

NOETHER'S THEOREM FOR NONCONSERVATIVE SYSTEMS IN QUASICOORDINATES

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ABSTRACT. In this paper the generalized Noether's theorem is given in quasicordinates for the systems of particles, the motion of which can be presented in quasicordinates and quasivelocities. After a systematic review of the calculus with quasicordinates and the corresponding Boltzmann–Hamel's equations of motion, the total variation of action is given in quasicordinates. Then, the corresponding generalized Noether's theorem is formulated, valid for nonconservative systems as well, which is obtained from the total variation of action and corresponding Boltzmann–Hamel's equations.

So formulated Noether's theorem in quasicordinates is valid for all conservative and nonconservative systems without any limitation. It is applied to obtain the corresponding energy integrals in quasicordinates for conservative and nonconservative systems, in the latter case these are energy integrals in broader sense. The obtained results are illustrated by a characteristic example, where the corresponding energy integral is found.

This generalized Noether's theorem is equivalent, but not in the form and with some limitation, to the corresponding Noether's theorem formulated by Dj. Djukić [13], which is obtained from the invariance of total variation only of element of action $\Delta(L dt)$. However, for nonconservative systems the Lagrangian L , appearing in this relations, represents not the usual, but an equivalent Lagrangian, which completely determines the considered system, including the influence of nonpotential forces. Therefore, the cited Noether's theorem is valid only for these nonconservative systems for which it is possible to find such equivalent Lagrangian, (what for the natural systems is mostly possible).

1. Introduction

As it is known, Emmy Noether has formulated a theorem, well known under her name, where the problem of invariants in physical theories is analyzed, utilizing theory of groups [1]. In this way she has given a general algorithm for finding a complete set of the invariants of any physical theory represented in terms of Lagrangian or Hamiltonian formalism. This theorem has been adopted and applied to the classical mechanics and theory of fields by E. Hill [2] (see also Dobronravov [3]).

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Later this theorem was formulated also in modern mathematical language by W. Sarlet and F. Cantrijn [4], and applied in the quantum mechanics and quantum theory of fields as well. Afterwards, many studies are dedicated to this theorem in order to deeper and generalize it, presented in the classical and modern mathematical language.

Restricting oneself to the systems of particles in classical mechanics, Noether's theorem can be formulated in the following way. If only the potential forces act on the particles of system, then for every transformation of the generalized coordinates and time which conserves the action $W = \int_{t_0}^{t_1} L(q^i, \dot{q}^i, t) dt$ invariant, or changes it up to so-called calibration term, there is an integral (or constant) of motion. So formulated Noether's theorem is limited to the systems with potential forces, and only a small number of papers consider its generalization to the nonconservative systems, but only in special cases (for example L. Bahar and H. Kwanty [5], L. Duan [6]).

A complete generalization of Noether's theorem to the nonconservative systems was done by B. Vujanović and Dj. Djukić [7–9]. This generalized Noether's theorem is obtained by generalization of the transformation functions of generalized coordinates and time, and by transformation of the total variation of action element $\Delta(L dt)$, or by transformation of d'Alembert–Lagrange's principle into suitable form, expressed in form of total variations. Using this theorem, it is demonstrated that certain nonconservative systems have energy integrals in broader sense, so-called energy-like conservation laws. They have the form of product of an exponential factor, which expresses the influence of allowed nonpotential forces, and the sum of energy of the system and some additional term. Later, this Noether's theorem was extended to the systems with variable mass by L. Cvetičanin [10, 11] and the obtained energy integrals have the analogous form. The equivalent results can be obtained by introduction of pseudoconservative systems (Dj. Mušicki [12]), which are defined as such nonconservative systems, whose Lagrangian equations can be reduced to Euler–Lagrange's equations by introduction of a new Lagrangian.

In one of these papers Dj. Djukić [13] has formulated Noether's theorem in quasicordinates, which is of special interest for this considered problem. It is obtained from invariance of total variation only of element of action $\Delta(L dt)$ in respect to transformation of generalized coordinates and time up to a total differential. However, here is implicitly supposed (what is evident from examples) that for nonconservative systems L is not usual Lagrangian, but an equivalent Lagrangian, which completely determines the considered system of particles, including influence of nonpotential force. Then, such formulated Noether's theorem and its inverse one are applied to certain conservative and nonconservative systems to obtain the corresponding energy integrals.

In this paper we shall formulate generalized Noether's theorem in quasicordinates in direct way, by analogy with corresponding proof in the usual formulation (Dj. Mušicki [14]). It is obtained starting from the multiparametric transformation of quasicordinates and time, and from the corresponding total variation of action, where Boltzmann–Hamel's equations of motions are applied. So obtained Noether's

theorem is formulated in the variables (π^i, ω^i) , where π^i are quasicoordinates and ω^i quasivelocities, but it is always possible to connect all the quantities which figure in these relations with the real, defined quantities. This is in essence equivalent to the cited Djukić's Noether's theorem [13], but only for such nonconservative systems for which it is possible to find such Lagrangian which completely determines the motion of system, including the influence of nonpotential forces.

2. Quasicoordinates and quasivelocities. Boltzmann–Hamel's equations

Introduction of quasicoordinates. Let us consider a set of generalized coordinates q^i ($i = 1, 2, \dots, n$), which determines the position of a mechanical system of particles and let us form a linear combination of differentials of these generalized coordinates, supposing that it cannot be presented as differential of some function (see e.g. Lur'e [15, p. 22–26], Teodorescu [16, p. 423–427])

$$d'\pi^i = a_k^i dq^k \quad (i = 1, 2, \dots, n),$$

where the summation over the repeated indices is understood. Let us still suppose that the coefficients a_k^i depend on the generalized coordinates, but not explicitly on time and here the symbol d' denotes that the quantity on which this symbol is applied is not a total differential (what is not usual). If we divide this quantity by dt

$$(2.1) \quad \omega^i = \frac{d'\pi^i}{dt} = \dot{\pi}^i = a_k^i \dot{q}^k \quad (i = 1, 2, \dots, n),$$

so obtained quantity is not a derivative, it is only quotient of $d'\pi^i$ and dt . However, it can be considered formally as a symbolic (or quasi) derivative with respect to time, denoted here by the symbol \circ over π^i . So introduced quantities π^i , defined only through $d'\pi^i$, are named quasicoordinates, and the corresponding quantities ω^i quasivelocities. If the determinant $|a_k^i| \neq 0$, the system of equations (2.1) can be resolved with respect to the quantities \dot{q}^k ,

$$(2.2) \quad \dot{q}^k = b_i^k \omega^i \Leftrightarrow dq^k = b_i^k d'\pi^i \quad (k = 1, 2, \dots, n)$$

If we insert this expression into the relation (2.1), we have

$$\omega^i = a_k^i (b_j^k \omega^j) = (a_k^i b_j^k) \omega^j$$

and this relation will be satisfied only if

$$(2.3) \quad a_k^i b_j^k = \delta_j^i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

where δ_j^i is Kronecker's symbol.

A typical example of such quantities is the angular velocity of the rigid body, defined as quotient of the vector of elementary rotation $d'\vec{\alpha} = d\alpha \vec{\omega}_0$, and the corresponding time interval dt

$$\vec{\omega} = \frac{d'\vec{\alpha}}{dt} = \frac{d\alpha}{dt} \vec{\omega}_0.$$

Here $d\alpha$ is the rotation angle of the rigid body about the instantaneous axis of rotation in time interval $(t, t + dt)$ and $\vec{\omega}_0$ is the unit vector of this axis of rotation.

Symbolic operations. In order to establish a complete accordance with the usual Lagrangian formulation, where all the relations and equations must be expressed in the variables (q^i, \dot{q}^i) , by using the quasicordinates all relations and equations should be expressed in the variables $(\pi^i, \dot{\pi}^i = \omega^i)$. However, one employs often an equivalent formulation in the variables (q^i, w^i) , since they are immediately determined quantities. In this paper we shall employ the first formulation, but with comprehension or formulation of main results in the variables (q^i, w^i) as well.

In this aim it is necessary to introduce some symbolic operations, which can be defined in the following way. If we have the function $\varphi(q^1, q^2, \dots, q^n)$, its differential according to (2.2) can be written in the form

$$d\varphi = \frac{\partial\varphi}{\partial q^i} dq^i = \frac{\partial\varphi}{\partial q^i} b_k^i d'\pi^k = \left(b_k^i \frac{\partial\varphi}{\partial q^i} \right) d'\pi^k$$

and if we desire to present this expression in the variables (π^i, ω^i) , we shall define the expression in parentheses as $\partial\varphi/\partial\pi^k$

$$(2.4) \quad \frac{\partial\varphi}{\partial\pi^k} \stackrel{def}{=} b_k^i \frac{\partial\varphi}{\partial q^i} \Rightarrow d\varphi = \frac{\partial\varphi}{\partial\pi^k} d'\pi^k$$

This relation defines partial derivative of any function with respect to quasicordinate, and if we multiply it by a_j^k and sum over the repeated index, according to (2.3) we obtain

$$(2.5) \quad \frac{\partial\varphi}{\partial q^j} = a_j^k \frac{\partial\varphi}{\partial\pi^k}$$

Then, by utilizing (2.4) the relation (2.2) can be written in the form

$$\dot{q}^k = b_i^k w^i = b_i^j \delta_j^k w^i = b_i^j \frac{\partial q^k}{\partial q^j} w^i = \frac{\partial q^k}{\partial \pi^i} w^i,$$

which can be interpreted as the decomposition of the generalized velocity \dot{q}^k in the base of vector fields $\partial q^k/\partial\pi^i$ with the corresponding components w^i .

If we consider again this function $\varphi(q^1, q^2, \dots, q^n)$, its total variation as the difference between corresponding values on the varied and real trajectory in the near instants $t + \Delta t$ and t can be presented in the form

$$\Delta\varphi = \bar{\varphi}(t + \Delta t) - \varphi(t) = \bar{\varphi}(t) + \Delta t \left(\frac{\partial \bar{\varphi}}{\partial t} \right)_0 + \dots - \varphi(t) \approx \delta\varphi + \frac{\partial\varphi}{\partial t} \Delta t,$$

where the first term is developed into Taylor's series and higher order terms neglected. If we now introduce the dependences of the function φ on the variables q^i , we have

$$\delta\varphi = \frac{\partial\varphi}{\partial q^i} \delta q^i, \quad \frac{\partial\varphi}{\partial t} = \frac{\partial\varphi}{\partial q^i} \frac{\partial q^i}{\partial t}$$

and the previous relation obtains the form

$$\Delta\varphi = \frac{\partial\varphi}{\partial q^i} \left(\delta q^i + \frac{\partial q^i}{\partial t} \Delta t \right) = \frac{\partial\varphi}{\partial q^i} \Delta q^i$$

Here the derivative $\partial\varphi/\partial q^i$ can be substituted by the expression (2.5)

$$\Delta\varphi \approx \frac{\partial\varphi}{\partial q^i} \Delta q^i = a_i^k \frac{\partial\varphi}{\partial \pi^k} \Delta q^i = \frac{\partial\varphi}{\partial \pi^k} (a_i^k \Delta q^i)$$

and if we again desire to present this expression in the variables (π^i, ω^i) , we shall define the expression in parentheses as $\Delta\pi^k$

$$(2.6) \quad \Delta\pi^k \stackrel{\text{def}}{=} a_i^k \Delta q^i \quad \Rightarrow \quad \delta\pi^k = a_i^k \delta q^i \quad (\text{for } \Delta t = 0)$$

Since the operations of differentiating and varying are independent, we consider that this relation must be treated as a definition, independent from other relations in the calculus with quasicordinates (what is not usual), similarly to the definition of partial derivative with respect to quasicordinates (2.4). In this way, there would be two fundamental symbolic operations: symbolic partial derivative with respect to quasicordinate defined by (2.4), and symbolic total variation (including partial one) of quasicordinate defined by (2.6).

These relations define the variations of quasicordinate, and if we multiply the first one by b_k^j and sum over the repeated index, using again (2.3) we get

$$(2.7) \quad \Delta q^j = b_k^j \Delta\pi^k \quad \Rightarrow \quad \delta q^j = b_k^j \delta\pi^k \quad (\text{for } \Delta t = 0)$$

As a consequence, the elementary work of the active forces can be presented in the form

$$\delta A = Q_i \delta q^i = Q_i b_k^i \delta\pi^k = (b_k^i Q_i) \delta\pi^k,$$

where $Q_\nu = \vec{F}_\nu(\partial\vec{r}_\nu/\partial q^i)$, or more concisely

$$(2.8) \quad \delta A = \Pi_k \delta\pi^k, \quad \Pi_k = b_k^i Q_i$$

This elementary work can be presented in the other form as well, putting $\delta\vec{r}_\nu = (\partial\vec{r}_\nu/\partial\pi^k) \delta\pi^k$

$$\delta A = \vec{F}_\nu \delta\vec{r}_\nu = \left(\vec{F}_\nu \cdot \frac{\partial\vec{r}_\nu}{\partial\pi^k} \right) \delta\pi^k$$

or more concisely

$$(2.9) \quad \delta A = \Pi_k \delta\pi^k, \quad \Pi_k = \vec{F}_\nu \frac{\partial\vec{r}_\nu}{\partial\pi^k}$$

So formulated generalized forces in quasicordinates Π_k by the first relation are connected with the real generalized forces Q_i , and the second relation shows that Π_k can be interpreted as the generalized force which corresponds to the quasicordinate π^k .

One of the main characteristics of quasicordinates is the noncommutativity of the operations δ and d/dt . Let us form the derivative of simultaneous variations of quasicordinate $\delta\pi^i = a_k^i \delta q^k$ with respect to time, and afterwards these operations in the inverse sequence: the simultaneous variation of symbolic time derivative of quasicordinate $\dot{\pi}^i = \omega^i = a_k^i \dot{q}^k$. Their difference can be presented in the following form, bearing in mind that coefficients a_k^i depend only on the generalized coordinates (see e.g. Lur'e [15, p. 32–33])

$$(2.10) \quad (\delta\pi^i)^\cdot - \delta\dot{\pi}^i = \gamma_{jl}^i \omega^j \delta\pi^l \quad (\dot{\pi} = \omega^i),$$

where so-called Boltzmann's threeindices symbol is given by

$$(2.11) \quad \gamma_{jl}^i = b_j^m b_l^k \left(\frac{\partial a_k^i}{\partial q^m} - \frac{\partial a_m^i}{\partial q^k} \right)$$

From here we see that $(\delta\pi^i)^\bullet \neq \delta\pi^i$, i.e. the operations δ and d/dt for quasicordinates are not commutative and so introduced symbols γ_{jl}^i depend only on the liasion between quasicordinates and true generalized coordinates.

On the basis of this relation (2.9) one can find total variation of quasivelocity. If we start from the liasion between the total and simultaneous variation of quasivelocity and substitute simultaneous variation $\delta\omega = \delta\pi^i$ by corresponding expression from (2.10), we have

$$\Delta\omega^i = \bar{\omega}^i(t + \Delta t) - \omega^i(t) = \delta\omega^i + \dot{\omega}^i \Delta t = (\delta\pi^i)^\bullet - \gamma_{jl}^i \omega^j \delta\pi^l + \dot{\omega}^i \Delta t$$

Let us now pass from simultaneous to total variations

$$\Delta\omega^i = (\Delta\pi^i - \omega^i \Delta t)^\bullet - \gamma_{jl}^i \omega^j (\Delta\pi^l - \omega^l \Delta t) + \dot{\omega}^i \Delta t$$

and since due to the property $\gamma_{lj}^i = -\gamma_{jl}^i$ the term $\gamma_{jl}^i \omega^j \omega^l$ is equal to zero, the previous relation will be reduced to

$$(2.12) \quad \Delta\omega^i = \frac{d}{dt}(\Delta\pi^i) - \omega^i \frac{d}{dt}(\Delta t) - \gamma_{jl}^i \omega^j \Delta\pi^l$$

Boltzmann–Hamel's equations. In order to obtain the corresponding differential equations in the Lagrangian form in quasicordinates, let us start from d'Alembert–Lagrange's principle

$$(\vec{F}_\nu - m_\nu \vec{a}_\nu) \cdot \delta\vec{r}_\nu = 0$$

with the usual notations. Let us now transform this principle into the generalized coordinates putting $\delta\vec{r}_\nu = (\partial\vec{r}_\nu/\partial q^i)\delta q^i$, and in this way one obtains

$$\left(Q_i - \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^i} + \frac{\partial T}{\partial q^i} \right) \Delta q^i = 0,$$

where $Q_i = \vec{F}_\nu (\partial\vec{r}_\nu/\partial q^i)$ and T is the kinetic energy of system. Afterwards, let us express this principle in the independent variables (π^i, ω^i) ($i = 1, 2, \dots, n'$), where n' is the number of independent quasicordinates, and in this aim substitute δq^i by the expression (2.7) (compare with Lur'e [15, p. 253–259, 363–372], Whittaker [17, p. 41–44])

$$(2.13) \quad \left(b_k^i Q_i - b_k^i \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^i} + b_k^i \frac{\partial T}{\partial q^i} \right) \delta\pi^k = 0,$$

The first term in the parentheses according to (2.8) represents generalized force Π_k expressed in quasicordinates, and the other terms must be transformed expressing kinetic energy into the form $T^*(q^i, \omega^i, t)$. In this transformation we must bear in mind that T depends on \dot{q}^i through all the variables ω^k , and depends on q^i or directly from the explicit dependence of components of metric tensor or through the variables $\omega^k = a_j^k \dot{q}^j$, where coefficients a_i^k depend on q^i

$$\frac{\partial T}{\partial \dot{q}^i} = \frac{\partial T^*}{\partial \omega^k} \frac{\partial \omega^k}{\partial \dot{q}^i}, \quad \frac{\partial T}{\partial q^i} = \frac{\partial T^*}{\partial q^i} + \frac{\partial T^*}{\partial \omega^k} \frac{\partial (a_j^k \dot{q}^j)}{\partial q^i}$$

After expressing the terms $b_k^i(d/dt)(\partial T/\partial \dot{q}^i)$ and $b_k^i(\partial T/\partial q^i)$ in the variables (π^i, ω^i) , the d'Alembert–Lagrange's principle (2.13) obtains the form

$$\left(\Pi_k - \frac{d}{dt} \frac{\partial T^*}{\partial \omega^k} - \gamma_{lk}^j \omega^l \frac{\partial T^*}{\partial \omega^j} + \frac{\partial T^*}{\partial \pi^k} \right) \delta \pi^k = 0,$$

where the symbols γ_{lk}^j are given by (2.11). Because of the independence of variations $\delta \pi^k$ from here immediately follows

$$\frac{d}{dt} \frac{\partial T^*}{\partial \omega^k} + \gamma_{lk}^j \omega^l \frac{\partial T^*}{\partial \omega^j} - \frac{\partial T^*}{\partial \pi^k} = \Pi_k \quad (k = 1, 2, \dots, n')$$

If if we still decompose the generalized forces Π_k into the potential and nonpotential ones

$$\Pi_k = -\frac{\partial U}{\partial \pi^k} + \tilde{\Pi}_k, \quad \tilde{\Pi}_k = \Pi_k^{pot} = b_k^i \tilde{Q}_i,$$

the previous equations by grouping the similar terms obtain the form

$$(2.14) \quad \frac{d}{dt} \frac{\partial L^*}{\partial \omega^k} + \gamma_{lk}^j \omega^l \frac{\partial L^*}{\partial \omega^j} - \frac{\partial L^*}{\partial \pi^k} = \tilde{\Pi}_k \quad (k = 1, 2, \dots, n'),$$

where

$$L^*(q^i, \omega^i, t) = T^*(q^i, \omega^i, t) - U(q^i, t)$$

These are seeked differential equations of motion expressed in quasicordinates, obtained by L. Boltzmann [18] and G. Hamel [19], in a more general form, when the coefficients a_k^i depend explicitly on time as well (see Lure's [15, p. 32–33, 368–372]). These equations are consequently formulated in the variables (π^i, ω^i) , where the symbolic terms $\partial L^*/\partial \pi^k$ and $\tilde{\Pi}_k$ must be comprehended in the sense of their definitions (2.4) and (2.6). For the holonomic systems the number of these equations is equal to the number of independent quasicordinates π^i , i.e. to the number of degrees of freedom $n' = n$, and if the motion of system is limited by l nonholonomic constraints, the number of these equations will be $n' = n - l$.

3. Total variation of action

As a first step, in the corresponding proof of Noether's theorem, let us formulate total variation of action in quasicordinates. In this aim, let us form this quantity in the variables (q^i, ω^i)

$$\Delta W = \int_{\bar{t}_0}^{\bar{t}_1} L^*(\bar{q}^i, \bar{\omega}^i, \bar{t}) d\bar{t} - \int_{t_0}^{t_1} L^*(q^i, \omega^i, t) dt,$$

where

$$\bar{q}^i = q^i + \Delta q^i, \quad \bar{\omega}^i = \omega^i + \Delta \omega^i, \quad \bar{t} = t + \Delta t,$$

and decompose the first integral into three parts

$$\int_{\bar{t}_0}^{\bar{t}_1} L^*(\bar{q}^i, \bar{\omega}^i, \bar{t}) d\bar{t} = \int_{t_0 + \Delta t_0}^{t_0} \bar{L}^* d\bar{t} + \int_{t_0}^{t_1} \bar{L}^* d\bar{t} + \int_{t_1}^{t_1 + \Delta t_1} \bar{L}^* d\bar{t}$$

Because of little intervals of integration the first and the third integral can be neglected with respect to the second one, on condition that Lagrangian in these intervals is a regular function. In this way the total variation of action is reduced to

$$(3.1) \quad \Delta W = \int_{t_0}^{t_1} \bar{L}^*(\bar{q}^i, \bar{\omega}^i, \bar{t}) d\bar{t} - \int_{t_0}^{t_1} \bar{L}^*(q^i, \omega^i, t^i) dt$$

If we put in the first integral

$$d\bar{t} = d(t + \Delta t) = dt + \frac{d}{dt}(\Delta t) dt$$

and develop function $L^*(\bar{q}^i, \bar{\omega}^i, \bar{t})$ in Taylor's series

$$\begin{aligned} L^*(q^i + \Delta q^i, \omega^i + \Delta \omega^i, t + \Delta t) &\approx L^*(q^i, \omega^i, t) + \Delta q^i \left(\frac{\partial L^*}{\partial q^i} \right)_0 \\ &\quad + \Delta \omega^i \left(\frac{\partial L^*}{\partial \omega^i} \right)_0 + \Delta t \left(\frac{\partial L^*}{\partial t} \right)_0 \end{aligned}$$

neglecting the terms of higher order, the total variation of action (3.1) obtains the form

$$\Delta W \approx \int_{t_0}^{t_1} \left[\frac{\partial L^*}{\partial q^i} \Delta q^i + \frac{\partial L^*}{\partial \omega^i} \Delta \omega^i + \frac{\partial L^*}{\partial t} \Delta t + L^* \frac{d}{dt}(\Delta t) \right] dt$$

If we here substitute $\Delta \omega^i$ by the corresponding expression (2.12) and put according to (2.7) and (2.4)

$$\frac{\partial L^*}{\partial q^i} \Delta q^i = \frac{\partial L^*}{\partial q^i} (b_k^i \Delta \pi^k) = \left(b_k^i \frac{\partial L^*}{\partial q^i} \right) \Delta \pi^k = \frac{\partial L^*}{\partial \pi^k} \Delta \pi^k,$$

the previous relation becomes

$$(3.2) \quad \Delta W = \int_{t_0}^{t_1} \left\{ \frac{\partial L^*}{\partial \pi^k} \Delta \pi^k + \frac{\partial L^*}{\partial \omega^i} \left[\frac{d}{dt}(\Delta \pi^i) - \omega^i \frac{d}{dt}(\Delta t) - \gamma_{ik}^i \omega^l \Delta \pi^k \right] + \frac{\partial L^*}{\partial t} \Delta t + L^* \frac{d}{dt}(\Delta t) \right\} dt$$

Let us now set apart these terms in this relation which can be partially written in the form of a total derivative with respect to time

$$\begin{aligned} \frac{\partial L^*}{\partial \omega^i} \frac{d}{dt}(\Delta \pi^i) &= \frac{d}{dt} \left(\frac{\partial L^*}{\partial \pi^i} \Delta \pi^i \right) - \frac{d}{dt} \left(\frac{\partial L^*}{\partial \pi^i} \right) \Delta \pi^i \\ \frac{\partial L^*}{\partial \omega^i} \omega^i \frac{d}{dt}(\Delta t) &= \frac{d}{dt} \left(\frac{\partial L^*}{\partial \omega^i} \omega^i \Delta t \right) - \frac{d}{dt} \left(\frac{\partial L^*}{\partial \omega^i} \right) \omega^i \Delta t - \frac{\partial L^*}{\partial \omega^i} \dot{\omega}^i \Delta t \\ L^* \frac{d}{dt}(\Delta t) &= \frac{d}{dt} (L^* \Delta t) - \frac{dL^*}{dt} \Delta t = \frac{d}{dt} (L^* \Delta t) - \left(\frac{\partial L^*}{\partial \pi^i} \dot{\pi}^i + \frac{\partial L^*}{\partial \omega^i} \dot{\omega}^i + \frac{\partial L^*}{\partial t} \right) \Delta t \end{aligned}$$

If we insert these expressions into relation (3.2), we have

$$\begin{aligned} \Delta W &= \int_{t_0}^{t_1} \left\{ \frac{\partial L^*}{\partial \pi^i} \Delta \pi^i + \frac{d}{dt} \left(\frac{\partial L^*}{\partial \omega^i} \Delta \pi^i \right) - \frac{d}{dt} \left(\frac{\partial L^*}{\partial \omega^i} \right) \Delta \pi^i - \frac{d}{dt} \left(\frac{\partial L^*}{\partial \omega^i} \omega^i \Delta t \right) \right. \\ &\quad \left. + \frac{d}{dt} \left(\frac{\partial L^*}{\partial \omega^i} \right) \omega^i \Delta t + \frac{\partial L^*}{\partial \omega^i} \dot{\omega}^i \Delta t + \frac{\partial L^*}{\partial t} \Delta t + \frac{d}{dt} (L^* \Delta t) \right\} dt \end{aligned}$$

$$- \left(\frac{\partial L^*}{\partial \pi^i} \dot{\omega}^i + \frac{\partial L^*}{\partial \omega^i} \dot{\omega}^i + \frac{\partial L^*}{\partial t} \right) \Delta t - \gamma_{ik}^i \omega^l \frac{\partial L^*}{\partial \omega^i} \Delta \pi^k \Big\}$$

and after grouping similar terms, this expression can be written in the form

$$(3.3) \quad \Delta W = \int_{t_0}^{t_1} \left\{ \frac{d}{dt} \left[\frac{\partial L^*}{\partial \omega^i} (\Delta \pi^i - \omega^i \Delta t) + L^* \Delta t \right] \right. \\ \left. + \left(\frac{\partial L^*}{\partial \pi^i} - \frac{d}{dt} \frac{\partial L^*}{\partial \omega^i} \right) (\Delta \pi^i - \omega^i \Delta t) - \gamma_{ik}^i \omega^l \frac{\partial L^*}{\partial \omega^i} \Delta \pi^k \right\} dt$$

This formula represents total variation of action in quasicordinates presented consequently in variables (π^i, ω^i) , where the symbolic expression $\partial L^* / \partial \pi^i$ and $\Delta \pi^i$ have to be comprehended in the sense of their definitions (2.4) and (2.6), by which they are connected with the real, defined quantities.

4. Generalized Noether's theorem in quasicordinates

Formulation of the problem. If the motion of the considered system of particles can be presented in the variables (q^i, ω^i) or (π^i, ω^i) , let us formulate the corresponding Emmy Noether's theorem in quasicordinates. Here we shall generalize the usual proof of this theorem to the nonconservative systems, by analogy with the corresponding proof in the usual Lagrangian formulation (Dj. Mušicki [14]), starting from corresponding total variation of action and applying Boltzmann–Hamel's equations of motion.

Let us choose the total variations of quasicordinates and time $\Delta \pi^i$ and Δt in total variation of action (3.3) in the following form with r infinitesimal parameters \mathcal{E}^m ($m = 1, 2, \dots, r$),

$$(4.1) \quad \Delta \pi^i = a_k^i \Delta q^k = \mathcal{E}^m \xi_m^i(q^k, \omega^k, t), \quad \Delta t = \mathcal{E}^m \xi_m^0(q^k, \omega^k, t)$$

Let us now formulate in a usual way the problem: find such transformations of quasicordinates and time (4.1) which conserve the Hamilton's action invariant or change it up to so-called calibration term

$$(4.2) \quad \Delta W = 0 \quad \text{or} \quad \Delta W = \int_{t_0}^{t_1} \frac{d}{dt} \Lambda(q^i, \omega^i, t) dt,$$

where Λ , so-called calibration function is an arbitrary function of the variables q^i, ω^i and t . So formulated second condition is more general and includes the first one for $\Lambda = 0$

Generalized Noether's theorem in quasicordinates. . Let us start from the second condition (4.2) and in order to include the speciality of the considered problem, let us substitute variational derivative in the expression (3.3) for total variation of action by the corresponding expression from Boltzmann–Hamel's equations (2.14)

$$\int_{t_0}^{t_1} \left\{ \frac{d}{dt} \left[\frac{\partial L^*}{\partial \omega^i} (\Delta \pi^i - \omega^i \Delta t) + L^* \Delta t \right] + \left(\gamma_{li}^j \omega^l \frac{\partial L^*}{\partial \omega^j} - \tilde{\Pi}_i \right) \right. \\ \left. \times (\Delta \pi^i - \omega^i \Delta t) - \gamma_{li}^j \omega^l \frac{\partial L^*}{\partial \omega^j} \Delta \pi^i \right\} dt = \int_{t_0}^{t_1} \frac{d\Lambda}{dt} dt$$

Here, because of the property $\gamma_{kl}^i = -\gamma_{lk}^i$ we have $\gamma_{lk}^i \omega^l \omega^i \equiv 0$ and this relation is reduced to an equivalent form

$$(4.3) \quad \int_{t_0}^{t_1} \left\{ \frac{d}{dt} \left[\frac{\partial L^*}{\partial \omega^i} \Delta \pi^i + \left(L^* - \frac{\partial L^*}{\partial \omega^i} \omega^i \right) \Delta t - \Lambda \right] - \tilde{\Pi}_i (\Delta \pi^i - \omega^i \Delta t) \right\} dt = 0$$

On the other side, we can transform ΔW in the following way, bearing in mind that $\Delta(dt) = d(\Delta t)$

$$\Delta W = \Delta \int_{t_0}^{t_1} L^* dt = \int_{t_0}^{t_1} \Delta(L^* dt) = \int_{t_0}^{t_1} \left[\Delta L^* + L^* \frac{d}{dt}(\Delta t) \right] dt$$

and let us insert this expression into the second condition (4.2)

$$(4.4) \quad \int_{t_0}^{t_1} \left[\Delta L^* + L^* \frac{d}{dt}(\Delta t) - \frac{d\Lambda}{dt} \right] dt = 0$$

If we now develop ΔL^* , substituting $\Delta \omega^i$ by the corresponding expression (2.12), we have

$$\Delta L^* = \frac{\partial L^*}{\partial \pi^k} \Delta \pi^k + \frac{\partial L^*}{\partial \omega^i} \left[\frac{d}{dt}(\Delta \pi^i) - \omega^i \frac{d}{dt}(\Delta t) - \gamma_{lk}^i \omega^l \Delta \pi^k \right] + \frac{\partial L^*}{\partial t} \Delta t$$

and if we insert this expression into (4.4), we get

$$(4.5) \quad \int_{t_0}^{t_1} \left[\left(\frac{\partial L^*}{\partial \pi^k} - \gamma_{lk}^i \omega^l \frac{\partial L^*}{\partial \omega^i} \right) \Delta \pi^k + \frac{\partial L^*}{\partial \omega^i} \frac{d}{dt}(\Delta \pi^i) + \frac{\partial L^*}{\partial t} \Delta t + \left(L^* - \frac{\partial L^*}{\partial \omega^i} \omega^i \right) \frac{d}{dt}(\Delta t) - \frac{d\Lambda}{dt} \right] dt = 0$$

Let us now subtract relation (4.5) from relation (4.3), what can be presented in form

$$\begin{aligned} & \int_{t_0}^{t_1} \left\{ \frac{d}{dt} \left[\frac{\partial L^*}{\partial \omega^i} \Delta \pi^i + \left(L^* - \frac{\partial L^*}{\partial \omega^i} \omega^i \right) \Delta t - \Lambda \right] \right. \\ & \quad - \left[\left(\frac{\partial L^*}{\partial \pi^k} - \gamma_{lk}^i \omega^l \frac{\partial L^*}{\partial \omega^i} \right) \Delta \pi^k + \frac{\partial L^*}{\partial \omega^i} \frac{d}{dt}(\Delta \pi^i) + \frac{\partial L^*}{\partial t} \Delta t \right. \\ & \quad \left. \left. + \left(L^* - \frac{\partial L^*}{\partial \omega^i} \omega^i \right) \frac{d}{dt}(\Delta t) + \tilde{\Pi}_i (\Delta \pi^i - \omega^i \Delta t) - \frac{d\Lambda}{dt} \right] \right\} dt = 0 \end{aligned}$$

and if we substitute $\Delta \pi^i$ and Δt by the corresponding expressions (4.1) and put $\Lambda = \mathcal{E}^m \Lambda_m$, we obtain

$$(4.6) \quad \int_{t_0}^{t_1} \mathcal{E}^m \left\{ \frac{d}{dt} \left[\frac{\partial L^*}{\partial \omega^i} \xi_m^i + \left(L^* - \frac{\partial L^*}{\partial \omega^i} \omega^i \right) \xi_m^0 - \Lambda_m \right] \right. \\ \quad - \left[\left(\frac{\partial L^*}{\partial \pi^k} - \gamma_{lk}^i \omega^l \frac{\partial L^*}{\partial \omega^i} \right) \xi_m^k + \frac{\partial L^*}{\partial \omega^i} \dot{\xi}_m^i + \frac{\partial L^*}{\partial t} \xi_m^0 \right. \\ \quad \left. \left. + \left(L^* - \frac{\partial L^*}{\partial \omega^i} \omega^i \right) \dot{\xi}_m^0 + \tilde{\Pi}_i (\xi_m^i - \omega^i \xi_m^0) - \dot{\Lambda}_m \right] \right\} dt = 0$$

From here we can conclude: if the following condition is satisfied

$$(4.7) \quad \left(\frac{\partial L^*}{\partial \pi^k} - \gamma_{lk}^i \omega^l \frac{\partial L^*}{\partial \omega^i} \right) \xi_m^k + \frac{\partial L^*}{\partial \omega^i} \dot{\xi}_m^i + \frac{\partial L^*}{\partial t} \xi_m^0 + \left(L^* - \frac{\partial L^*}{\partial \omega^i} \omega^i \right) \dot{\xi}_m^0 + \tilde{\Pi}_i (\xi_m^i - \omega^i \xi_m^0) - \dot{\Lambda}_m = 0,$$

the relation (4.6) will be reduced to

$$\int_{t_0}^{t_1} \mathcal{E}^m \frac{d}{dt} \left[\frac{\partial L^*}{\partial \omega^i} \xi_m^i + \left(L^* - \frac{\partial L^*}{\partial \omega^i} \omega^i \right) \xi_m^0 - \Lambda_m \right] dt = 0$$

and because of the arbitrariness of the parameters \mathcal{E}^m and the limits of integration from here immediately follows

$$(4.8) \quad \mathcal{I}_m = \frac{\partial L^*}{\partial \omega^i} \xi_m^i + \left(L^* - \frac{\partial L^*}{\partial \omega^i} \omega^i \right) \xi_m^0 - \Lambda_m = \text{const.} \quad (m = 1, 2, \dots, r)$$

These results can be expressed by means of immediately determined quantities as well with aid of (2.4) and (2.8). In this way the condition for existence of energy integrals (4.7) obtains the form

$$(4.9) \quad \left(b_k^i \frac{\partial \tilde{L}}{\partial q^i} - \gamma_{lk}^i \omega^l \frac{\partial \tilde{L}}{\partial \omega^i} \right) \xi_m^k + \frac{\partial \tilde{L}}{\partial \omega^i} \dot{\xi}_m^i + \frac{\partial \tilde{L}}{\partial t} \xi_m^0 + \left(\tilde{L} - \frac{\partial \tilde{L}}{\partial \omega^i} \omega^i \right) \dot{\xi}_m^0 + b_i^k \tilde{Q}_k (\xi_m^i - \omega^i \xi_m^0) - \tilde{\Lambda}_m = 0,$$

and the form of integrals of motion (4.8) remains unchanged.

Conclusion. For every transformation of the quasicordinates and time (4.1), for which there is at least one set of particular solutions $(\xi_m^i, \xi_m^0, \Lambda_m)$ which satisfies the condition (4.7) or (4.9), there are r mutually independent integrals (or constants) of motion of the form (4.8). So formulated Noether's theorem in quasicordinates is valid in general, for all the conservative and nonconservative systems without any limitation. It is formulated in the variables (π^i, ω^i) as well as in the variables (q^i, ω^i) , starting from the multiparametric transformation of these variables, in the form where the influence of nonpotential forces appears explicitly.

This generalized Noether's theorem is in essence (but not in the form) equivalent to the corresponding Noether's theorem given by Dj. Djukić [13], but with some limitation. Namely, this equivalence exists only for such nonconservative systems for which it is possible to find some equivalent Lagrangian, which completely determines the motion of considered system, including the influence of nonpotential forces (as can be seen from his examples), what for the natural systems is mostly possible.

5. Energy integrals for the conservative systems in quasicordinates

Energy integrals as a consequence of the translation of time. Let us demonstrate how from this generalized Noether's theorem one can obtain the energy integrals of any system whose motion can be presented in the quasicordinates (for example the rigid body) of the considered conservative systems, i.e. corresponding energy conservation laws. In this aim, let us consider the translation of time, i.e. choose the functions ξ_m^i , ξ_m^0 and Λ_m in form

$$(5.1) \quad \xi_m^i = 0, \quad \xi_m^0 = A = \text{const.}, \quad \Lambda_m = 0.$$

Then the condition for the existence of the integrals of motion (4.7) obtain form

$$(5.2) \quad \frac{\partial L^*}{\partial t} \cdot A - \tilde{\Pi}_i \omega^i A = 0,$$

which is analogous to the corresponding condition $\partial L/\partial t - \tilde{Q}_i \dot{q}^i = 0$ in the usual Lagrangian formulation. This condition can be expressed in terms of the usual quantities as well, according to (2.7) in variables (q^i, ω^i)

$$(5.3) \quad \frac{\partial L^*}{\partial t} - b_i^k \tilde{Q}_k \omega^i = 0.$$

Then the corresponding integral of motion (4.8) on the basis of (5.1) will be

$$(5.4) \quad \mathcal{I}_m = \left(L^* - \frac{\partial L^*}{\partial \omega^i} \omega^i \right) A = \text{const.} \quad \Leftrightarrow \quad \mathcal{E} = \frac{\partial L^*}{\partial \omega^i} \omega^i - L^* = \text{const.}$$

and so defined quantity \mathcal{E} represents generalized enery expressed in quasicordinates.

Therefore, as a consequence of translation of time, which according to (5.2) and (5.3) is equivalent to the satisfied conditions $\partial L^*/\partial t = 0$ and $\tilde{\Pi}_i \omega^i = 0$ or $b_k^i \tilde{Q}_k \omega^i = 0$, the energy integral (5.4) appears, i.e. energy conservation law in the course of time.

Energy integrals in the usual Lagrangian formulation. In order to present the energy integrals of such systems in the variables (q^i, ω^i) , let us write their kinetic energy in the general case as

$$T = \frac{1}{2} d_{jl} \dot{q}^j \dot{q}^l + e_j \dot{q}^j + f$$

and putting according to (2.2) $\dot{q}^j = b_i^j \omega^i$ and $\dot{q}^l = b_k^l \omega^k$, we can present this kinetic energy in the quasivelocities ω^i

$$(5.5) \quad T^*(q^i, \omega^i, t) = \frac{1}{2} a_{ik} \omega^i \omega^k + b_i \omega^i + c = T_2^* + T_1^* + T_0^*,$$

where

$$a_{ik} = b_i^j b_k^l d_{jl}, \quad b_i = b_i^j e_j, \quad c = f.$$

In the case of the free or scleronomic systems, where time does not figures explicitly, the kinetic energy (5.5) has only the first term and their Lagrangian for the usual forces, dependant only on generalized coordinates and time, will be

$$L^*(q^i, \omega^i, t) = \frac{1}{2} a_{ik} \omega^i \omega^k - U_{(q^i, t)} = T_2 - U,$$

where U is the potential energy of the system. If we choose again the transformation functions in the form (5.1), the corresponding integral of motion (4.8), which now becomes energy integral obtains the form

$$\mathcal{I}_m = \left(L^* - \frac{\partial L^*}{\partial \omega^i} \omega^i \right) A = \left(T_2^* - U - \frac{\partial T_2^*}{\partial \omega^i} \omega^i \right) A = \text{const.}$$

and by application of Euler's theorem for homogeneous functions we get energy integral of such systems in quasicordinates

$$\mathcal{E}^* = -\mathcal{I}_m = \frac{\partial L^*}{\partial \omega^i} \omega^i - L^* = T_2^* + U = \text{const.}$$

But, for rheonomic systems the kinetic energy (5.5) has all three terms, and Lagrangian is equal to

$$L^*(q^i, \omega^i, t) = \frac{1}{2}a_{ik}\omega^i\omega^k + b_i\omega^i + c - U_{(q^i, \omega^i)} = T_2^* + T_1^* + T_0^* - U.$$

Then the corresponding integral of motion (4.8), i.e. the corresponding energy integral will be

$$\mathcal{I}_m = \left(L^* - \frac{\partial L^*}{\partial \omega^i} \omega^i \right) A = \left(T_2^* + T_1^* + T_0^* - U - \frac{\partial T_2^*}{\partial \omega^i} \omega^i - \frac{\partial T_1^*}{\partial \omega^i} \omega^i \right) A = \text{const.},$$

i.e., applying the cited Euler's theorem

$$\mathcal{E}^* = -\mathcal{I}_m = \frac{\partial L^*}{\partial \omega^i} \omega^i - L^* = T_2^* - T_0^* + U = \text{const.},$$

this is Painlevé's energy integral for such systems in quasicordinates.

From here we see that the specific term $\gamma_{ik}^i \omega^l (\partial L^* / \partial \omega^i)$ is absent in the energy integrals, they have the same form as in the case of true generalized coordinates, this term figures only in the condition for existence of energy integrals.

6. One example

Formulation of the problem. Let us consider the motion of a rigid body in a viscous medium without influence of the exterior forces about a fixed point, which coincides with the centre of masses. The position of this rigid body can be determined by Euler's angles (ψ, θ, φ) , which then represent the generalized coordinates. As it is known (see e.g. Goldstein [20, p.198–205], it is the most suitable to study the motion of rigid body in a coordinate system whose origin is in the centre of masses and whose axes are the principal axes of inertia. In this frame of reference the motion of the rigid body can be determined by Euler's equations

$$\begin{aligned} I_1 \left(\frac{d\omega^1}{dt} \right)_{\text{rel}} - (I_2 - I_3) \omega^2 \omega^3 &= L_1^{(c)} \\ I_2 \left(\frac{d\omega^2}{dt} \right)_{\text{rel}} - (I_3 - I_1) \omega^3 \omega^1 &= L_2^{(c)} \\ I_3 \left(\frac{d\omega^3}{dt} \right)_{\text{rel}} - (I_1 - I_2) \omega^1 \omega^2 &= L_3^{(c)}, \end{aligned}$$

where I_1, I_2 and I_3 are the moments of inertia with respect to the principal axes of inertia, ω^1, ω^2 and ω^3 the corresponding contravariant components of angular velocity, and $L_1^{(c)}, L_2^{(c)}$ and $L_3^{(c)}$ the components of the moment of all the forces which act on the rigid body in respect to the center of masses. In this frame of reference the kinetic energy of rigid body has the form

$$T^*(q^i, \omega^i, t) = \frac{1}{2} I_i g_{ik} \omega^i \omega^k = \frac{1}{2} I_i \omega_i \omega^i,$$

where g_{ik} is the corresponding metric tensor, and between the components angular momentum and angular velocity, there are the simple relations

$$M_1^{(c)} = I_1 \omega_1, \quad M_2^{(c)} = I_2 \omega_2, \quad M_3^{(c)} = I_3 \omega_3.$$

Let us suppose that the components of resistance force, by which the viscous medium opposes to the motion of rigid body, are proportional to the corresponding components of angular momentum

$$(6.1) \quad \begin{aligned} \tilde{F}_1 &= -\mu M_1^{(c)} = -\mu I_1 \omega_1, \\ \tilde{F}_2 &= -\mu M_2^{(c)} = -\mu I_2 \omega_2, \\ \tilde{F}_3 &= -\mu M_3^{(c)} = -\mu I_3 \omega_3. \end{aligned}$$

So formulated components of resistance force are proportional to the corresponding components of angular velocity, similarly to the corresponding resistance force by which a viscous medium opposites to the motion of a particle $\tilde{F}_i = -\mu m v_i$ ($i = 1, 2, 3$).

The manner of representation of all results in this paper is given in the variables (π^i, ω^i) or (q^i, ω^i) and all the quantities and relations must be presented in this way. So the Lagrangian, which because absence of the active forces is reduced to kinetic energy is equal to

$$(6.2) \quad L^*(q^i, \omega^i, t) = \frac{1}{2} I_i \omega_i \omega^i \quad (U = 0)$$

and it is immediately given in these variables, but without explicit appearance q^i or π^i . In a similar way, the quantities \tilde{F}_i in the relations (6.1) can be considered as the components $\tilde{\Pi}_i$ of nonpotential part of generalized forces in quasicordinates, defined by (2.7), but without explicitly figuring π^i or q^i

$$\tilde{\Pi}_1 = \tilde{F}_1 = -\mu I_1 \omega_1, \quad \tilde{\Pi}_2 = \tilde{F}_2 = -\mu I_2 \omega_2, \quad \tilde{\Pi}_3 = \tilde{F}_3 = -\mu I_3 \omega_3.$$

Application of generalized Noether's theorem. Let us now apply this generalized Noether's theorem to this problem. In this aim, let us start from the condition for existence of integrals of motion (4.7) and choose the functions ξ_m^i, ξ_m^0 and Λ_m in the form

$$\xi_m^i = 0, \quad \xi_m^0 = \varphi(t), \quad \Lambda_m = 0.$$

Since here generalized coordinates are Euler' angles (ψ, θ, φ) and Lagrangian (6.2) does not depend on Euler's angles, the term $\partial L^*/\partial \pi^k$ in (4.7) according (2.4) is equal to

$$\frac{\partial L^*}{\partial \pi^k} = b_k^i \frac{\partial L^*}{\partial q^i} = b_k^1 \frac{\partial L^*}{\partial \psi} + b_k^2 \frac{\partial L^*}{\partial \theta} + b_k^3 \frac{\partial L^*}{\partial \varphi} = 0$$

Bearing in mind that the energy of the rigid body is $(\partial L^*/\partial \omega^i) \omega^i - L^* = \frac{1}{2} I_i \omega_i \omega^i$ and $\partial L^*/\partial t = 0$, the cited condition (4.7) gets the form

$$-\frac{1}{2} I_i \omega_i \omega^i \cdot \frac{d\varphi}{dt} - \mu I_i \omega_i (-\omega^i \varphi) = 0$$

and this equation gives immediately one particular solution

$$\frac{d\varphi}{dt} - 2\mu\varphi = 0 \quad \Rightarrow \quad \varphi(t) = e^{2\mu t}.$$

In this way, the condition for existence of integrals of motion is satisfied, and the corresponding integral of motion (4.8) will be

$$\mathcal{I}_m = \left(L^* - \frac{\partial L^*}{\partial \omega^i} \omega^i \right) \varphi(t) = -\frac{1}{2} I_i \omega_i \omega^i e^{2\mu t} = \text{const.}$$

and this is one energy integral in a broader sense

$$\mathcal{I}'_m = \frac{1}{2} e^{2\mu t} I_i \omega_i \omega^i = \text{const.}$$

This result is in full accordance with the result obtained by Dj. Djukić [13], by means of his generalized Noether's theorem, starting from the corresponding invariance only of element of action $\Delta(L dt)$ up to a total differential, but with the Lagrangian $L = e^{2\mu t} I_i(\omega^i)^2$.

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НЕТЕРИНА ТЕОРЕМА ЗА НЕКОНЗЕРВАТИВНЕ СИСТЕМЕ У КВАЗИКООРДИНАТАМА

РЕЗИМЕ. У овом раду дато је уопштење Нетерине теореме за системе материјалних тачака, чије кретање може бити представљено у квазикоординатама и квазибрзинама. Након детаљног подсећања рачуна у квазикоординатама и одговарајућих Болцман-Хамелих једначина кретања, дата је тотална варијација дејства у квазикоординатама. Затим је из тоталне варијације дејства и Болцман-Хамелових једначина изведено одговарајуће уопштење Нетерине теореме.

Тако формулисана Нетерина теорема у квазикоординатама важи за све конзервативне и неконзервативне системе без икаквих ограничења. Теорема је примењена на извођењу одговарајућег интеграла енергије у квазикоординатама за конзервативне и неконзервативне системе. У другом случају то су интеграл енергије у ширем смислу. Добијени резултати су илустровани карактеристичном примером, где је пронађен одговарајући интеграл енергије.

Овако уопштена Нетерина теорема је еквивалентна, са извесним ограничењима, са одговарајућом Нетерином теоремом формулисаном од стране Ђ. Ђукића [13], која се добија из инваријантности укупне варијације само за елемент дејства $\Delta(L dt)$. Међутим, за неконзервативне системе Лагражијан L који се појављује у овим релацијама, представља не уобичајени, већ еквиваленти Лагранжијан, који у потпуности одређује посматрани систем, укључујући утицај непотенцијалних сила. Стога, наведена Нетерина теорема важи само за оне неконзервативне системе за које је могуће да се нађе еквивалентни Лагранжијан (што је за природне механичке системе углавном могуће).

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