

OPTIMAL SHAPE OF A COLUMN WITH CLAMPED-ELASTICALLY SUPPORTED ENDS POSITIONED ON ELASTIC FOUNDATION

Branislava Novaković

ABSTRACT. We determine optimal shape of an elastic column positioned on elastic foundation of Winkler type. The Euler-Bernoulli model of beam is considered. The column is loaded by a compressive force and has one clamped end and the other elastically supported end. In deriving the optimality conditions, the Pontryagin's principle was used. The optimality conditions for the case of bimodal optimization are derived. Optimal cross-sectional area is obtained from the solution of a non-linear boundary value problem. A first integral (Hamiltonian) is used to monitor accuracy of integration. This system is solved by using standard Math CAD procedure. New numerical results are obtained.

1. Introduction

The problem of determining the shape of a rod of a given volume that is the strongest against buckling was first formulated by J.-L. Lagrange in 1773 (see [1]) and is now known as the Lagrange problem. However, the solution obtained by him proved to be incorrect. Clausen in [2] found the first optimal solution for the case of a cantilever column analytically. The optimization of the column with simply supported ends was derived in latter works (see [3, 4]). Optimal solutions for clamped-clamped and clamped-hinged columns was obtained analytically in [5]. All the mentioned solutions are unimodal, i.e. possessing a single buckling mode. Olhoff and Rasmussen (see [6]) found that the solution obtained in [5] for the clamped-clamped case is incorrect and determined the bimodal solution to the problem i.e. there are two buckling modes of the rod at the same buckling load. They obtained the optimal solution by numerical procedure. The same problem was treated in [7]. In both works it was assumed that second moment of inertia I is proportional to the square of the cross-sectional area A , that is $I = \alpha A^2$, $\alpha = \text{const}$. The bimodal optimality conditions were derived in [8, 9] for the column with clamped-clamped ends. It was found that analytical expressions of

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these conditions are in the form of elliptic integrals. The optimal shapes of elastic columns on elastic foundation of Winkler type was treated in [10–12] with the minimum compliance as optimization criteria. The columns on elastic foundation for different boundary conditions were treated in [13]. The optimization led to unimodal and bimodal solution. The optimal shape of an elastic column with clamped ends positioned on elastic foundation of Winkler type was determined in [14]. The optimal shape of a column on elastic foundation subjected to restrictions on minimum and maximum cross-sectional area was treated in [15]. It is shown that in this case the optimization can be both bimodal and unimodal.

2. Mathematical formulation

Consider an elastic rod of length L loaded by an axial force F with the action line coinciding with the x axis of a rectangular coordinate system x - B - y (see Figure 1). The column is positioned on a Winkler type of foundation has one clamped end and the other elastically supported end. We use the following notation: H and V are components of the contact force (i.e. the resultant force in an arbitrary cross-section) along x and y axes, respectively, M is the bending moment, θ is the angle between the tangent to the column axis and the x axis, S is the arc-length of the column axis measured from the origin of the coordinate system B , E is the modulus of elasticity.

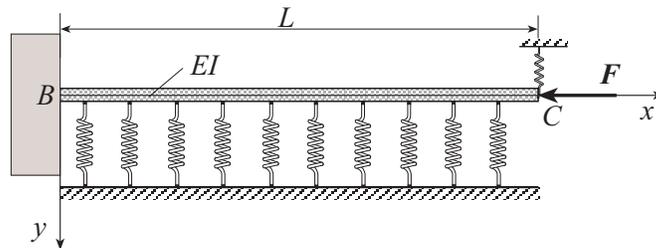


FIGURE 1. Coordinate system and load configuration

Also we assume that the axial moment of inertia I and the cross-sectional area A are connected as $I = \alpha_n A^n$ where α_n is a constant that depends of n and $n = 1, 2, 3$. The governing equations: equilibrium equations, geometrical and constitutive relations (see [16]) are

$$(2.1) \quad \begin{aligned} \frac{dH}{dS} &= 0, & \frac{dV}{dS} &= -q_y, & \frac{dM}{dS} &= -V \cos \theta + H \sin \theta, \\ \frac{d\bar{x}}{dS} &= \cos \theta, & \frac{d\bar{y}}{dS} &= \sin \theta, & M &= EI \frac{d\theta}{dS}. \end{aligned}$$

where $q_y = -\mu y$ and $\mu > 0$ is a constant stiffness of the foundation. In (2.1)_{4,5} we use \bar{x} and \bar{y} to denote coordinates of an arbitrary point on the column axis. Boundary conditions are

$$\bar{y}(0) = 0, \quad \theta(0) = 0, \quad M(L) = 0, \quad V(L) = -c\bar{y}(L), \quad H(L) = -F,$$

where c is a spring constant of the support.

The volume of the column is

$$W = \int_0^L A(S) dS,$$

where $A(S)$ is the cross-sectional area. By introducing the dimensionless quantities

$$\begin{aligned} t = \frac{S}{L}, \quad a = \frac{A}{L^2}, \quad \zeta = \frac{\bar{x}}{L}, \quad \eta = \frac{\bar{y}}{L}, \quad w = \frac{W}{L^3}, \quad b = \frac{c}{\alpha_n EL^{2n-3}} \\ \lambda_1 = \frac{\mu}{\alpha_n EL^{2n-4}}, \quad \lambda_2 = \frac{F}{\alpha_n EL^{2n-2}}, \quad v = \frac{V}{\alpha_n EL^{2n-2}}, \quad m = \frac{M}{\alpha_n EL^{2n-1}}, \end{aligned}$$

and after linearization of equations (2.1) we have (see [17])

$$(2.2) \quad \dot{v} = \lambda_1 \eta, \quad \dot{m} = -v - \lambda_2 \theta, \quad \dot{\zeta} = 0, \quad \dot{\eta} = \theta, \quad \dot{\theta} = \frac{m}{a^n},$$

subject to

$$(2.3) \quad \eta(0) = 0, \quad \theta(0) = 0, \quad m(1) = 0, \quad v(1) = -b\eta(1),$$

where $(\dot{\cdot}) = \frac{d}{dt}(\cdot)$.

The dimensionless volume becomes

$$(2.4) \quad w = \int_0^1 a(t) dt.$$

The multiplicity of an eigenvalue for the system (2.2) and (2.3) can be at most two (see [14]). We assume that the cross-sectional area $a(t)$ belongs to the set \mathbf{U} called the set of *admissible* cross-sectional area functions.

Suppose now that $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ is given (for chosen b). We define the optimal *compressed column on an elastic foundation with clamped-elastically supported ends* as the column so shaped that any other column of same length (in our case equal to one) and smaller volume will buckle under load and foundation characterized by (λ_1, λ_2) . Thus, the problem of determining the shape of the optimal column may be stated as an optimal control problem as: given λ_1, λ_2 find $a^* \in \mathbf{U}$ such that the integral (2.4) is minimal when the system is subjected to constraints (2.2) and (2.3).

3. Solution of the problem

In order to apply Pontryagin's maximum principle, we introduce new dependent variables as

$$x_1 = \eta, \quad x_2 = \theta, \quad x_3 = m, \quad x_4 = v.$$

Then, the system (2.2), (2.3) becomes

$$(3.1) \quad \begin{aligned} \dot{x}_1 &= x_2, & \dot{x}_2 &= \frac{x_3}{a^n}, & \dot{x}_3 &= -x_4 - \lambda_2 x_2, & \dot{x}_4 &= \lambda_1 x_1, \\ x_1(0) &= 0, & x_2(0) &= 0, & x_4(1) &= -bx_1(1), & x_3(1) &= 0. \end{aligned}$$

In terms of the optimal control, the problem now becomes: Given (λ_1, λ_2) , find the control $a^*(t) \in \mathbf{U}$ such that

$$\min_{a \in \mathbf{U}} I = \min_{a \in \mathbf{U}} \int_0^1 a(t) dt = \int_0^1 a^*(t) dt.$$

under the state equations (3.1). We assume that $a(t)$ is continuous function, i.e., $a \in C(0, 1)$.

Suppose now that for given (λ_1, λ_2) and for the optimal $a(t) = a^*(t)$ the linear boundary value problem (3.1) has two linearly independent solutions, $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$ and $(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4)$, corresponding to two buckling modes. Since both solutions correspond to the same (λ_1, λ_2) and $a(t) = a^*(t)$ we have (see [18])

$$(3.2) \quad \begin{aligned} \dot{\bar{x}}_1 &= \bar{x}_2, & \dot{\bar{x}}_2 &= \frac{\bar{x}_3}{a^n}, & \dot{\bar{x}}_3 &= -\bar{x}_4 - \lambda_2 \bar{x}_2, & \dot{\bar{x}}_4 &= \lambda_1 \bar{x}_1, \\ \dot{\hat{x}}_1 &= \hat{x}_2, & \dot{\hat{x}}_2 &= \frac{\hat{x}_3}{a^n}, & \dot{\hat{x}}_3 &= -\hat{x}_4 - \lambda_2 \hat{x}_2, & \dot{\hat{x}}_4 &= \lambda_1 \hat{x}_1, \end{aligned}$$

satisfying

$$(3.3) \quad \begin{aligned} \bar{x}_1(0) &= 0, & \bar{x}_2(0) &= 0, & \bar{x}_4(1) &= -b\bar{x}_1(1), & \bar{x}_3(1) &= 0, \\ \hat{x}_1(0) &= 0, & \hat{x}_2(0) &= 0, & \hat{x}_4(1) &= -b\hat{x}_1(1), & \hat{x}_3(1) &= 0. \end{aligned}$$

The Pontryagin's function \mathcal{H} , taking into account that differential constraints are given by (3.2) reads

$$(3.4) \quad \begin{aligned} \mathcal{H} &= a + \bar{p}_1 \bar{x}_2 + \bar{p}_2 \frac{\bar{x}_3}{a^n} + \bar{p}_3 (-\bar{x}_4 - \lambda_2 \bar{x}_2) + \bar{p}_4 \lambda_1 \bar{x}_1 \\ &\quad + \hat{p}_1 \hat{x}_2 + \hat{p}_2 \frac{\hat{x}_3}{a^n} + \hat{p}_3 (-\hat{x}_4 - \lambda_2 \hat{x}_2) + \hat{p}_4 \lambda_1 \hat{x}_1, \end{aligned}$$

where the co-state variables $\bar{p}_i, \hat{p}_i, i = 1, \dots, 4$ satisfy

$$\begin{aligned} \dot{\bar{p}}_1 &= -\frac{\partial \mathcal{H}}{\partial \bar{x}_1} = -\bar{p}_4 \lambda_1, & \dot{\bar{p}}_2 &= -\frac{\partial \mathcal{H}}{\partial \bar{x}_2} = -\bar{p}_1 + \lambda_2 \bar{p}_3, \\ \dot{\bar{p}}_3 &= -\frac{\partial \mathcal{H}}{\partial \bar{x}_3} = -\frac{\bar{p}_2}{a^n}, & \dot{\bar{p}}_4 &= -\frac{\partial \mathcal{H}}{\partial \bar{x}_4} = \bar{p}_3, \\ \dot{\hat{p}}_1 &= -\frac{\partial \mathcal{H}}{\partial \hat{x}_1} = -\hat{p}_4 \lambda_1, & \dot{\hat{p}}_2 &= -\frac{\partial \mathcal{H}}{\partial \hat{x}_2} = -\hat{p}_1 + \lambda_2 \hat{p}_3, \\ \dot{\hat{p}}_3 &= -\frac{\partial \mathcal{H}}{\partial \hat{x}_3} = -\frac{\hat{p}_2}{a^n}, & \dot{\hat{p}}_4 &= -\frac{\partial \mathcal{H}}{\partial \hat{x}_4} = \hat{p}_3, \end{aligned}$$

subject to

$$\begin{aligned} \bar{p}_4(0) &= 0, & \bar{p}_3(0) &= 0, & \bar{p}_1(1) &= b\bar{p}_4(1), & \bar{p}_2(1) &= 0, \\ \hat{p}_4(0) &= 0, & \hat{p}_3(0) &= 0, & \hat{p}_1(1) &= b\hat{p}_4(1), & \hat{p}_2(1) &= 0. \end{aligned}$$

The optimality condition is

$$\frac{\partial \mathcal{H}}{\partial a} = 1 - n\bar{p}_2 \frac{\bar{x}_3}{a^{n+1}} - n\hat{p}_2 \frac{\hat{x}_3}{a^{n+1}} = 0,$$

or

$$a = a^*(t) = [n(\bar{p}_2 \bar{x}_3 + \hat{p}_2 \hat{x}_3)]^{1/(n+1)}.$$

In order to reduce the dimension of the system, we proposed in [17, 18] the identification of state and co-state variables as

$$(3.5) \quad \begin{aligned} \bar{p}_1 &= \beta_{11}\bar{x}_4 + \beta_{12}\hat{x}_4, & \bar{p}_2 &= \beta_{11}\bar{x}_3 + \beta_{12}\hat{x}_3, \\ \bar{p}_3 &= -\beta_{11}\bar{x}_2 - \beta_{12}\hat{x}_2, & \bar{p}_4 &= -\beta_{11}\bar{x}_1 - \beta_{12}\hat{x}_1, \\ \hat{p}_1 &= \beta_{21}\bar{x}_4 + \beta_{22}\hat{x}_4, & \hat{p}_2 &= \beta_{21}\bar{x}_3 + \beta_{22}\hat{x}_3, \\ \hat{p}_3 &= -\beta_{21}\bar{x}_2 - \beta_{22}\hat{x}_2, & \hat{p}_4 &= -\beta_{21}\bar{x}_1 - \beta_{22}\hat{x}_1, \end{aligned}$$

where β_{ij} , $i, j = 1, 2$ are constants.

Note that with (3.5) cross-sectional area becomes

$$(3.6) \quad a = a^*(t) = [n(\gamma_{11}(\bar{x}_3)^2 + 2\gamma_{12}\bar{x}_3\hat{x}_3 + \gamma_{22}(\hat{x}_3)^2)]^{1/(n+1)},$$

where $\gamma_{11} = \beta_{11}$, $\gamma_{12} = (\beta_{12} + \beta_{21})/2$, $\gamma_{22} = \beta_{22}$. The relevant system of equations is

$$(3.7) \quad \begin{aligned} \dot{\bar{x}}_1 &= \bar{x}_2, & \dot{\bar{x}}_2 &= \frac{\bar{x}_3}{[n(\gamma_{11}(\bar{x}_3)^2 + 2\gamma_{12}\bar{x}_3\hat{x}_3 + \gamma_{22}(\hat{x}_3)^2)]^{n/(n+1)}}, \\ \dot{\bar{x}}_3 &= -\bar{x}_4 - \lambda_2\bar{x}_2, & \dot{\bar{x}}_4 &= \lambda_1\bar{x}_1, \\ \dot{\hat{x}}_1 &= \hat{x}_2, & \dot{\hat{x}}_2 &= \frac{\hat{x}_3}{[n(\gamma_{11}(\bar{x}_3)^2 + 2\gamma_{12}\bar{x}_3\hat{x}_3 + \gamma_{22}(\hat{x}_3)^2)]^{n/(n+1)}}, \\ \dot{\hat{x}}_3 &= -\hat{x}_4 - \lambda_2\hat{x}_2, & \dot{\hat{x}}_4 &= \lambda_1\hat{x}_1, \end{aligned}$$

subject to (3.3). Note that doesn't depend on t explicitly therefore on the solution of (3.2), (3.3) we have $\mathcal{H} = \text{const}$. We substitute (3.6) and (3.5) into (3.4) to get

$$\begin{aligned} \mathcal{H} &= [n(\gamma_{11}(\bar{x}_3)^2 + 2\gamma_{12}\bar{x}_3\hat{x}_3 + \gamma_{22}(\hat{x}_3)^2)]^{1/(n+1)} \\ &+ \frac{\gamma_{11}(\bar{x}_3)^2 + 2\gamma_{12}\bar{x}_3\hat{x}_3 + \gamma_{22}(\hat{x}_3)^2}{[n(\gamma_{11}(\bar{x}_3)^2 + 2\gamma_{12}\bar{x}_3\hat{x}_3 + \gamma_{22}(\hat{x}_3)^2)]^{n/(n+1)}} \\ &+ (\beta_{11}\bar{x}_4 + \beta_{12}\hat{x}_4)\bar{x}_2 + (\beta_{11}\bar{x}_2 + \beta_{12}\hat{x}_2)(\bar{x}_4 + \lambda_2\bar{x}_2) \\ &- (\beta_{11}\bar{x}_1 + \beta_{12}\hat{x}_1)\lambda_1\bar{x}_1 + (\beta_{21}\bar{x}_4 + \beta_{22}\hat{x}_4)\hat{x}_2 \\ &+ (\beta_{21}\bar{x}_2 + \beta_{22}\hat{x}_2)(\hat{x}_4 + \lambda_2\hat{x}_2) - (\beta_{21}\bar{x}_1 + \beta_{22}\hat{x}_1)\lambda_1\hat{x}_1 \end{aligned}$$

4. Numerical results

1. First, we consider the column clamped on one end and simply supported on the other on elastic foundation. Thus, we take in this example $n = 2$, $\lambda_1 = 300$, $\lambda_2 = 51.34$ and $\gamma_{11} = 1$, $\gamma_{22} = 3.6$, $\gamma_{12} = 1$. The value of dimensionless volume is $w = 1$. We assume first that the optimization is bimodal (we use system (3.7)) because we can not decide, which optimization procedure (uni or bi modal) leads to optimal shape. Buckling modes are shown in Figure 2. The cross-sectional area is shown on the right side of Figure 2 and the maximum value is $a_{\max} = 1.4138645281$. In this case the first integral is $\mathcal{H} = 1.1315387174$ (for $\beta_{12} = 1.4$, $\beta_{21} = 0.6$) in the whole interval $t \in (0, 1)$ to within 10^{-10} .

2. Next we treat the column ($n = 2$) with parameter $\lambda_1 = 300$ and $\lambda_2 = 51.34$. The column has one clamped end and the other elastically supported end. We

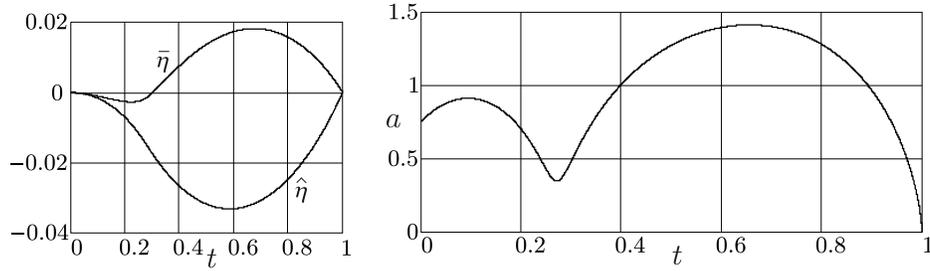


FIGURE 2. Buckling modes and cross-sectional area of the column
 $n = 2$, $\lambda_1 = 300$, $\lambda_2 = 51.34$

used $\gamma_{11} = 1$, $\gamma_{22} = 3.6$, $\gamma_{12} = 1$, $b = 10^3$. The value of dimensionless volume is $w = 1.034537$. Optimization is bimodal and buckling modes are shown in Figure 3. Displacements on the right end of the rod are $\bar{\eta}(1) = -1.350469271 \cdot 10^{-3}$, $\hat{\eta}(1) = 3.3881631463 \cdot 10^{-3}$. The cross-sectional area is shown in right side of Figure 3. Maximum value of the cross-sectional area is $a_{\max} = 1.4606307$. In this case the

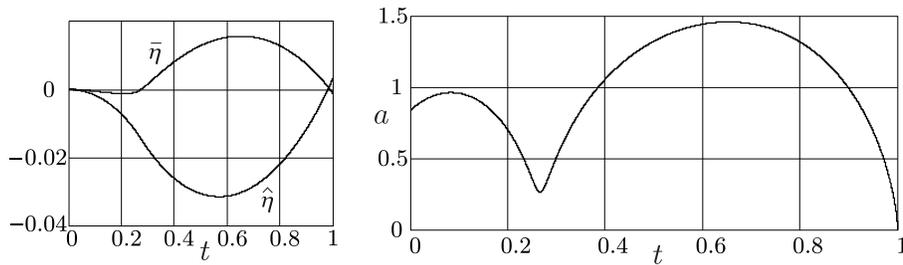


FIGURE 3. Buckling modes and cross-sectional area of the column
 $n = 2$, $\lambda_1 = 300$, $\lambda_2 = 51.34$, $b = 10^3$

first integral is $\mathcal{H} = 1.2641907826$ (for $\beta_{12} = 1.4$, $\beta_{21} = 0.6$) in the whole interval $t \in (0, 1)$ to within 10^{-10} .

3. Next we treat the same column ($n = 2$) but now we have different parameter of foundation $\lambda_1 = 450$ and parameter of the axial force $\lambda_2 = 51.34$. The column has one clamped end and the other elastically supported end. We used $\gamma_{11} = 1$, $\gamma_{22} = 3.6$, $\gamma_{12} = 1$, $b = 10^3$. Optimization is bimodal and buckling modes are shown in Figure 4. Displacements on the right end of the rod are $\bar{\eta}(1) = -6.0627949841 \cdot 10^{-3}$, $\hat{\eta}(1) = -3.0732283646 \cdot 10^{-4}$. The value of dimensionless volume is $w = 0.8513223241$. The cross-sectional area is shown in Figure 4 and maximum value of the cross-sectional area is $a_{\max} = 1.2156145$. In this case the first integral is $\mathcal{H} = 0.8026847591$ (for $\beta_{12} = 1.4$, $\beta_{21} = 0.6$) in the whole interval $t \in (0, 1)$ to within 10^{-10} .

4. In next example we treated the column ($n = 2$) on elastic foundation for $\lambda_1 = 450$ and the parameter of axial force $\lambda_2 = 66$. We used $\gamma_{11} = 1$, $\gamma_{22} = 3.6$, $\gamma_{12} = 1$, $b = 10^3$. Optimization is bimodal and buckling modes are shown in

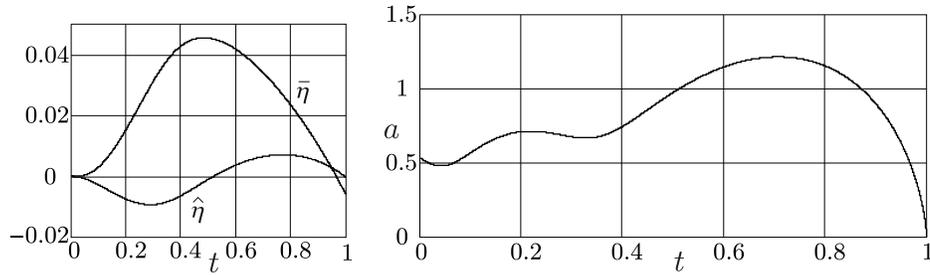


FIGURE 4. Buckling modes and cross-sectional area of the column
 $n = 2$, $\lambda_1 = 450$, $\lambda_2 = 51.34$, $b = 10^3$

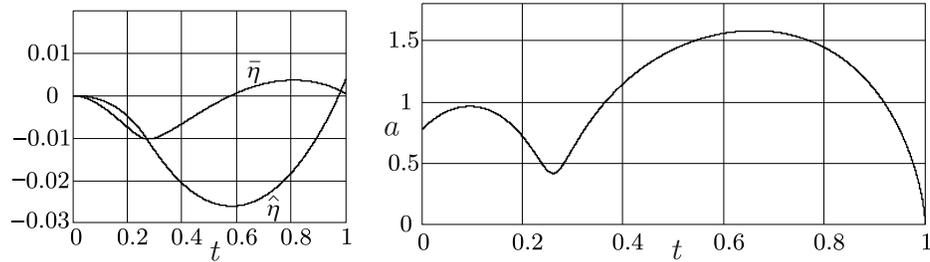


FIGURE 5. Buckling modes and cross-sectional area of the column
 $n = 2$, $\lambda_1 = 450$, $\lambda_2 = 66$, $b = 10^3$

Figure 5. Displacements on the right end of the rod are $\bar{\eta}(1) = 4.1663590135 \cdot 10^{-4}$, $\hat{\eta}(1) = 3.969859862 \cdot 10^{-3}$. The value of dimensionless volume is $w = 1.1126706682$. The cross-sectional area is shown in Figure 5. The cross-sectional area maximum is $a_{\max} = 1.5790753$. The first integral $\mathcal{H}(t)$ was constant with the value $\mathcal{H} = 1.1609679684 \pm 10^{-10}$ (for $\beta_{12} = 1.4$, $\beta_{21} = 0.6$).

5. In the last example we treated the column on elastic foundation for $\lambda_1 = 300$ and parameters are $n = 2$, $\lambda_2 = 51.34$. We used $\gamma_{11} = 1$, $\gamma_{22} = 3.6$, $\gamma_{12} = 1$, $b = 500$. Optimization is bimodal and buckling modes are shown in Figure 6.

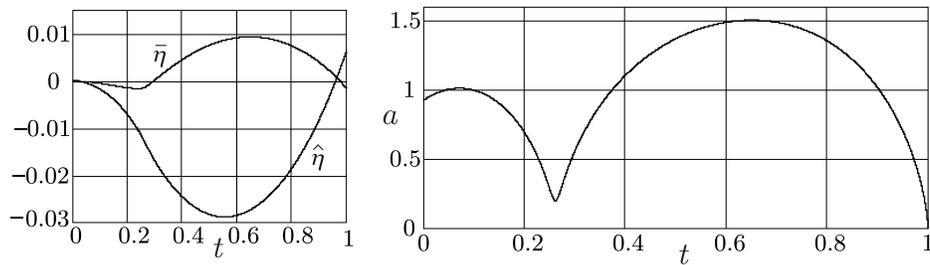


FIGURE 6. Buckling modes and cross-sectional area of the column
 $n = 2$, $\lambda_1 = 300$, $\lambda_2 = 51.34$, $b = 500$

Displacements on the right end of the rod are $\bar{\eta}(1) = -1.5325193879 \cdot 10^{-3}$, $\hat{\eta}(1) = 6.4062502507 \cdot 10^{-3}$. The value of dimensionless volume is $w = 1.0678660003$. The cross-sectional area is shown in Figure 6 and $a_{\max} = 1.50521745$. The first integral is $\mathcal{H} = 1.3912644928$ (for $\beta_{12} = 1.4$, $\beta_{21} = 0.6$) in the whole interval $t \in (0, 1)$ to within 10^{-10} .

5. Conclusions

We analyzed the optimization problem for an elastic rod on elastic foundation which is clamped on one end and elastically supported on the other. The system of equations (3.7) and (3.3) has a solution which determines the cross-sectional area of the optimal column through equation (3.6). We found optimal shapes for columns, for different values of the stiffness of foundation and the spring constant. In all cases we have bimodal optimization.

We concluded that by increasing the stiffness of foundation for the same axial force we have decreasing of the volume and the maximum cross-sectional area.

By increasing the value of the spring constant we have decreasing of the volume and the maximum cross-sectional area, for the same value of the axial force and the same value of constants γ_{11} , γ_{12} , γ_{22} .

By increasing the value of the axial force we have increasing of the volume and the maximum cross-sectional area for the same value of foundation stiffness and the same value of constants γ_{11} , γ_{12} , γ_{22} .

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**ОПТИМАЛНИ ОБЛИК ШТАПА НА ЕЛАСТИЧНОЈ ПОДЛОЗИ
УКЛЕШТЕНОГ НА ЈЕДНОМ И ЕЛАСТИЧНО ОСЛОЊЕНОГ
НА ДРУГОМ КРАЈУ**

РЕЗИМЕ. Одређен је оптимални облик еластичног штапа који се налази на еластичној подлози Винклеровог типа. Разматран је Ојлер–Бернулијев модел штапа. Штап је оптерећен силом притиска и на једном крају је уклештен, а на другом је еластично ослоњен. У извођењу услова оптималности коришћен је Понтријагинов принцип. Добијени су услови оптималности за случај бимодалне оптимизације. Оптимална површина попречног пресека је одређена из решења нелинеарног проблема граничних вредности. Први интеграл (Хамилтонијан) је коришћен за проверу тачности интеграције. Систем је решен коришћењем стандардног Math CAD поступка. Добијени су нови нумерички резултати.

Department of Mechanics
University of Novi Sad
Novi Sad
Serbia
nbrana@uns.ac.rs

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