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STRESS INTEGRATION OF THE DRUCKER–PRAGER MATERIAL MODEL WITH KINEMATIC HARDENING

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Abstract. This paper presents a method for implicit stress integration of the Drucker–Prager material model with kinematic hardening. The stress integration of the material model is conducted using the incremental plasticity method, while the kinematic hardening of material is defined using nonlinear Armstrong–Frederick hardening. This type of granular material hardening occurs as a consequence of the cyclic loading effects, such as the seismic load. For this reason, this material model is used for the earthquake analysis in the soil mechanics. Yield surface of the material model changes its position under the cyclic loads in the stress space, whereas there is no change in the size of the yield surface in deviatoric plane. The developed algorithm of the material model has been implemented in the software package PAK.

Nomenclature

$\boldsymbol{\sigma}$	stress tensor
σ_m	mean stress
$\mathbf{e}$	total strain
$\mathbf{e}^E$	elastic strain
$\mathbf{e}^P$	plastic strain
f	yield function
g	plastic potential function
$d\lambda$	plastic parameter
$\mathbf{C}^E$	elastic constitutive matrix
$\mathbf{C}^{EP}$	elastic-plastic constitutive matrix
α	Drucker–Prager material parameter
β	Drucker–Prager material parameter
k	Drucker–Prager material parameter
I_1	first stress invariant
J_{2D}	second deviatoric stress invariant
t	time
$\mathbf{s}$	deviatoric stress

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$\hat{\mathbf{s}}$	stress radius
$\boldsymbol{\alpha}$	back stress
$\mathbf{m}$	identity matrix
$\mathbf{C_1}$	Armstrong–Frederick material parameter
$\mathbf{C_2}$	Armstrong–Frederick material parameter
$d\bar{e}^P$	equivalent plastic strain increment

1. Introduction

Stress integration represents calculation of stress change during an incremental step, corresponding to strain increments in the step. The essence of the incremental integration of inelastic constitutive relations is to trace the history of material deformation. The stress integration is an important ingredient in the overall finite element inelastic analysis of structures. It is crucial that the integration algorithm accurately reproduces the material behaviour since the accuracy of the numerical results directly depends on accuracy of the stress integration algorithm. The algorithm should be also computationally efficient because the stress integration is performed at all integration points. For general applications, this computational procedure should be robust, providing reliable results under all possible loading conditions. This paper presents a formulation of the computational algorithm for the Drucker–Prager (DP) constitutive model [1] with kinematic hardening feature, using incremental plasticity method (IPM) [2].

2. Elastic-plastic constitutive matrix using incremental plasticity method

Elastic-plastic constitutive models are described using elastic-plastic constitutive relations. In theory of incremental plasticity, stress is directly proportional to strain up to reaching yield stress. After reaching yield stress, in case of small strains, strain increment can be divided into elastic and plastic part [3]:

$$d\mathbf{e} = d\mathbf{e}^E + d\mathbf{e}^P \tag{2.1}$$

Only elastic part of strain causes the stress change thus the stress increment can be formulated as:

$$d\boldsymbol{\sigma} = \mathbf{C}^E d\mathbf{e}^E \tag{2.2}$$

where $\mathbf{C}^E$ is elastic constitutive matrix. Substituting (2.1) in (2.2), the following is obtained:

$$d\boldsymbol{\sigma} = \mathbf{C}^E(d\mathbf{e} - d\mathbf{e}^P) \tag{2.3}$$

In the case of elastic-plastic constitutive models, yield function is the stress state function, therefore the increment of its change can be formulated as:

$$f = 0 \quad \text{and} \quad df = \frac{\partial f^T}{\partial \boldsymbol{\sigma}} d\boldsymbol{\sigma} = 0 \tag{2.4}$$

In incremental plasticity theory it is necessary that the failure function is in every time step less than or equal to zero (neutral loading condition). Implicit stress integration implies the increment of plastic strain in the normal direction on the plastic potential surface, which can be formulated as:

$$d\mathbf{e}^P = d\lambda \frac{\partial g}{\partial \boldsymbol{\sigma}} \tag{2.5}$$

where $d\lambda$ is positive scalar, which is to be calculated, and plastic potential function g is the stress state function. Substituting the plastic strain increment (2.5) in (2.3) and using (2.4), it is obtained:

$$df = \frac{\partial f^T}{\partial \boldsymbol{\sigma}}(\mathbf{C}^E d\mathbf{e} - d\lambda \mathbf{C}^E \frac{\partial g}{\partial \boldsymbol{\sigma}}) = 0 \tag{2.6}$$

Plastic parameter $d\lambda$ can be calculated from equation (2.6) as:

$$d\lambda = \frac{\frac{\partial f}{\partial \boldsymbol{\sigma}}^T \mathbf{C}^E d\mathbf{e}}{\frac{\partial f}{\partial \boldsymbol{\sigma}}^T \mathbf{C}^E \ \frac{\partial g}{\partial \boldsymbol{\sigma}}} \tag{2.7}$$

Finally, using parameter from (2.7), stress increment $\boldsymbol{\sigma}$ is obtained using (2.5) and (2.3) in the function of total strain increment:

$$d\boldsymbol{\sigma} = \mathbf{C}^{EP} d\mathbf{e}$$

where term $\mathbf{C}^{EP}$ represents elastic-plastic constitutive matrix:

$$\mathbf{C}^{EP} = \mathbf{C}^E - \frac{\mathbf{C}^E \frac{\partial g}{\partial \boldsymbol{\sigma}} \frac{\partial f}{\partial \boldsymbol{\sigma}}^T \mathbf{C}^E}{\frac{\partial f}{\partial \boldsymbol{\sigma}}^T \mathbf{C}^E \ \frac{\partial g}{\partial \boldsymbol{\sigma}}}$$

where g represents plastic potential function, which is introduced below.

3. Stress integration of the Drucker–Prager material model with kinematic hardening

Drucker–Prager material model is one of the oldest material models in soil mechanics [1, 4]. In the principal stress space (σ_1, σ_2, σ_3), this surface represents a cone whose axis matches the space diagonal in the principal stresses space, as shown in Figure 1.

The yield surface equation of this model is a function of the stress state and defined as:

$$f = \alpha I_1 + \sqrt{J_{2D}} - k \tag{3.1}$$

In the case of non-associated yield condition the plastic potential surface is defined through the expression:

$$g = \beta I_1 + \sqrt{J_{2D}}$$

where I_1 represents the first stress invariant and J_{2D} represents second stress deviatoric invariant. Terms α, k and β represent parameters of the material model which can be calculated indirectly using parameters of the Mohr-Coulomb model [5].

In the analysis of the mechanical behaviour of soil exposed to cyclic loading such as earthquake, due to the effect of material hardening, models with kinematic hardening are often in use [6]. Yield surface of these models under the loads changes position in principal stresses space, whereas the size of the yield surface remains unchanged. Yield surface of the Drucker–Prager material model with kinematic hardening in deviatoric plane is shown in Figure 2.

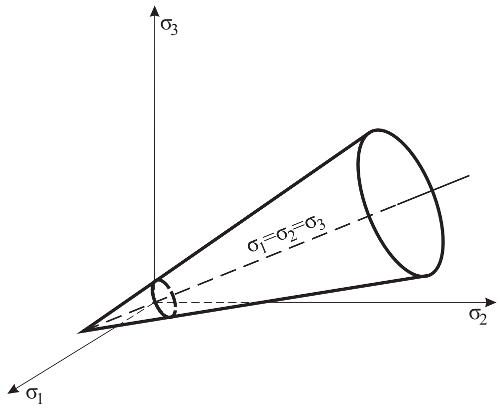

FIGURE 1. Drucker–Prager yield surface in principal stress space

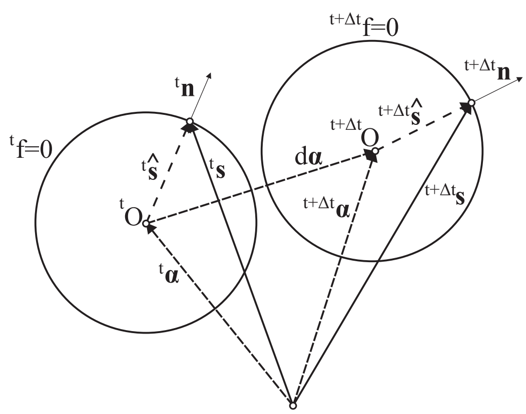

FIGURE 2. Yield surface in the deviatoric plain in case of kinematic hardening

Deviatoric stress in $t + \Delta t$ in case of kinematic hardening, according to Figure 2, can be expressed as:

$$ {}^{t+\Delta t}\mathbf{s} = {}^{t+\Delta t}\hat{\mathbf{s}} + {}^{t+\Delta t}\boldsymbol{\alpha} \tag{3.2} $$

where ${}^{t+\Delta t}\hat{\mathbf{s}}$ represents stress radius, whereas ${}^{t+\Delta t}\boldsymbol{\alpha}$ is the tensor internal variable termed back stress:

$$ {}^{t+\Delta t}\boldsymbol{\alpha} = {}^{t}\boldsymbol{\alpha} + d\boldsymbol{\alpha} $$

In this case, second stress invariant using stress radius has a form:

$$ J_{2D} = \frac{1}{2}\hat{\mathbf{s}}\,\hat{\mathbf{s}} = \frac{1}{2}(\mathbf{s} - \boldsymbol{\alpha})(\mathbf{s} - \boldsymbol{\alpha}) \tag{3.3} $$

Equation of the Drucker–Prager yield surface is a composite function of the stress, so using the chain rule, derivative of this function is:

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{\partial f}{\partial I_1}\frac{\partial I_1}{\partial \boldsymbol{\sigma}} + \frac{\partial f}{\partial J_{2D}}\frac{\partial J_{2D}}{\partial \boldsymbol{\sigma}} \tag{3.4}$$

Derivative of the yield surface equation (3.1) with respect to stress invariant is:

$$\frac{\partial f}{\partial I_1} = \alpha$$

whereas his derivative with respect to second stress deviatoric invariant has a form:

$$\frac{\partial f}{\partial J_{2D}} = \frac{1}{2\sqrt{J_{2D}}}$$

Derivative of the second deviatoric stress invariant with respect to stress from (3.4) according to (3.3) can be calculated using the chain rule as:

$$\frac{\partial J_{2D}}{\partial \boldsymbol{\sigma}} = \frac{\partial J_{2D}}{\partial \hat{\mathbf{s}}}\frac{\partial \hat{\mathbf{s}}}{\partial \boldsymbol{\sigma}} \tag{3.5}$$

Second deviatoric stress invariant according to [7] using stress radius can be calculated as:

$$J_{2D} = \hat{s}_1\hat{s}_2 + \hat{s}_2\hat{s}_3 + \hat{s}_3\hat{s}_1 - \hat{s}_4^2 - \hat{s}_5^2 - \hat{s}_6^2$$

so first term in equation (3.5) has a form:

$$\frac{\partial J_{2D}}{\partial \hat{\mathbf{s}}} = \begin{bmatrix}(\hat{s}_2+\hat{s}_3) & (\hat{s}_1+\hat{s}_3) & (\hat{s}_1+\hat{s}_2) & -2\hat{s}_4^2 & -2\hat{s}_5^2 & -2\hat{s}_6^2\end{bmatrix}$$

Using equation (3.2), stress radius can be written as:

$$^{t+\Delta t}\hat{\mathbf{s}} = {}^{t+\Delta t}\boldsymbol{\sigma} - \mathbf{m}\,{}^{t+\Delta t}\sigma_m - {}^{t+\Delta t}\boldsymbol{\alpha} \tag{3.6}$$

which implies second member of equation (3.5) in the form:

$$\frac{\partial \hat{\mathbf{s}}}{\partial \boldsymbol{\sigma}} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Term $\mathbf{m}$ in (3.6) represents identity matrix:

$$\mathbf{m} = \begin{bmatrix}1 & 1 & 1 & 0 & 0 & 0\end{bmatrix}$$

For mechanical behaviour analysis of the granular materials exposed to non-proportional static or cyclic loading more than one type of kinematic hardening has been in use. The simplest type of kinematic hardening represents Drucker's linear kinematic hardening [8], whereas the increment of back stress $d\mathbf{e}^P$ is collinear to the plastic strain increment and has the form:

$$d\boldsymbol{\alpha} = \frac{2}{3}C_1 d\mathbf{e}^P$$

where C_1 is material parameter, whereas $d\mathbf{e}^P$ represents plastic strain increment. However, significantly better definition of kinematic hardening has Armstrong–Fredrick hardening [9] due to the introduction of the member which represents dynamic relaxation. Back stress increment in this case has a form:

$$d\boldsymbol{\alpha} = \frac{2}{3}C_1 d\mathbf{e}^P - C_2 {}^t\boldsymbol{\alpha} d\bar{e}^P$$

where C_2 represents material parameter, whereas $d\bar{e}^P$ is the equivalent plastic strain increment:

$$d\bar{e}^P = \sqrt{\frac{2}{3}d\mathbf{e}^P d\mathbf{e}^P}$$

Kinematic hardening can be implemented in the same way in other material models for mechanical behaviour analysis of the granular materials. Algorithm for stress integration using Drucker–Prager material model with kinematic hardening is shown in Table 1.

Table 1. Algorithm for stress integration

A. Known ${}^{t+\Delta t}\mathbf{e}$, ${}^t\mathbf{e}$, ${}^t\boldsymbol{\sigma}$, ${}^t\mathbf{e}^P$, ${}^t\boldsymbol{\alpha}$
Trial (elastic) solution:
$d\boldsymbol{\sigma} = \mathbf{C}^E d\mathbf{e}^E = \mathbf{C}^E({}^{t+\Delta t}\mathbf{e} - {}^t\mathbf{e})$;
${}^{t+\Delta t}\boldsymbol{\sigma} = {}^t\boldsymbol{\sigma} + d\boldsymbol{\sigma}$; ${}^{t+\Delta t}\mathbf{s} = {}^{t+\Delta t}\boldsymbol{\sigma} - \mathbf{m}\,{}^{t+\Delta t}\sigma_m$
B. Check the yield condition:
IF $(f \leqslant 0)$ elastic solution ${}^{t+\Delta t}\mathbf{e}^P = {}^t\mathbf{e}^P$ and ${}^{t+\Delta t}\boldsymbol{\alpha}^P = {}^t\boldsymbol{\alpha}^P$ (GOTO **E**)
IF $(f > 0)$ elastic-plastic solution (CONTINUE)
$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{\partial f}{\partial I_1}\frac{\partial I_1}{\partial \boldsymbol{\sigma}} + \frac{\partial f}{\partial J_{2D}}\frac{\partial J_{2D}}{\partial \boldsymbol{\sigma}}$; $\frac{\partial g}{\partial \boldsymbol{\sigma}} = \frac{\partial g}{\partial I_1}\frac{\partial I_1}{\partial \boldsymbol{\sigma}} + \frac{\partial g}{\partial J_{2D}}\frac{\partial J_{2D}}{\partial \boldsymbol{\sigma}}$
$d\lambda = \frac{\frac{\partial f}{\partial \boldsymbol{\sigma}}^T \mathbf{C}^E d\mathbf{e}}{\frac{\partial f}{\partial \boldsymbol{\sigma}}^T \mathbf{C}^E \frac{\partial g}{\partial \boldsymbol{\sigma}}}$
C. Correction $d\lambda$ (local iterations):
$d\mathbf{e}^P = d\lambda\frac{\partial g}{\partial \boldsymbol{\sigma}}$; $d\bar{e}^P = \sqrt{\frac{2}{3}d\mathbf{e}^P d\mathbf{e}^P}$; $d\boldsymbol{\alpha} = \frac{2}{3}C_1 d\mathbf{e}^P - C_2 {}^t\boldsymbol{\alpha} d\bar{e}^P$
$d\mathbf{e}^E = d\mathbf{e} - d\mathbf{e}^P$; $d\boldsymbol{\sigma} = \mathbf{C}^E d\mathbf{e}^E$; ${}^{t+\Delta t}\boldsymbol{\sigma} = {}^t\boldsymbol{\sigma} + d\boldsymbol{\sigma}$; ${}^{t+\Delta t}\boldsymbol{\alpha} = {}^t\boldsymbol{\alpha} + d\boldsymbol{\alpha}$
D. IF$(ABS\,(f) \geqslant tol)$ GOTO to **C** with new $d\lambda$:
${}^{t+\Delta t}\mathbf{e}^P = {}^t\mathbf{e}^P + d\mathbf{e}^P$
E. End ${}^{t+\Delta t}\mathbf{e}^P$, ${}^{t+\Delta t}\boldsymbol{\sigma}$, ${}^{t+\Delta t}\boldsymbol{\alpha}$

4. Verification of the material model

Verification of the developed algorithm for implicit stress integration of Drucker–Prager material model with kinematic hardening is performed using triaxial test with cyclic loading. The unit size FE model with boundary conditions and loads is used as shown in Figure 3a. Model was loaded with the same values of prescribed displacement in two coordinate directions, while the prescribed displacement (d_1 and d_2) in the third coordinate direction (d_3) has a variable value, which represents cyclic loading of the soil sample. This cyclic load should create nonproportional stress state that should cause the kinematic hardening of the material [6]. The load functions used in numerical simulation are shown in Figure 3b.

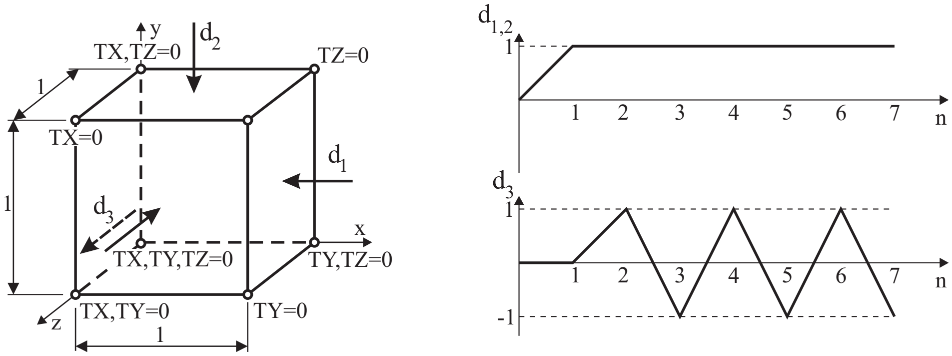

FIGURE 3. Model for cyclic loading simulation and functions of prescribed displacements

For numerical simulation of the specimen analysis loaded using uniaxial cyclic loading which represents a seismic load in a simplified form, the data for the compacted sand is used which is showed in Table 2 [3].

TABLE 2. Parameters of the material model

$E\ [KPa]$	ν	$k\ [kPa]$	α	C_1	C_2
100.0	0.25	10.0	0.0	20.0	1.4

Analysis results of the compacted sand sample are shown in Figure 4. The Figure 4a shows the results of the numerical simulations using the same load function in case when there is no kinematic hardening of the material while Figure 4b shows the results of the numerical simulation in the case where the model has a Armstrong–Frederick kinematic hardening.

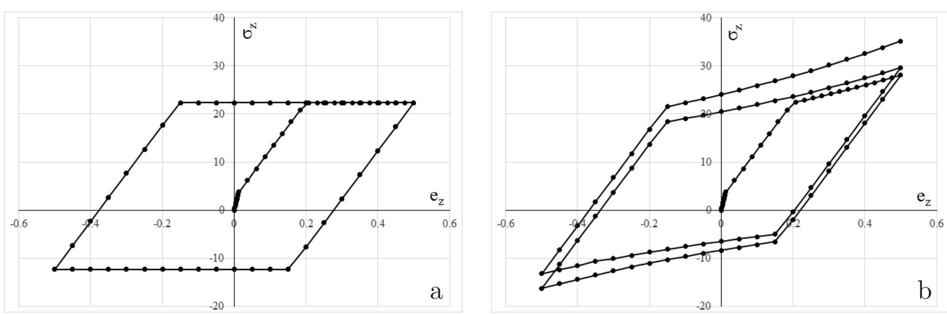

FIGURE 4. Axial stress vs. axial strain: a) without hardening, b) hardening model

The analysis of the results presented in Figure 4 shows the effect of using Armstrong–Frederick kinematic hardening in case of cyclic loading of the specimen.

Presented results shows effect of using Armstrong–Frederick kinematic hardening where hysteresis behaviour of the sample can be noted. It can be seen an increase of the axial stress in the direction of the cyclic load with the increasing of load cycles number. This behaviour represents the kinematic hardening of the material and this material model can be applied in the seismic analysis of soil material.

5. Conclusions

The results obtained using Drucker–Prager constitutive model with kinematic hardening are shown in the paper. The model for analyzing the mechanical behaviour of materials exposed to the seismic loads is created through introduction of Armstrong–Frederick kinematic hardening in the existing model for the analysis of the mechanical behaviour of granular materials. The advantage of the presented computational procedure is its general formulation which can be applied to similar material model. Verification example shows its hardening feature under cyclic loading as the main characteristics of the model.

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ИНТЕГРАЦИЈА НАПОНА ДРАКЕР–ПРАГЕРОВОГ МАТЕРИЈАЛНОГ МОДЕЛА СА КИНЕМАТСКИМ ОЈАЧАЊЕМ

Резиме. У раду је приказана примена методе имплицитне интеграције напона Дракер–Прагеровог материјалног модела са кинемацким ојачањем. Интеграција напона материјалног модела је извршена применом методе инкрементиалне пластичности, док је кинемацко ојачање материјала дефинисано применом нелинеарног Армстронг–Фредерик ојачања. Овај тип ојачања грануларних материјала се јавља као последица цикличног оптерећења какво је сеизмичко оптерећење. С тога, овај материјални модел има примену у анализи земљотреса у механици тла. Површ течења овог материјалног модела мења свој положај у простору главних напона током дејства цикличног оптерећења, док величина површи течења модела у девијаторској равни остаје непромењена. Развијени алгоритам материјалног модела је уграђен у софтверски пакет PAK.

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