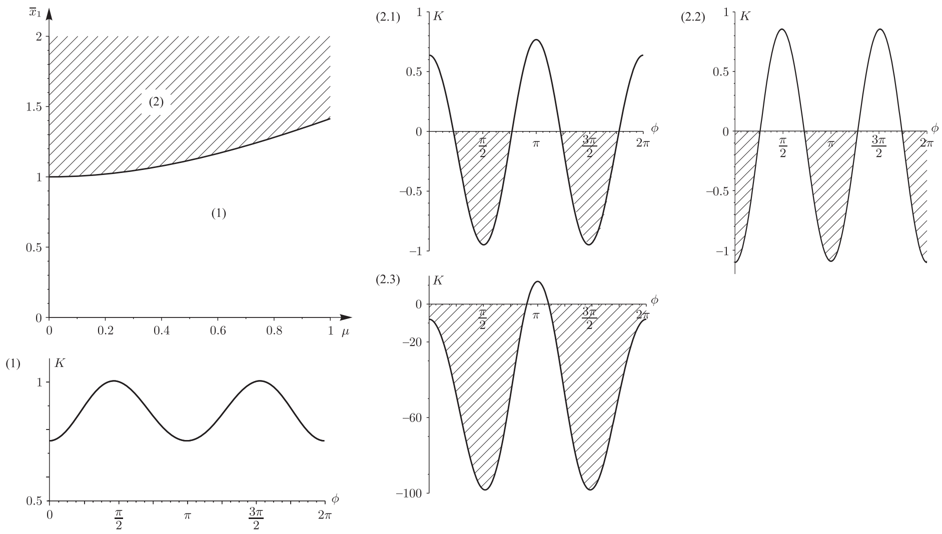

FIGURE 3. Dependence of the asymptotic stability of the translational motion of a tripod whose supports lie at the apices of an isosceles triangle on the parameters $\bar{x}_1, \mu$ (a). Graphs of the functions $K(\phi, \mu, \bar{x}_1, \bar{x}_2)$ for regions of asymptotically stable and unstable translational motions (b). In region (1) (when $\bar{x}_1 < \sqrt{\mu^2+1}$) *the translational motion is unstable* in any direction of sliding, ϕ. In region (2) (when $\bar{x}_1 > \sqrt{\mu^2+1}$) *the translational motion is asymptotically stable* in some directions of sliding, ϕ, the values of which can be determined from the graphs of the functions $K(\phi, \mu, \bar{x}_1, \bar{x}_2)$ under the condition $K(\phi, \mu, \bar{x}_1, \bar{x}_2) < 0$.

Figure 3 shows the dependence of the asymptotic stability of translational motion on the plane of parameters $(\bar{x}_1, \mu)$. In region (1) (when $\bar{x}_1 < \sqrt{\mu^2+1}$) *the translational motion is unstable* in any direction of sliding. A typical graph of the function $K(\phi)$ has been constructed for the parameters $\bar{x}_1 = 0.5, \bar{x}_2 = -0.5, \mu = 0.05$. $K(\phi) > 0$ for any value of ϕ. In region (2) (when $\bar{x}_1 > \sqrt{\mu^2+1}$) the stability of translational motion depends on the value of the parameter $\bar{x}_2$ and the orientation of the body relative to the direction of sliding, ϕ. The translational motion can be asymptotically stable both when it has the same direction as the axis Cx and when it has the opposite direction (see Fig. 3 (2.2) showing the graph

$K(\phi)$ for $\bar{x}_1 = 1.5, \bar{x}_2 = -1.4, \mu = 0.05$). Sliding can be stable in a neighborhood of an arbitrary direction ϕ (see Fig. 3 (2.1) showing the graph $K(\phi)$ for $\bar{x}_1 = 1.5, \bar{x}_2 = -0.2, \mu = 0.05$) and in the directions opposite to the direction of the axis Cx (see Fig. 3 (2.3) showing the graph $K(\phi)$ for $\bar{x}_1 = 10, \bar{x}_2 = 0, \mu = 1$).

2. Since the value of the parameter of effective friction is always positive, $\mu = f\frac{h}{\rho} > 0$, *the translational motion of the bodies whose supports lie inside the circle of the radius of inertia, that is,* $\bar{x}_1 < 1$, *will be unstable for any values of the parameters* $\bar{x}_1, \bar{x}_2, \mu$ *in any direction of sliding.*

As an additional graphical illustration of this statement and the conclusion about the stability of the translational motion of the tripod, one can construct three-dimensional graphs of the function $K(\bar{x}_2, \phi)$ depending on the parameter $\bar{x}_1$, as shown in Fig. 4:

(1) when $\bar{x}_1 < 1$, *the translational motion is always unstable*, $K(\bar{x}_2, \phi) > 0$ (Fig. 4a);

(2) when $\bar{x}_1 = 1$, *the translational motion is unstable*, however, when $\mu = 0$, the function $K(\phi) = 0$ in five limiting cases ($\bar{x}_2 = 0, \phi = \pm\frac{\pi}{2}$), ($\bar{x}_2 = -1, \phi = \pm\pi$) and ($\bar{x}_2 = -1, \phi = 0$) (Fig. 4b);

(3) when $\bar{x}_1 > 1$, *the translational motion is stable in some directions of motion* $\phi^*(\bar{x}_2)$ (Fig. 4c).

3. The asymptotic stability of the translational motion of the tripod for a particular case where its supports lie at the apices of an equilateral triangle is defined by the following conditions:

(1) when $\bar{x}_1 < \bar{x}_{12}$, the translational motion is unstable in any direction of sliding, ϕ (see Fig. 5, region 1), which corresponds to $K(\phi) > 0$ for any values of ϕ (the graph $K(\phi)$ has been constructed for the parameters $\bar{x}_1 = 0.9, \mu = 0.05$);

(2) when $\bar{x}_1 > \bar{x}_{23}$, the translational motion is stable in any direction of sliding, ϕ (see Fig. 5, region 3), which corresponds to $K(\phi) < 0$ for any value of ϕ (the graph $K(\phi)$ has been constructed for the parameters $\bar{x}_1 = 1.7, \mu = 0.1$);

(3) when $\bar{x}_{12} < \bar{x}_1 < \bar{x}_{23}$, the stability of the translational motion depends on the orientation of the tripod relative to the direction of sliding, ϕ (see Fig. 5, region 2), the corresponding graph of $K(\phi)$ has been constructed for the parameters $\bar{x}_1 = 1.5, \mu = 0.5$),

where $\bar{x}_{12}(\mu) = \frac{1}{2}(-\mu + \sqrt{9\,\mu^2 + 8}), \quad \bar{x}_{23}(\mu) = \frac{1}{2}(\mu + \sqrt{9\,\mu^2 + 8})$.

3.5. Analysis of the dynamics of a reduced system (regimes of asymptotic motion). We now examine the dynamics of the system (3.4) which corresponds to the asymptotic dynamics of the complete system (3.3) at final stages of motion using the method of constructing phase portraits on the plane (ϕ, ξ). We show that even for a particular case where the supports lie at the apices of an equilateral triangle the pattern of final motion of the tripod is determined by the position of the supports relative to the radius of inertia of the tripod, i.e., the parameter $\bar{x}_1 = \frac{x_1}{\rho}$, and by the value of the coefficient of effective friction $\mu = f\frac{h}{\rho}$.

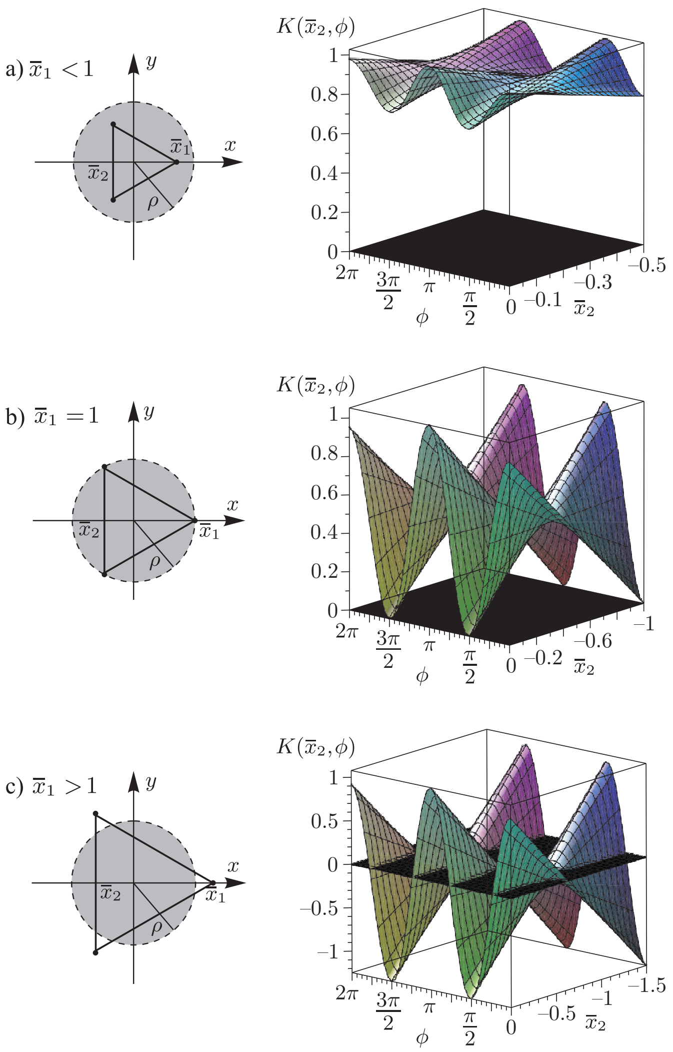

FIGURE 4. Schematic of the supports of the tripod relative to its radius of inertia ρ and the corresponding typical view of the function $K(\bar{x}_2, \phi)$. When $K(\bar{x}_2, \phi) < 0$, the translational motion in the direction ϕ is asymptotically stable.

The regime of asymptotic motion can be pure rotation, pure sliding, or sliding and rotation can cease simultaneously. We shall compare the results with the previously obtained conditions for existence of the asymptotic stability of translational motion.

At small values of $\bar{x}_1 \leqslant 0.5$ all trajectories tend to a degenerate family of singular points $(\phi, \zeta^* = 1)$, which corresponds to asymptotically pure rotation of

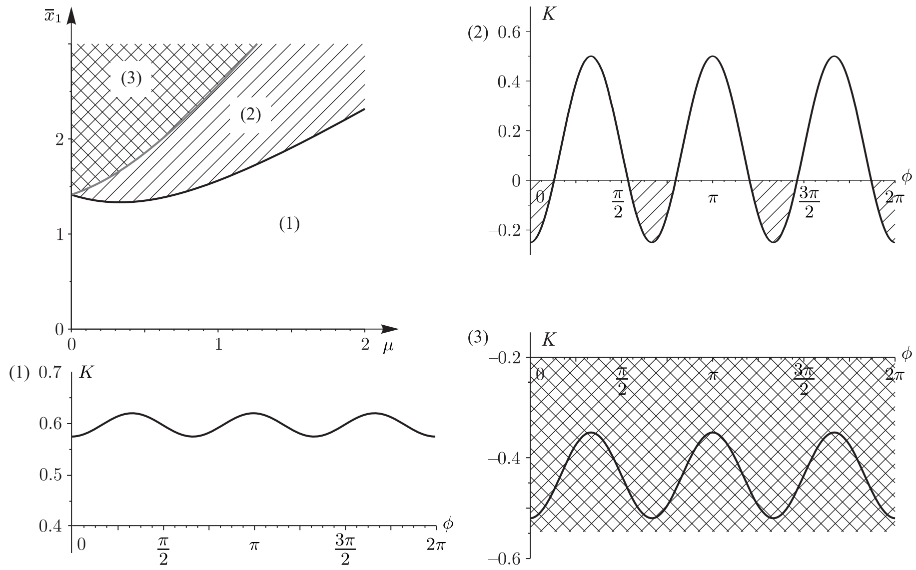

FIGURE 5. Dependence of the asymptotic stability of the translational motion of a tripod whose supports lie at the apices of an equilateral triangle on the parameters $\bar{x}_1, \mu$. Graphs of the functions $K(\phi, \mu, \bar{x}_1)$ for regions of asymptotically stable and unstable translational motions.

$$\bar{x}_{12} = \tfrac{1}{2}(-\mu + \sqrt{9\,\mu^2 + 8}) \quad \bar{x}_{23} = \tfrac{1}{2}(\mu + \sqrt{9\,\mu^2 + 8})$$

In region (1) (when $\bar{x}_1 < \bar{x}_{12}$) the translational motion is *unstable* for any value of the direction of sliding, ϕ. In region (2) (when $\bar{x}_{12} < \bar{x}_1 < \bar{x}_{23}$) the asymptotic stability of the translational motion depends on the direction of sliding of the tripod. In region (3) (when $\bar{x}_1 > \bar{x}_{23}$) the translational motion is asymptotically *stable* in all directions of sliding.

the tripod (see Fig. 6a). The translational motion is asymptotically unstable, which corresponds to region 1 of Fig. 5.

With further increase in the parameter $\bar{x}_1$ near $\zeta^* = 1$ there appear three stable and three unstable fixed points, which at $\bar{x}_1 = 1$ lie on the same straight line $\zeta^* = \frac{1}{\sqrt{2}}$ (see Fig. 6b). Analytically, it is straightforward to show that at $\bar{x}_1 = 1$ the value $\zeta^* = \frac{1}{\sqrt{2}}$ is a partial solution of the system of equations (3.4). In this case, the asymptotic motion of the tripod corresponds to simultaneous cessation of rotation and sliding, and $k = \frac{v}{\rho\omega} = 1$ at the instant of stop. Accordingly, the translational motion is asymptotically unstable, which corresponds to region 1 of Fig. 5.

With further increase in the parameters $\bar{x}_1$ and μ the fixed points shift and regions of asymptotic stable translational motion appear (see Fig. 6c), which for $\bar{x}_1 = 1.5, \mu = 0.5$ corresponds to region 2 of Fig. 5.

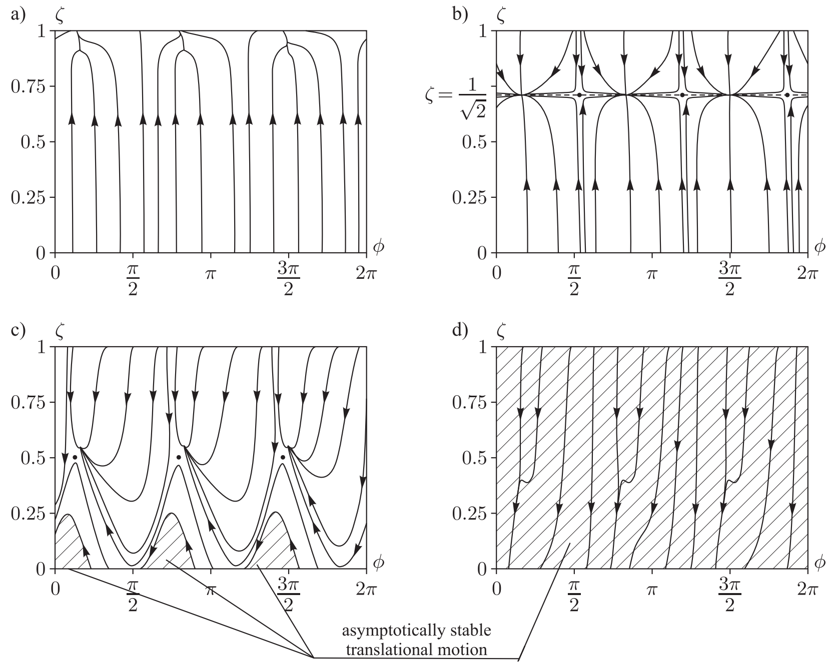

FIGURE 6. Phase trajectories of the system of asymptotic dynamics on the plane (ζ, ϕ) for the parameters a) $\bar{x}_1 = 0.5, \mu = 0.05$, b) $\bar{x}_1 = 1, \mu = 0.05$, c) $\bar{x}_1 = 1.5, \mu = 0.5$, d) $\bar{x}_1 = 2, \mu = 0.5$.

With further increase in the parameters $\bar{x}_1$ all trajectories tend to a degenerate familó of singular points $(\phi, \zeta^* = 0)$, which corresponds to asymptotically pure rotation of the tripod (see Fig. 6d). The translational motion is asymptotically stable in any direction of motion, which corresponds to region 3 of Fig. 5.

4. Motion of the center of mass of the body

Knowing the classification of final motion regimes of the tripod depending on the parameter of the position of its supports, $\bar{x}_1$, and the coefficient μ, we construct the corresponding trajectories of the system (2.8) in absolute space on the plane (X, Y). We show that the trajectory of the center of mass of the tripod and its final dynamics depend substantially on the position of the body's supports. As a tripod we consider a round solid cylinder of radius R which slides on three point supports (the radius of inertia is equal, accordingly, to $\rho = \frac{R}{\sqrt{2}}$). Let us compare the results with those obtained previously in [11] for this system and in [6] for a cylinder with a flat base (disk) under the assumption of linear pressure distribution of the body on the plane of sliding.

In all cases considered, the linear velocity vector of the center of mass at the initial instant of time is directed along the ordinate axis $\boldsymbol{v}(0) = (0, v_0)$ and the body rotates counterclockwise.

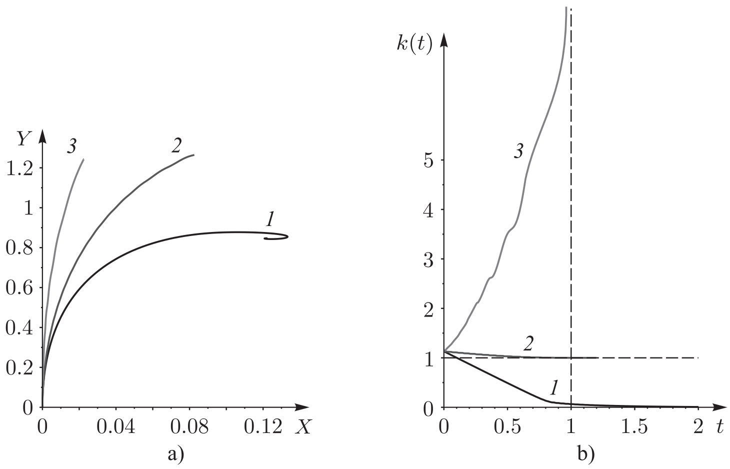

FIGURE 7. Trajectories of the center of mass of the tripod depending on the position of the supports relative to its radius of inertia (a) and the corresponding dependences $k(t)$ (b). The system parameters are $f = 0.23, h = 0.022\,m, R = 0.05\,m, \mu = 0.14, \boldsymbol{v}(0) = (0, 2\,m/s), \omega(0) = 50\,s^{-1}, k(0) = 1.13$. 1. The supports lie inside the circle of the radius of inertia $\bar{x}_1 = 0.1$, the asymptotic motion is pure rotation $k = \frac{v}{\rho\omega} \to 0$. 2. The supports lie on the circle of the radius of inertia $\bar{x}_1 = 1$, the asymptotic motion implies simultaneous cessation of sliding and rotation $k = \frac{v}{\rho\omega} \to 1$. 3. The supports lie outside the circle of the radius of inertia $\bar{x}_1 = 2$, the asymptotic motion is pure sliding $k = \frac{v}{\rho\omega} \to \infty$.

Figure 7a shows the trajectories of the center of mass of three tripods, which, all other factors being equal, differ only in the position of the supporting points relative to the radius of inertia. The trajectory (1) relates to the tripod whose supports lie inside the circle of the radius of inertia ($\bar{x}_1 = 0.1$), as for the trajectory (2), the supports lie on the circle of the radius of inertia ($\bar{x}_1 = 1$), and in the case of the trajectory (3) the supports lie outside the circle of the radius of inertia ($\bar{x}_1 = 2$). It can be seen from the graphs 7a that the trajectory of the center of mass deviates in all cases in the direction opposite to the direction of rotation. *The value of deviation depends substantially on the position of the supports relative to the radius of inertia: the closer the supports are to each other, the greater the deviation.*

Figure 7b shows the dependences of the parameter $k(t) = \frac{v(t)}{\rho\omega(t)}$ for these tripods. For the tripod (1), $k(t) \to 0$, which corresponds to asymptotic pure rotation of the tripod at the instant of stop. For the tripod (2), $k(t) \to 1$, which corresponds to simultaneous cessation of sliding and rotation at the instant of stop. For the tripod (3), $k(t) \to \infty$, which corresponds to asymptotic pure sliding of the tripod at the instant of stop. Depending on the parameters $\bar{x}_1$ and μ, these regimes of final motion of the tripod are in good agreement with the results of classification of asymptotic dynamics, which were obtained in the previous sections (see Fig. 6).

Let us construct the trajectories of the center of mass of the tripods. We shall compare the results with those obtained previously for a tripod in [11] and for a disk in [6].

Figure 8 shows the trajectories of the tripods for the case $k(0) \ll 1$, which corresponds to fast rotation at the initial instant of time. The tripods (1, 2, 3, 4) differ from each other only in the position of the supports relative to the radius of inertia. The dotted trajectory (5) relates to the motion of a disk under the assumption of the linear law of pressure distribution on the plane of sliding. The trajectory of the center of mass of the tripod (4) has been constructed for the case where the supports are arranged according to the conditions of the numerical experiment in [11] (the case $s = 10$ in Fig.4 in [11]), and coincides with it.

Comparing the trajectories in Fig. 8, it is easy to draw the conclusion that the length and the spiral twist at the final stage of motion depend substantially on the position of the supports of the tripod: the closer the supports are to each other, the greater the length and the deviation of the trajectory from the initial direction of motion.

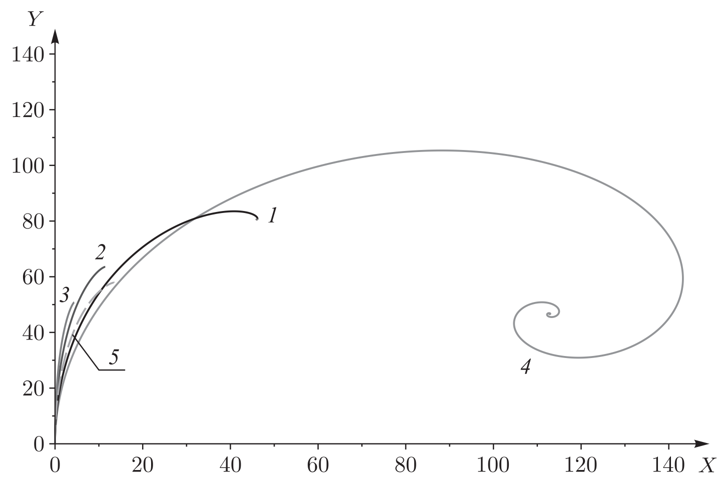

FIGURE 8. Trajectories of the center of mass of the tripod depending on the position of the supports relative to the radius of inertia for $k << 1$. The parameters of the system are $f = 0.5, h = 0.625\,m, R = 0.6\,m, \mu = 0.74, \boldsymbol{v}(0) = (0, 2.5\,m/s), \omega(0) = 400\,s^{-1}, k(0) = 0.015$. 1. The supports lie inside the circle of the radius of inertia $\bar{x}_1 = 0.5$. 2. The supports lie on the circle of the radius of inertia $\bar{x}_1 = 1$. 3. The supports lie on the circle of radius R, $\bar{x}_1 = \sqrt{2}$. 4. The experiment of Shegelski, the supports lie inside the circle of the radius of inertia $\bar{x}_1 \approx 0.12$. 5. Body with a flat base (dotted line).

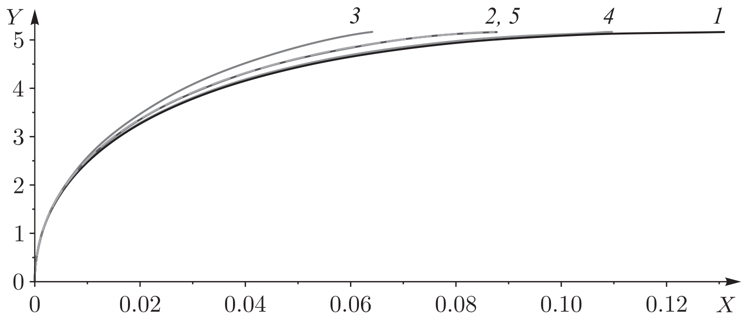

FIGURE 9. Trajectories of the center of mass of the tripod depending on the position of the supports relative to the radius of inertia for $k >> 1$. The parameters of the system are $f = 0.25, h = 0.0625\,m, R = 0.14\,m, \mu = 0.15, \boldsymbol{v}(0) = (0, 5\,m/s), \omega(0) = 8\,s^{-1}, k(0) = 6.3$ 1. The supports lie inside the circle of the radius of inertia $\bar{x}_1 = 0.5$. 2. The supports lie on the circle of the radius of inertia $\bar{x}_1 = 1$. 3. The supports lie on the circle of radius R, $\bar{x}_1 = \sqrt{2}$. 4. The experiment of Shegelski, the supports lie inside the circle of the radius of inertia $\bar{x}_1 \approx 0.63$. 5. Body with a flat base (dotted line).

Moreover, the trajectory of the disk (5) calculated under the condition of linear pressure distribution differs essentially from all trajectories and even from the trajectory (3) constructed for a tripod that slides on supports lying along the perimeter of the cylinder on the circle of radius R.

Figure 9 shows the trajectories of the center of mass of the tripod for the case $k(0) > 1$, which corresponds to slow rotation at the initial instant of time. The trajectory of the center of mass of the tripod (4) has been constructed for the case where the supports are arranged according to the conditions of the numerical experiment in Fig. 5 in [11]. As in the previous experiments, the length and the value of deviations of the trajectories of the center of mass from the initial direction decreases as the distance between the tripod's supports increases. We note that in this case the trajectory of the center of mass of the tripod with supports on the circle of the radius of inertia (2) coincides with the trajectory of the center of mass of the disk (5).

Of interest is the case of "serpantine" (undulatory) trajectory shown in Fig.6 in [11]. In this case, the center of mass can deviate in both directions in the course of motion. Numerical calculation of the trajectories for different positions of the supports relative to the radius of inertia (see Fig. 10a) has shown the presence of an undulatory effect irrespective of the configuration of supports. As expected, the trajectory of the disk (5) deviates in the direction opposite to rotation. We note that in case (1) the asymptotic final dynamics is a pure rotation of the tripod, and in cases (2), (3) and (4) we observe simultaneous cessation of rotation and sliding (see Fig. 10b).

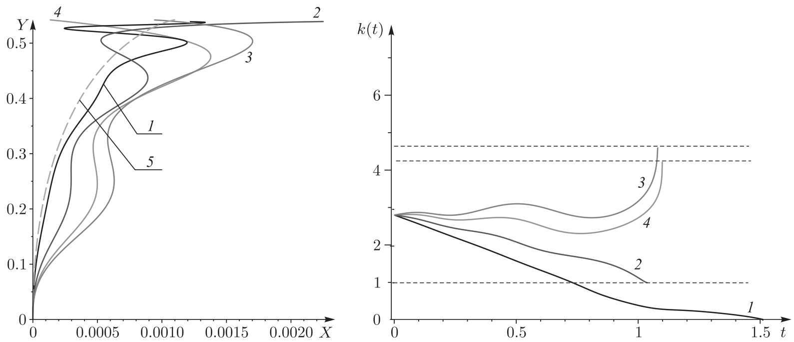

FIGURE 10. Undulatory trajectories of the center of mass of the tripod depending on the position of the supports relative to the radius of inertia of the body (a) and corresponding to the dependence $k(t)$ (b). The parameters of the system are $f = 0.1, h = 0.0044\,m, R = 0.069\,m, \mu = 0.009, \boldsymbol{v}(0) = (0, 1\,m/s), \omega(0) = 7.2\,s^{-1}, k(0) = 2.85$ 1. The supports lie inside the circle of the radius of inertia $\bar{x}_1 = 0.5$, the asymptotic motion is pure rotation $k = \frac{v}{\rho\omega} \to 0$. 2. The supports lie on the circle of the radius of inertia $\bar{x}_1 = 1$, the asymptotic motion implies simultaneous cessation of sliding and rotation $k = \frac{v}{\rho\omega} \to 1$. 3. The supports lie on the circle of radius R, $\bar{x}_1 = \sqrt{2}$, the asymptotic motion implies simultaneous cessation of sliding and rotation $k = \frac{v}{\rho\omega} \to 4.7$. 4. The experiment of Shegelski, the supports lie outside the circle of the radius of inertia $\bar{x}_1 \approx 1.3$, the asymptotic motion implies simultaneous cessation of sliding and rotation $k = \frac{v}{\rho\omega} \to 4.2$. 5. Body with a flat base (dotted line).

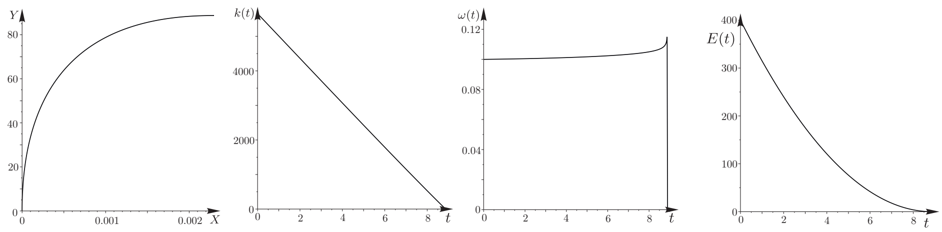

FIGURE 11. Example of an increase in the angular velocity of a tripod which asymptotically tends to pure rotation. The parameters of the system are $f = 0.23, h = 0.022\,m, R = 0.05\,m, \bar{x}_1 = 0.1, \mu = 0.14, \boldsymbol{v}(0) = (0, 20\,m/s), \omega(0) = 0.1\,s^{-1}, k(0) = 5656$.

Moreover, an interesting case of increase in the angular velocity was found for a tripod whose asymptotic motion regime is pure rotation (see Fig. 11). This case corresponds to the dynamics of a reduced system, as shown in Fig. 6a. The initial conditions were chosen to be close to pure sliding. In the course of motion the tripod deviated in the direction opposite to rotation, and the kinetic energy, linear velocity and the relation $k(t)$ decreased, but the angular velocity monotonically increased, albeit only slightly, and rapidly decreased to zero immediately before stopping. This fact contradicts the statement that the absolute value of the spinning velocity obtained in [2] decreases, no matter what the direction of sliding. Such an example of translational motion arising from rotational motion was shown for a bipod as far back as 1970 by Wittenburg [13].

5. Discussion

We have shown that the motions of a tripod *can be asymptotic stable translational motions*, which depends on the position of the tripod's supports relative to its radius of inertia ρ and the coefficient of effective friction $\mu = f\frac{h}{\rho}$. For the case where the supports lie at the apices of an equilateral triangle, the regime of asymptotic motion can be *pure rotation, pure sliding, or sliding and rotation can terminate simultaneously.* The final dynamics is determined by the position of the supports of the tripod relative to the radius of inertia and by the parameter μ. The conclusions qualitatively agree with the results obtained previously for composite disks and bodies with ring supports in [10, 12].

In absolute space, as a rule, *the trajectory of the center of mass of the tripod deviates in the direction opposite to rotation*, and its length and the value of deviation decrease as the distance between the tripod's supports increases. However, under some conditions the trajectories of the center of mass of the tripods can be undulatory, which was pointed out previously in [11].

Comparison of the trajectories of the tripods with those of the disk has not led to any satisfactory results. The coincidence observed in one of the experiments is accidental, and the calculated trajectories of the center of mass of the disk do not demonstrate undulatory motion, no matter what the conditions.

Nevertheless, many interesting problems remain open: explanation of the fact why undulatory trajectories of the tripod arise for certain system parameters, analysis of the dynamics of a bipod and a tripod on an inclined plane (an interesting trajectory of the bipod (dumbbell) on an inclined plane is presented in [13]), preparation and implementation of a natural experiment and comparison of its results with the results of this paper and those obtained in [6, 7].

Acknowledgments. The work was carried out within the framework of the state assignment for institutions of higher education and supported by the RFBR grant No 15-08-09261-a.

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ДИНАМИКА КЛИЗАЊА ТЕЛА ПО ХРАПАВОЈ ПОВРШИ КОЈЕ СЕ ОСЛАЊА У ТРИ ТАЧКЕ

Резиме. У раду се проучава кретање крутог тела (трипода) које се ослања у три тачке на хоризонталну површ и на које делују силе сувог трења. Показано је да режим асимптотског кретања трипода може бити чиста ротација, чисто клизање или ротација и клизање може да се десе истовремено, што је одређено позицијом ослонаца трипода у односу на радијус инерције. Проучавана је и зависност трајекторије центра маса од параметара система. Дата је и компаративна анализа са до сада познатим теоријским и експерименталним студијама кретања тела са равном основом.

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(Received 30.11.2016.)
(Revised 19.12.2016.)
(Available online 22.12.2016.)

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