NONLOCAL VIBRATION OF A FRACTIONAL ORDER VISCOELASTIC NANOBEAM WITH ATTACHED NANOPARTICLE

Milan Cajić, Danilo Karličić, and Mihailo Lazarević

ABSTRACT. We propose a novel mathematical framework to examine the free damped transverse vibration of a nanobeam by using the nonlocal theory of Eringen and fractional derivative viscoelasticity. The motion equation of a nanobeam with arbitrary attached nanoparticle is derived by considering the nonlocal viscoelastic constitutive equation involving fractional order derivatives and using the Euler–Bernoulli beam theory. The solution is proposed by using the method of separation of variables. Eigenvalues and mode shapes are determined for three typical boundary conditions. The fractional order differential equation in terms of a time function is solved by using the Laplace transform method. Time dependent behavior is examined by observing the time function for different values of fractional order parameter and different ratios of other parameters in the model. Validation study is performed by comparing the obtained results for a special case of our model with corresponding molecular dynamics simulation results found in the literature.

1. Introduction

With the advent of nanoengineering and nanotechnology, nanotube like structures are now potential design candidates which are likely to play key roles in many engineering devices or components at the nanometer scale. Nanobeams are nanostructures that can be recognized in many nanomaterials grown from zinc-oxide, boron nitride, carbon, silver or gold by various technological processes. Such nanomaterial's are having unique electrical, thermal and mechanical properties that make nanobeams superior candidates for application in nano-electromechanical (NEMS) and micro-electromechanical (MEMS) devices [1, 2]. Experimental research on nanobeams and other nanostructures mechanical bechaviour was reported in the literature [3]. However, it is not easy to perform such investigations on nanoscale level due to the weak control of experimental parameters. Further, molecular

dynamics (MD) simulation studies can be employed in order to investigate mechanical performance of nanosturctures [4]. Nevertheless, this method is computationally prohibitive for nanostructures with large number of atoms. An alternative to mentioned methodologies is theoretical analysis of nanostructures using continuum based theories. Still, classical continuum theories are not able to take into account small-size effects that cannot be neglected on the nano-scale level. Thus, nonlocal continuum field theory of Eringen, which is able to take into account nonlocal effects such as forces between atoms and the internal length scale via single material parameter, has been widely accepted among scientists [5–10].

To consider internal structural damping of nanobeams, we can use some of the available constitutive relations for viscoelastic bodies. Several definitions of viscoelastic models are commonly used in the literature such as Kelvin–Voigt, Maxwell and Zener model. It was proven that in some cases, for classical integer order derivative models of viscoelasticity to many parameters are needed to fit experimental curves. Accordingly, some authors suggested application of generalized models of viscoelasticity with fractional order derivatives [11–13]. Such models can represent the time dependent behavior of materials, which is in between the pure elastic and pure viscous behavior with less parameters needed to fit experimental curves due to the presence of fractional order derivative [14]. In numerous of papers [15–19], fractional calculus is often applied in vibration studies of rods, beams, plates as well as in other fields of mechanics.

Nonlocal nanobeam or nanoplate models representing the carbon nanotubes (CNT), zinc-oxide (ZnO) nanotubes, graphene sheets and other small-size beam like structures are widely used to examine vibration response, critical buckling forces and other physical phenomena's [20,21]. Such theoretical studies can lead to important data in the fabrication and exploitation of NEMS and MEMS devices. Vibration responses of beams and nanobeams with attached mass are also reported in the literature [22] often as mass sensors or for other applications [23]. Attached mass in nanosensors can represent e.g. carbon buckyballs or in the case of biosensors it can represent some proteins, enzymes or bacteria. In paper by Lee et al. [24], nonlocal elasticity theory is used to model a cantilever sensor with an arbitrary attached mass. Different effects of varying mass and nonlocal parameter on natural frequencies were examined. Murmu and Adhikari [25] derived explicit analytical expressions for frequencies of longitudinally vibrating single-walled carbon nanotubes (SWCNTs) with an attached buckyballs at the tip. In addition, the authors have investigated frequency shifts for changes of different types of buckyballs and changes in nonlocal parameter. Kiani et al. [26] have analyzed a more general case of SWCNT based mass sensor for arbitrary attached nano-objects. The authors modeled SWCNTs via nonlocal Rayleigh, Timoshenko, and higher-order beam theories, considering a wide range of boundary conditions and observing nano-objects as rigid solids. They obtained discrete governing equation for each model by employing the meshless technique and investigated the influence of parameters representing the nonlocal effect, mass weight of nano-objects, slendermass ratio of SWCNT and number of nanoparticles on the frequency shift ratio.

All listed works regarding to mass sensing nanostructures are done within the scope of elasticity and nonlocal elasticity theory. According to the best of authors knowledge there is no work in the literature considering the free vibration of fractional order viscoelastic model of nanobeam with attached mass. Since internal damping of nanostructures is significant for theoretical analysis and practical use of nanodevices, it is of great importance to apply improved viscoelastic models based on fractional derivatives such as fractional Kelvin–Voigt or Zener model [12]. Here, we limited our study to application of fractional order Kelvin–Voigt model, which is introduced into the motion equation of nonlocal Euler–Bernoulli beam. Arbitrary attached mass is taken into account via internal boundary conditions. Eigenvalues and mode shapes are explored for different boundary conditions, different mass positions and nonlocal parameter. Time dependent behavior is investigated by finding the solution of fractional order differential equation using the Laplace transform method and the effect of change of fractional order derivative parameter is also examined.

2. Preliminaries

2.1. Fractional order viscoelasticity. Fractional calculus is a branch of mathematical analysis that has found many applications in mechanics. The base of fractional calculus is study of an arbitrary real or complex order integrals and derivatives. Various authors have proposed many different definitions of fractional order integrals and derivatives. However, in our study we will consider only the Riemann–Liouville's definition of fractional derivative as follows

DEFINITION 2.1. (cf. [12]) If $x(\cdot)$ is an absolutely continuous function in $[a, b]$ and $0 < \alpha < 1$, then

a) The left Riemann–Liouville fractional derivative of order α is of the form

$$(2.1) \qquad {}_aD_t^\alpha = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{x(\tau)}{(t-\tau)^\alpha} d\tau, \quad t \in [a, b],$$

b) The right Riemann–Liouville fractional derivative of order α is of the form

$$ {}_aD_t^\alpha = \frac{1}{\Gamma(1-\alpha)} \left(-\frac{d}{dt} \right) \int_a^t \frac{x(\tau)}{(\tau-t)^\alpha} d\tau, \quad t \in [a, b],$$

The well-known application of the given Riemann–Liouville's definition of fractional derivative is in rheology as well as in structural mechanics. It is used for accurate modeling of structures when internal damping is considered via fractional order viscoelastic models. In [11], it was shown that classical viscoelastic models failed to describe damping of viscoelastic solid and that improved fractional derivative based models need to be considered. Such models have few advances. First, they are based on molecular theories [14]. Second, such models are satisfying the second law of thermodynamics. At last, they require just a few parameters to describe viscoelastic behavior that in some cases of classical viscoelastic models can reach very high number of parameters needed to fit experimental curves.

The first time to introduce the modified Kelvin–Voigt model of viscoelastic body with fractional order derivative was in the paper by Shermergor [31]. An

interesting chronology of available constitutive relations of viscoelastic bodies based on fractional order derivatives one may find in the review paper by Rossikhin [**32**]. For the one dimensional isothermal viscoelastic body, fractional Kelvin–Voigt type model [**32**] is given as

$$(2.2) \qquad t_{xx} = E_\infty \big(\varepsilon_{xx}(t) + \tau_\sigma^\alpha {}_0 D_t^\alpha (\varepsilon_{xx}(t)) \big),$$

where t_{xx} is the stress in terms of the strain ε_{xx}, E_∞ is the relaxed modulus of elasticity, τ_σ^α is the retardation or creep time, and ${}_0 D_t^\alpha$ is the operator of Riemann–Liouville derivative (2.1) of real order α where $0 < \alpha < 1$. The previous Eq. (2.2) can be rewritten as

$$(2.3) \qquad t_{xx} = E_\infty \varepsilon_{xx}(t) + E_\alpha {}_0 D_t^\alpha (\varepsilon_{xx}(t)),$$

where $E_\alpha = E_\infty \tau_\sigma^\alpha$. More about the restrictions on the value of coefficients E_∞ and E_α that follow from the second law of thermodynamics one can find in [**12**]. Even though fractional Kelvin–Voigt model is applicable in the low frequency range [**11**], it can be useful for the ''fast'' dynamics applications or to describe the creep behavior in materials.

2.2. Nonlocal theory. In the nonlocal elasticity theory the stress at a point x is a function of strains at all other points of an elastic body. The integral form of nonlocal constitutive relation for a three-dimensional structure is

$$\sigma_{ij}(x) = \int \alpha(|x - x'|, \tau) t_{ij}(x') dV(x'), \quad \forall x \in V,$$

where σ_{ij} is the nonlocal stress tensor, t_{ij} is the local or classical stress tensors at a point x', $\alpha(|x - x'|, \tau)$ denotes attenuation function which incorporates nonlocal effects into the constitutive equation, $|x - x'|$ is a distance in Euclidean norm and $\tau = e_0 a / l$ is the nonlocal parameter where l is the external characteristic length (crack length or wave length), a is internal characteristic length (lattice parameter, granular etc.) and e_0 is a material constant that can be determined from molecular dynamics simulations or by using dispersive curve of the Born-Karman model of lattice dynamics. Later, by adopting an appropriate kernel function in the previous integral form of equation, Eringen [**6**] proposed a differential form of constitutive relation (see also [**33**]) as

$$(2.4) \qquad (1 - \tau^2 l^2 \nabla^2) \sigma_{ij} = t_{ij},$$

For one dimensional case and according to the Hooke's law, the local stress t_{xx} at a point x' is related to the strain ε_{xx} at that point as

$$(2.5) \qquad t_{xx}(x') = E \varepsilon_{xx}(x'),$$

that yields the following differential form of nonlocal constitutive equation

$$\sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E \varepsilon_{xx},$$

where E denotes elastic modulus of the elastic body, $\mu = (e_0 a)^2$ is the nonlocal parameter and σ_{xx} is the nonlocal stress. In order to obtain constitutive relation for a nonlocal viscoelastic body we can combine elasticity and viscoelasticity

theory [27]. Therefore, for one-dimensional nonlocal viscoelastic solids, constitutive relation for a classical integer order Kelvin-Voight viscoelastic model is given by

$$\sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E\varepsilon_{xx} + \tau_d \dot{\varepsilon}_{xx},$$

where τ_d is the viscous damping coefficient of nanobeam. In order to extend this definition to the nonlocal fractional Kelvin-Voight model of viscoelastic body with fractional derivative, based on Eq. (2.3) we propose the following form of constitutive equation

$$(2.6) \qquad \sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E_\infty \varepsilon_{xx} + E_{\alpha 0} D_t^\alpha \varepsilon_{xx},$$

Based on Eqs. (2.4)–(2.6) it can be concluded that behavior of nonlocal fractional Kelvin–Voigt model in time domain is equivalent to the behavior of the local fractional Kelvin–Voigt model (2.3). This will be more discussed in the validation study. In addition, it can be observed that fractional derivative operator is a nonlocal operator that can be used not only for non-locality in the time domain as in our case but also to describe non-locality in the spatial domain (e.g. see [29]).

3. Governing equation

3.1. Derivation of the governing equation. Here, we review basic equations of nonlocal viscoelastic Euler–Bernoulli beam of length L, cross-sectional area A, density ρ and transverse deflection $w(x, t)$ in z direction (see Figure 1). We assume that cross-sectional area is constant along x coordinate and that material of a nanobeam is homogeneous. According to the Euler–Bernoulli beam theory that assumes small deflection, the axial strain is given as

$$(3.1) \qquad \varepsilon_{xx} = -z \frac{\partial^2 w}{\partial x^2}$$

After substituting Eq. (3.1) into Eq. (2.6), we obtain the following equation

$$(3.2) \qquad \sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = -E_\infty z \frac{\partial^2 w}{\partial x^2} - E_{\alpha 0} D_t^\alpha \Big(z \frac{\partial^2 w}{\partial x^2} \Big),$$

The equation of motion for the transverse vibration of nanobeam can be obtained from the second Newton law as

$$(3.3) \qquad \rho A \frac{\partial^2 w}{\partial t^2} = \frac{\partial V}{\partial x} + q(x)$$

where V is resultant shear force on the cross section and $q(x)$ is distributed transverse force along x axis which is assumed to be zero in case of the free transverse vibration. From the moment equilibrium equation of Euler–Bernoulli beam we have

$$V = \frac{\partial M}{\partial x},$$

where M is resultant bending moment which is defined as $M = \int_A z\sigma_{xx} dA$. Taking into account Eq. (3.2) and that the second moment of inertia is $I = \int_A z^2 dA$, we
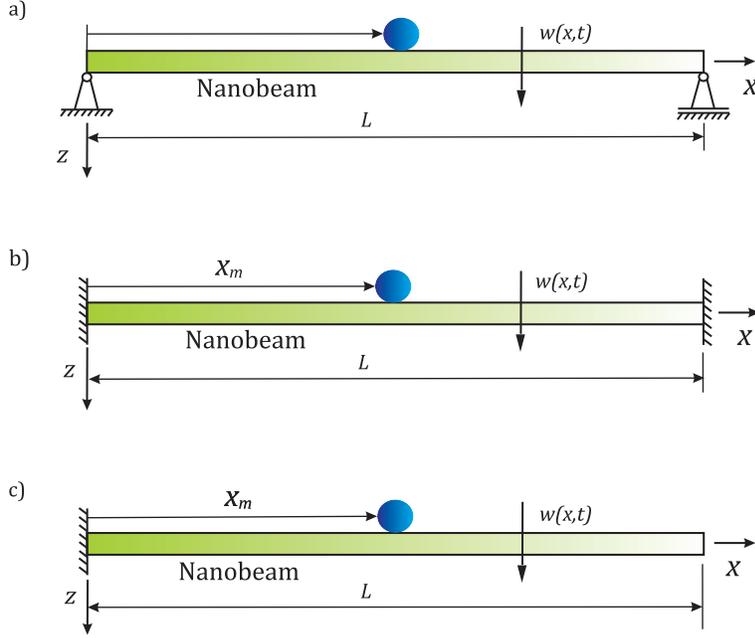
FIGURE 1. Nanobeam with arbitrary attached nanoparticle
a) Simply supported nanobeam; b) Clamped-clamped nanobeam;
c) Cantilever nanobeam.

obtain moment equation expressed as

$$(3.4) \qquad M = \mu\rho A\frac{\partial^2 w}{\partial t^2} - E_\infty I\frac{\partial^2 w}{\partial x^2} - E_\alpha I_0 D_t^\alpha\Big(\frac{\partial^2 w}{\partial x^2}\Big)$$

and the shear force V as

$$V = \mu\rho A\frac{\partial^3 w}{\partial x\partial t^2} - E_\infty I\frac{\partial^3 w}{\partial x^3} - E_\alpha I_0 D_t^\alpha\Big(\frac{\partial^3 w}{\partial x^3}\Big)$$

Finally, using Eq. (3.3) and (3.4) we can obtain equation of motion expressed in terms of the deflection $w(x,t)$ as

$$(3.5) \qquad \rho A\frac{\partial^2 w}{\partial t^2} - \mu\rho A\frac{\partial^4 w}{\partial x^2\partial t^2} + E_\infty I\frac{\partial^4 w}{\partial x^4} + E_\alpha I_0 D_t^\alpha\Big(\frac{\partial^4 w}{\partial x^4}\Big) = 0.$$

In follow, we utilize Eq. (3.5) for the development of a mathematical model of nonlocal fractional viscoelastic nanobeam with attached nanoparticle.

3.2. Solution of the governing equation. To investigate the free transverse vibration of the nanobeam with attached nanoparticle it is necessary to introduce boundary conditions properly into the solution. By introducing internal boundary conditions [22] at the position of attached nanoparticle we split nanobeam on two segments for which we have separate mode shape functions but with continuous transition from one to another. By this concept, each new attached nanoparticle

on the nanobeam introduces a new internal boundary and leads to twice as many motion eqations and four new boundary conditions where each side of internal boundary is observed as separate nanobeam. For the sake of simplicity, we will limit our study only to the nanobeam system with a single arbitrary attached nanoparticle where we have only one internal boundary. Following this, we have two motion equations of the same form as Eq. (3.5) but with different deflections $w_1(x,t)$ and $w_2(x,t)$ that are given in dimensionless form as

$$(3.6) \qquad \frac{\partial^2 \bar{w}_1}{\partial \tau^2} - \bar{\mu}\frac{\partial^4 \bar{w}_1}{\partial \bar{x}^2 \partial \tau^2} + \frac{\partial^4 \bar{w}_1}{\partial \bar{x}^4} + \gamma \cdot {}_0D_t^{\alpha}\Big(\frac{\partial^4 \bar{w}_1}{\partial \bar{x}^4}\Big) = 0.$$

$$(3.7) \qquad \frac{\partial^2 \bar{w}_2}{\partial \tau^2} - \bar{\mu}\frac{\partial^4 \bar{w}_2}{\partial \bar{x}^2 \partial \tau^2} + \frac{\partial^4 \bar{w}_2}{\partial \bar{x}^4} + \gamma \cdot {}_0D_t^{\alpha}\Big(\frac{\partial^4 \bar{w}_2}{\partial \bar{x}^4}\Big) = 0.$$

and the following dimensionless internal boundary conditions

$$\bar{w}_1 = \bar{w}_2,$$
$$\frac{\partial \bar{w}_1}{\partial x} = \frac{\partial \bar{w}_2}{\partial x},$$
$$\bar{M}_1 = \bar{M}_2,$$
$$\bar{V}_1 - \bar{V}_2 + \delta\frac{\partial^2 \bar{w}_2}{\partial \tau^2} = 0,$$

where dimensionless quantities are

$$\bar{w}_1 = \frac{w_1}{L}, \bar{w}_2 = \frac{w_2}{L}, \bar{x} = \frac{x}{L}, \bar{x}_m = \frac{x_m}{L}, c = L^2\sqrt{\frac{\rho A}{E_\infty I}},$$
$$\bar{\mu} = \frac{\mu}{L}, \gamma = \frac{E_\alpha}{E_\infty c^\alpha}, \tau = \frac{t}{c}, \delta = \frac{mAL}{E_\infty Ic}.$$

External boundary conditions are prescribed depending of the type of supports. Here, we consider three characteristic types of nanobeam supports: simply supported nanobeam, clamped-clamped and cantilever nanobeam. However, before further consideration of nanobeam boundary conditions, we first suggest the solution of motion equations (3.6) and (3.7) by separating the variables on time and amplitude functions in the form

$$(3.8) \qquad \bar{w}_1(\bar{x}, \tau) = W_1(\bar{x})T_1(\tau),$$

$$(3.9) \qquad \bar{w}_2(\bar{x}, \tau) = W_2(\bar{x})T_2(\tau),$$

where in the follow, it will be assumed that time functions are equal for both parts of nanobeam $T_1 = T_2 = T$. If we substitute Eqs. (3.8) and (3.9) into the Eqs. (3.6) and (3.7), respectively, then we obtain the following differential equations in terms of time function T and amplitude functions W_1 and W_2 as

$$(3.10) \qquad \ddot{T} + \gamma k^4 {}_0D_t^{\alpha}(T) + k^4 T = 0,$$

$$(3.11) \qquad W_1^{(IV)} + \bar{\mu}k^4 W_1^{(II)} - k^4 W_1 = 0,$$

$$(3.12) \qquad W_2^{(IV)} + \bar{\mu}k^4 W_2^{(II)} - k^4 W_2 = 0,$$

where k is an arbitrary constant and $W_i^{(II)} = d^2W_i/d\bar{x}^2$, $W_i^{(IV)} = d^4W_i/d\bar{x}^4$ for $i = 1, 2$. In order to solve fourth order differential equations (3.11) and (3.12) for amplitude functions, we propose solutions of the form $W_i = Ae^{\lambda\bar{x}}$, $i = 1, 2$. Substituting assumed solutions into Eqs. (3.11) and (3.12) we obtain the characteristic equation as

$$\lambda^4 + \bar{\mu}k^4\lambda^2 - k^4 = 0,$$

We have four solutions of characteristic equation which are $\lambda_{1,2} = \pm\lambda_e$ and $\lambda_{3,4} = \pm\lambda_f$ where λ_e and λ_f are

$$(3.13) \qquad \lambda_e = \pm k\sqrt{\frac{\sqrt{4 + \bar{\mu}^2 k^4} - \bar{\mu}k^2}{2}},$$

$$(3.14) \qquad \lambda_f = \pm k\sqrt{\frac{\sqrt{4 + \bar{\mu}^2 k^4} + \bar{\mu}k^2}{2}},$$

Rearranging constants, we can write solutions of Eqs. (3.11) and (3.12) as

$$(3.15) \qquad W_1 = C_1 \sinh(\lambda_e \bar{x}) + C_2 \sin(\lambda_f \bar{x}) + C_3 \cosh(\lambda_e \bar{x}) + C_4 \cos(\lambda_f \bar{x})$$

$$(3.16) \qquad W_2 = C_5 \sinh(\lambda_e \bar{x}) + C_6 \sin(\lambda_f \bar{x}) + C_7 \cosh(\lambda_e \bar{x}) + C_8 \cos(\lambda_f \bar{x})$$

where C_i, $i = 1, 2, \ldots, 8$ are unknown constants. In the above equations, k denotes non-dimensional flexural wave number of the classical Euler–Bernoulli beam and it is related to λ_f and λ_e through the Eqs. (3.13) and (3.14). When the influence of nonlocal parameter is neglected $\bar{\mu} = 0$, than we have $\lambda_f = \lambda_e = k$. Further, we find eigenvalues of nanobeam with arbitrary attached nanoparticle for three different external boundary conditions.

3.3. Case I: Simply supported nanobeam. For the simply supported boundary conditions we assume external boundary conditions at $\bar{x} = 0$ and $\bar{x} = 1$ as

$$(3.17) \qquad W_1 = W_2 = 0,$$

$$(3.18) \qquad W_1^{(II)} + \bar{\mu}k^4 W_1 = W_2^{(II)} + \bar{\mu}k^4 W_2 = 0,$$

and rewrite internal boundary conditions in terms of amplitude function at $\bar{x} = \bar{x}_m$ as

$$(3.19) \qquad W_1 = W_2,$$

$$(3.20) \qquad W_1^{(I)} = W_2^{(I)},$$

$$(3.21) \qquad W_1^{(II)} + \bar{\mu}k^4 W_1 = W_2^{(II)} + \bar{\mu}k^4 W_2,$$

$$(3.22) \qquad W_1^{(III)} + \bar{\mu}k^4 W_1^{(I)} - (W_2^{(III)} + \bar{\mu}k^4 W_2^{(I)}) + \delta k^4 W_1 = 0,$$

where $W_i^{(I)} = dW_i/d\bar{x}$ and $W_i^{(III)} = d^3W_i/d\bar{x}^3$ for $i = 1, 2$. By taking into account the solutions from Eqs. (3.15) and (3.16) and boundary conditions from Eqs. (3.17) and (3.18), we obtain eight equations in terms of eight unknown constants C_i, $i = 1, 2, \ldots, 8$. These equations can be rewritten in the matrix form as

$$[A]\{C\} = 0,$$

or in the form

$$
\begin{bmatrix}
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{23} & a_{24} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{35} & a_{36} & a_{37} & a_{38} \\
0 & 0 & 0 & 0 & a_{45} & a_{46} & a_{47} & a_{48} \\
a_{51} & a_{52} & a_{53} & a_{54} & -a_{51} & -a_{52} & -a_{53} & -a_{54} \\
\lambda_e a_{53} & \lambda_f a_{54} & \lambda_e a_{51} & -\lambda_f a_{52} & -\lambda_e a_{53} & -\lambda_f a_{54} & -\lambda_e a_{51} & \lambda_f a_{52} \\
a_{71} & a_{72} & a_{73} & a_{74} & -a_{71} & -a_{72} & -a_{73} & -a_{74} \\
a_{81} & a_{82} & a_{83} & a_{84} & -\lambda_e a_{73} & -\lambda_f a_{74} & -\lambda_e a_{71} & \lambda_f a_{72}
\end{bmatrix}
\begin{bmatrix}
C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8
\end{bmatrix} = 0,
$$

$a_{23} = \lambda_e^2 + \bar{\mu}k^4, a_{24} = \bar{\mu}k^4 - \lambda_f^2, a_{35} = \sinh(\lambda_e), a_{36} = \sin(\lambda_f), a_{37} = \cosh(\lambda_e),$

$a_{38} = \cos(\lambda_f), a_{45} = a_{23}a_{35}, a_{46} = a_{24}a_{36}, a_{47} = a_{23}a_{37}, a_{48} = a_{24}a_{38},$

$a_{51} = \sinh(\lambda_e \bar{x}_m), a_{52} = \sin(\lambda_f \bar{x}_m), a_{53} = \cosh(\lambda_e \bar{x}_m), a_{54} = \cos(\lambda_f \bar{x}_m),$

$a_{51} = \sinh(\lambda_e \bar{x}_m), a_{52} = \sin(\lambda_f \bar{x}_m), a_{53} = \cosh(\lambda_e \bar{x}_m), a_{54} = \cos(\lambda_f \bar{x}_m),$

$a_{71} = a_{23}a_{51}, a_{72} = a_{24}a_{52}, a_{73} = a_{23}a_{53}, a_{74} = a_{24}a_{54},$

$a_{81} = (\lambda_e^3 + \lambda_e \bar{\mu}k^4)a_{53} + \delta k^4 a_{51}, a_{82} = (\lambda_f \bar{\mu}k^4 - \lambda_f^3)a_{54} + \delta k^4 a_{52},$

$a_{83} = (\lambda_e^3 + \lambda_e \bar{\mu}k^4)a_{51} + \delta k^4 a_{53}, a_{84} = -(\lambda_f \bar{\mu}k^4 - \lambda_f^3)a_{51} + \delta k^4 a_{54},$

In that way, we can obtain characteristic transcendental equation $N(\delta, k)$ by finding the determinant of matrix A as

(3.23) $$N(\delta, k) = \det[A] = 0.$$

From Eq. (3.23) we can obtain an infinite number of roots k_n, $n = 1, 2, \ldots, \infty$ for the nonlocal viscoelastic nanobeam with attached nanoparticle of certain mass. Using the known algebra rule that ratios of unknown constants C_i, $i = 1, 2, \ldots, \infty$ and cofactors K_{8i} of matrix A are equal to some constant C_n as

$$\frac{C_1}{K_{81}} = \frac{C_2}{K_{82}} = \frac{C_3}{K_{83}} = \frac{C_4}{K_{84}} = \frac{C_5}{K_{85}} = \frac{C_6}{K_{86}} = \frac{C_7}{K_{87}} = \frac{C_8}{K_{88}} = C_n,$$

we can write mode shape functions in the following forms

(3.24) $$W_{1n} = C_n\left(\sinh(\lambda_e \bar{x}) + \frac{K_{82}}{K_{81}}\sin(\lambda_f \bar{x}) + \frac{K_{83}}{K_{81}}\cosh(\lambda_e \bar{x}) + \frac{K_{84}}{K_{81}}\cos(\lambda_f \bar{x})\right)$$

(3.25) $$W_{2n} = C_n\left(\frac{K_{85}}{K_{81}}\sinh(\lambda_e \bar{x}) + \frac{K_{86}}{K_{81}}\sin(\lambda_f \bar{x}) + \frac{K_{87}}{K_{81}}\cosh(\lambda_e \bar{x}) + \frac{K_{88}}{K_{81}}\cos(\lambda_f \bar{x})\right)$$

where λ_e and λ_f are calculated from Eqs. (3.13) and (3.14) for the corresponding n-th eigenvalue k_n.

3.4. Case II: Cantilever nanobeam. For the cantilever boundary conditions, we assume external boundary conditions at $\bar{x} = 0$ and $\bar{x} = 1$ as

(3.26) $$W_1 = W_1^{(I)} = 0,$$

(3.27) $$W_2^{(II)} + \bar{\mu}k^4 W_2 = W_2^{(III)} + \bar{\mu}k^4 W_2^{(I)},$$

Internal boundary conditions are of the same form as in Eqs. (3.19)–(3.22). Using the solutions from Eqs. (3.15) and (3.16), internal boundary conditions from Eqs. (3.19)–(3.22) and boundary conditions of cantilever nanobeam (3.26) and (3.27), we obtain the matrix form of equation as

$$[B]\{C\} = 0,$$

or in matrix form as

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \lambda_e & \lambda_f & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{35} & b_{36} & b_{37} & b_{38} \\ 0 & 0 & 0 & 0 & \lambda_e b_{37} & \lambda_f b_{38} & \lambda_e b_{35} & \lambda_f b_{36} \\ b_{51} & b_{52} & b_{53} & b_{54} & -b_{51} & -b_{52} & -b_{53} & -b_{54} \\ \lambda_e b_{53} & \lambda_f b_{54} & \lambda_e b_{51} & -\lambda_f b_{52} & -\lambda_e b_{53} & -\lambda_f b_{54} & -\lambda_e b_{51} & \lambda_f b_{52} \\ b_{71} & b_{72} & b_{73} & b_{74} & -b_{71} & -b_{72} & -b_{73} & -b_{74} \\ b_{81} & b_{82} & b_{83} & b_{84} & -\lambda_e b_{73} & -\lambda_f b_{74} & -\lambda_e b_{71} & \lambda_f b_{72} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{bmatrix} = 0,$$

where elements of matrix B are

$b_{35} = (\lambda_e^2 + \bar{\mu}k^4)\sinh(\lambda_e), b_{36} = -(\bar{\mu}k^4 - \lambda_f^2)\sin(\lambda_f), b_{37} = (\lambda_e^2 + \bar{\mu}k^4)\cosh(\lambda_e),$

$b_{38} = (\bar{\mu}k^4 - \lambda_f^2)\cos(\lambda_f), b_{51} = \sinh(\lambda_e \bar{x}_m), b_{52} = \sin(\lambda_f \bar{x}_m), b_{53} = \cosh(\lambda_e \bar{x}_m),$

$b_{54} = \cos(\lambda_f \bar{x}_m), b_{71} = (\lambda_e^2 + \bar{\mu}k^4)b_{51}, b_{72} = (\bar{\mu}k^4 - \lambda_f^2)b_{52}, b_{73} = (\lambda_e^2 + \bar{\mu}k^4)b_{53},$

$b_{74} = (\bar{\mu}k^4 - \lambda_f^2)b_{54}, b_{81} = (\lambda_e^3 + \lambda_e \bar{\mu}k^4)b_{53} + \delta k^4 b_{51}, b_{82} = (\lambda_f \bar{\mu}k^4 - \lambda_f^3)b_{54} + \delta k^4 b_{52},$

$b_{83} = (\lambda_e^3 + \lambda_e \bar{\mu}k^4)b_{51} + \delta k^4 b_{53}, b_{84} = -(\lambda_f \bar{\mu}k^4 - \lambda_f^3)b_{51} + \delta k^4 b_{54},$

In this case, characteristic transcendental equation $M(\delta, k)$ is obtained by finding the determinant of matrix B as

$$(3.28) \qquad\qquad M(\delta, k) = \det[B] = 0.$$

Further, mode shape functions are obtained from Eqs. (3.17) and (3.25) by considering the new eigenvalues k_n determined from Eq. (3.28).

3.5. Case III: Clamped-clamped nanobeam. For the clamped-clamped boundary conditions, external boundary conditions at $\bar{x} = 0$ and $\bar{x} = 1$ are

$$(3.29) \qquad\qquad W_1 = W_1^I = 0.$$

$$(3.30) \qquad\qquad W_2 = W_2^I = 0.$$

Internal boundary conditions are of the same form as in Eqs. (3.19)–(3.22). Considering the assumed solutions and boundary conditions from Eq. (3.29) and (3.30) in the same manner as in two previous cases, we obtain the following matrix form of equation

$$[D]\{C\} = 0,$$

or in the form

$$
\begin{bmatrix}
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
\lambda_e & \lambda_f & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_{35} & d_{36} & d_{37} & d_{38} \\
0 & 0 & 0 & 0 & \lambda_e d_{37} & \lambda_f d_{38} & \lambda_e d_{35} & \lambda_f d_{36} \\
d_{51} & d_{52} & d_{53} & d_{54} & -d_{51} & -d_{52} & -d_{53} & -d_{54} \\
\lambda_e d_{53} & \lambda_f d_{54} & \lambda_e d_{51} & -\lambda_f d_{52} & -\lambda_e d_{53} & -\lambda_f d_{54} & -\lambda_e d_{51} & \lambda_f d_{52} \\
d_{71} & d_{72} & d_{73} & d_{74} & -d_{71} & -d_{72} & -d_{73} & -d_{74} \\
d_{81} & d_{82} & d_{83} & d_{84} & -\lambda_e d_{73} & -\lambda_f d_{74} & -\lambda_e d_{71} & \lambda_f d_{72}
\end{bmatrix}
\begin{bmatrix}
C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8
\end{bmatrix} = 0,
$$

where elements of matrix B are

$d_{35} = \sinh(\lambda_e), d_{36} = -\sin(\lambda_f), d_{37} = \cosh(\lambda_e), d_{38} = \cos(\lambda_f),$

$d_{51} = \sinh(\lambda_e \bar{x}_m), d_{52} = \sin(\lambda_f \bar{x}_m), d_{53} = \cosh(\lambda_e \bar{x}_m), d_{54} = \cos(\lambda_f \bar{x}_m),$

$d_{71} = (\lambda_e^2 + \bar{\mu}k^4)d_{51}, d_{72} = (\bar{\mu}k^4 - \lambda_f^2)d_{52}, d_{73} = (\lambda_e^2 + \bar{\mu}k^4)d_{53},$

$d_{74} = (\bar{\mu}k^4 - \lambda_f^2)d_{54}, d_{81} = (\lambda_e^3 + \lambda_e \bar{\mu}k^4)d_{53} + \delta k^4 d_{51}, d_{82} = (\lambda_f \bar{\mu}k^4 - \lambda_f^3)d_{54} + \delta k^4 d_{52},$

$d_{83} = (\lambda_e^3 + \lambda_e \bar{\mu}k^4)d_{51} + \delta k^4 d_{53}, d_{84} = -(\lambda_f \bar{\mu}k^4 - \lambda_f^3)d_{51} + \delta k^4 d_{54},$

Finally, characteristic transcendental equation $R(\delta, k)$ is obtained by finding the determinant of matrix B as

(3.31) $$R(\delta, k) = \det[D] = 0,$$

where again, mode shape functions can be obtained from Eqs. (3.24) and (3.25) taking into account eigenvalues k_n determined from Eq. (3.31).

4. Time dependent behavior of damped nanobeam

In the literature [**12**,**15**–**17**], there are many analytical and numerical methods available to find the solution of fractional order differential equations. We can notice that solution of Eq. (3.10) for an infinitive number of modes with corresponding eigenvalues k_n and time functions T_n is the part of assumed solution given in the form of infinite series

$$
w_j = \sum_{n=1}^{\infty} W_{jn} T_n, \quad j = 1, 2.
$$

In the follow, we will examine the solution of fractional order differential equation. Let as rewritten Eq. (3.10) as

(4.1) $$\ddot{T}_n + \beta T_n^\alpha + \omega^2 T_n = 0$$

where $\beta = \gamma k^4$, $\omega^2 = k^4$ and $T_n^\alpha = {}_0 D_t^\alpha T_n$ is a fractional derivative of time function. Using the methodology similar to those applied in [**12**] to solve the fractional order differential equation (4.1) when $0 < \alpha < 1$, we use the Laplace transform method. Laplace transform of a function $T_n(\tau)$ is denoted by $\mathcal{L}(T_n)$. First, we write Laplace transform of fractional derivative of a function $T_n(\tau)$ as

$$
\mathcal{L}(T_n^\alpha) = s^\alpha \, \mathcal{L}(T_n) - T_n^{\alpha-1}(0) = 0,
$$

where for the initial condition of fractional order we have

$$
T_n^{\alpha-1}(0) = 0,
$$

if T_n is bounded in the interval $[0, \epsilon]$ for $\epsilon > 0$ [**12**]. Then, after performing the Laplace transform of Eq. (4.1) we obtain

$$(4.2) \qquad \mathcal{L}(T_n) = \frac{sT_n(0) + \dot{T}_n(0)}{s^2 + \beta s^\alpha + \omega^2}.$$

In order to expand Eq. (4.2) into trigonometric series we rewrite the equation in the following form

$$\mathcal{L}(T_n) = \frac{1}{s} \frac{T_n(0)}{1 + \frac{\beta}{s^2}\left(s^\alpha + \frac{\omega^2}{\beta}\right)} + \frac{1}{s^2} \frac{\dot{T}_n(0)}{1 + \frac{\beta}{s^2}\left(s^\alpha + \frac{\omega^2}{\beta}\right)}.$$

After writing previous equation in terms of series as

$$\mathcal{L}(T_n) = T_n(0) \sum_{k=0}^{\infty} \frac{(-1)^k \beta^k}{s^{2k+1-\alpha j}} \left(s^\alpha + \frac{\omega^2}{\beta}\right)^k + \dot{T}_n(0) \sum_{k=0}^{\infty} \frac{(-1)^k \beta^k}{s^{2k+2}} \left(s^\alpha + \frac{\omega^2}{\beta}\right)^k,$$

and using binomial coefficients we can write solution in the form

$$(4.3) \qquad \mathcal{L}(T_n) = T_n(0) \sum_{k=0}^{\infty} (-1)^k \sum_{j=0}^{k} \binom{k}{j} \frac{\beta^j \omega^{2(k-j)}}{s^{2k+1-\alpha j}}$$

$$+ \dot{T}_n(0) \sum_{k=0}^{\infty} (-1)^k \sum_{j=0}^{k} \binom{k}{j} \frac{\beta^j \omega^{2(k-j)}}{s^{2(k+1)-\alpha j}},$$

Finally, assuming that expansion leads to convergent series and applying the inverse Laplace transform on Eq. (4.3), yields

$$(4.4) \qquad T_n(\tau) = T_n(0) \sum_{k=0}^{\infty} (-1)^k \sum_{j=0}^{k} \binom{k}{j} \frac{\beta^j \omega^{2(k-j)} \tau^{2k-\alpha j}}{\Gamma[2k+1-\alpha j]}$$

$$+ \dot{T}_n(0) \sum_{k=0}^{\infty} (-1)^k \sum_{j=0}^{k} \binom{k}{j} \frac{\beta^j \omega^{2(k-j)} \tau^{2k+1-\alpha j}}{\Gamma[2(k+1)-\alpha j]}.$$

Solution (4.4) is of the same form as the solution of linear vibrations equation with the fractional dissipation term given in [**12**]. In addition, we should note that we have two special cases of the solution of Eq. (4.4) like in [**12**], which for $\alpha = 0$ and $\alpha = 1$ are

$$(4.5) \ \ T_n = \begin{cases} T_n(0) \cos\left(\tau\sqrt{\omega^2 + \beta}\right) + \dot{T}_n(0) \frac{\sin\left(\tau\sqrt{\omega^2+\beta}\right)}{\sqrt{\omega^2+\beta}}, \alpha = 0 \\ e^{-\frac{\beta\tau}{2}} \left[T_n(0) \cos\left(\tau\sqrt{\omega^2 - \frac{\beta^2}{4}}\right) + \frac{\dot{T}_n(0) + T_n(0)\frac{\beta}{2}}{\sqrt{\omega^2 - \frac{\beta^2}{4}}} \sin\left(\tau\sqrt{\omega^2 + \frac{\beta^2}{4}}\right) \right], \alpha = 1 \end{cases}$$

The first solution is the case of harmonic vibration when no damping occurs in the system. The second case is the solution for damped vibration when classical Kelvin–Voigt viscoelastic model is considered.

5. Numerical results and discussion

In this section, validation of the obtained results is given by comparison with the corresponding results in the literature. Numerical experiments are performed for the transverse vibration of a viscoelastic nanobeam with arbitrary attached nanoparticle of certain mass. Two different types of numerical analysis are performed. Firstly, we give mode shapes of the nanobeam with attached nanoparticle and examine the influence of the nonlocal parameter on them. Secondly, the effect of fractional order parameter on time dependent behavior of the observed system is analyzed. In this study, numerical analysis is carried out for three types of boundary conditions, simply supported, clamped–clamped and cantilever nanobeam.

5.1. Validation with MD simulations. In order to validate our model, we compare the results for complex roots of characteristic equation for the free vibrating nanobeam without attached mass with the results from MD simulations given in [35], in the similar manner as it was done in [34]. To find complex roots of the characteristic equation obtained from governing equation (3.4), we use the methodology described in [17] and assume the solution for the simply supported nanobeam in the form

$$(5.1) \qquad w(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(k_n x), \quad k_n = \frac{n\pi}{L}.$$

After substituting Eq. (5.1) into (3.4), taking into account orthogonality conditions and performing Laplace transformation yields the following equation

$$(5.2) \qquad \bar{T}_n(s) = \frac{sT(0) + \dot{T}(0)}{f_n(s)}.$$

where

$$f_n(s) = s^2 + \kappa s^\alpha + \omega_0^2, \quad \omega_0^2 = \frac{E_\infty I k_n^2}{\rho A(1 + \mu k_n^2)}, \quad \kappa = \omega_0^2 \tau_\sigma^\alpha.$$

with over-bar denoting the Laplace transform of the corresponding function and τ_σ denoting the retardation time . It can be noticed that Eq. (5.2) is of the equivalent form as dimensionless Eq. (5.2) with only difference in frequency of nonlocal elastic system ω_0 influenced by nonlocal parameter μ. By taking that $f_n = 0$, we obtain the characteristic equation which is of the same form as characteristic equation of the fractional Kelvin–Voigt single mass oscillator presented in [17], whose complex roots can be found by carrying out the following substitution $s = re^{i\psi}$. Than, separating the real and imaginary parts and adopting new variables $x_1 = r^2$ and $x_2 = \kappa r^\alpha$ we obtain

$$x_1 = \frac{\omega_0^2 \sin(\alpha\psi)}{\sin(2-\alpha)\psi}, \quad x_2 = -\frac{\omega_0^2 \sin(2\psi)}{\sin(2-\alpha)\psi}$$

where $r = x_1^{1/2}$ and $\kappa = x_2 r^{-\alpha}$ can be calculate by choosing ψ in the range $\pi/2$–$\pi/(2-\alpha)$ that gives one root of the characteristic equation (see also [17]). Choosing $-\psi$ instead of ψ, one can obtain complex conjugate root, $s_{1,2} = re^{\pm i\psi} = -\delta \pm i\Omega$, where the imaginary part Ω_0 represents system's natural frequency and the real part

δ represents damping ratio of the system. Further, we determine complex roots by adopting the following values of parameters of armchair [8, 8] SWCNT from Ansari et al. [35]: Young modulus $E_\infty = 1.1$ [TPa], density $\rho = 2300$[kg/m^3], thickness of nanotube $h = 0.34$ [nm] and characteristic length $a = 1.5$ [nm]. In addition, to obtain results in Table 1 we adopted angle $\psi = 1.572$ in order to determine complex roots for different length to diameter ratios $\xi = L/d$, fractional parameters α and nonlocal material constant e_0.

From Table 1 it is obvious that imaginary parts of complex roots representing natural frequencies of viscoelastic nanobeam are approaching to the frequencies obtained from MD simulations in [35] by choosing the corresponding nonlocal parameter. In Figure 2, we plotted complex roots only in the upper half of complex plane for changes of parameter κ from 0 to ∞ ($\kappa = 0$ corresponds to $s_{1,2} = \pm i\omega_0$) by changing the angle ψ from $\psi_0 = \pi/2$ to $\psi_\infty = \pi/(2 - \alpha)$. One can notice that imaginary parts of complex roots i.e. frequencies are in good agreement but slightly damped compared to nonlocal frequencies of elastic system ω_0 when higher i.e. integer and close to integer values of fractional parameter α are adopted (see Figure 2). However, lower values of fractional parameter results in higher imaginary parts i.e. natural frequencies of nanobeam. Such results for fractional Kelvin–Voigt model are in line with the results presented in [17] where "abnormal" behavior of the model was observed for $\alpha \leqslant 0.85$, where frequency increases for an increase of damping coefficient κ from 0 to ∞. In spite of their deficiencies, in the literature one can find both types of viscoelastic Kelvin-Voight models, the nonlocal integer order [27] and fractional order one [36]. For nonlocal fractional Kelvin–Voigt model we can say that even for some values of parameters "normal" results of decreasing frequency with increasing damping coefficient can be obtained, mentioned "anomaly" still does not have any rational physical explanation. Since such behavior of fractional Kelvin–Voigt model is visible in complex domain but not clearly in the time domain, both types of analysis should be performed [17].

5.2. Comparison of dimensionless eigenvalues. Here, we compare the results for dimensionless eigenvalues obtained for a nanobeam with neglected mass of nanoparticle, with the results for dimensionless eigenvalues given in Ref. [28]. The results are compared for three different types of boundary conditions. From Table 2, one can notice that our results for dimensionless eigenvalues are in excellent agreement with the results [28] for all three boundary conditions and different values of nonlocal parameter. It can be noticed that an increase of nonlocal parameter causes a decrease of eigenvalues. Further, it is obvious that for the "weaker" constraints such as cantilever and simply supported nanobeams, eigenvalues are lower than in the case of "stronger" constraints such as clamped-clamped nanobeam.

5.3. Mode shapes and eigenvalues. From Table 2 it is obvious that eigenvalues of the nanobeam without attached nanoparticle decreases for an increase of the value of nonlocal parameter. The same is true for the nanobeam with attached nanoparticle. Thus, in follow we examine the influence of the nonlocal parameter and corresponding eigenvalues on mode shapes of the nanobeam with nanoparticle

TABLE 1. Comparison of complex roots of nonlocal fractional Kelvin–Voigt simply supported nanobeam with natural frequencies [THz] of SWCNT obtained from MD simulations.

| ξ	MD [35]	Undamped frequencies		Complex roots	
		$e_0 = 0$	$e_0 = 0.691$	$e_0 = 0$	$e_0 = 0.691$
			$\alpha = 1.00$		
8.3	0.5299	0.5497	0.5299	$-0.00066 + 0.54966i$	$-0.00063 + 0.52995i$
10.1	0.3618	0.3712	0.3620	$-0.00045 + 0.37120i$	$-0.00043 + 0.36205i$
			$\alpha = 0.98$		
8.3	0.5299	0.5497	0.5299	$-0.00066 + 0.54968i$	$-0.00064 + 0.52997i$
10.1	0.3618	0.3712	0.3620	$-0.00045 + 0.37122i$	$-0.00044 + 0.36207i$
			$\alpha = 0.86$		
8.3	0.5299	0.5497	0.5299	$-0.00066 + 0.54981i$	$-0.00064 + 0.53009i$
10.1	0.3618	0.3712	0.3620	$-0.00045 + 0.37130i$	$-0.00044 + 0.36215i$
			$\alpha = 0.6$		
8.3	0.5299	0.5497	0.5299	$-0.00066 + 0.55014i$	$-0.00064 + 0.53041i$
10.1	0.3618	0.3712	0.3620	$-0.00045 + 0.37153i$	$-0.00044 + 0.36237i$
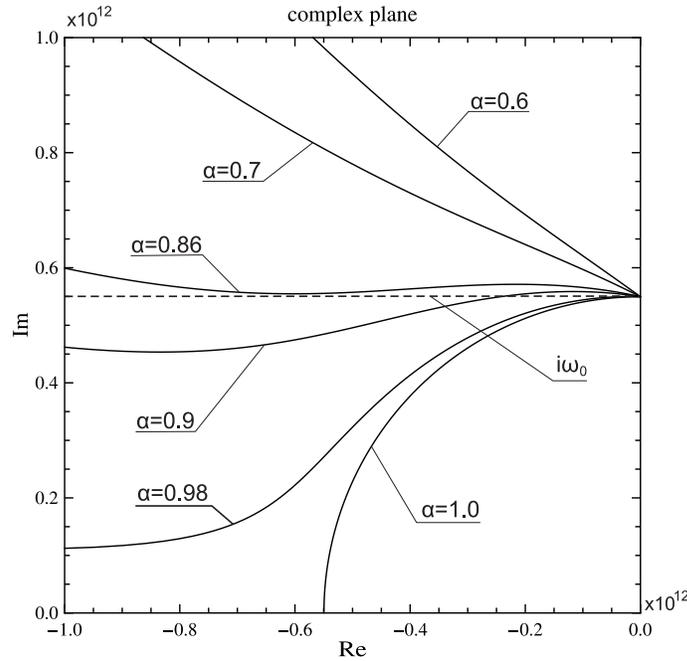
FIGURE 2. Complex roots of the nonlocal fractional Kelvin–Voigt model for changes of fractional parameter α and parameter κ from 0 to ∞.

attached at characteristic positions. Figures 3 to 5 shows the mode shapes for simply supported, clamped-clamped and cantilever boundary conditions, respectively.

In the case of the first two boundary conditions, nanoparticles are attached at the midpoint of nanobeam. In the case of cantilever boundary conditions, nanoparticle is attached at the free end. It can be noticed that for the simply supported boundary conditions, normalized mode shapes of nanobeam are the same for classical and nonlocal beam theory i.e., mode shapes of the local and nonlocal model are overlaping since there is no influence of nonlocal parameter. For clamped-clamped boundary conditions, this is not the case and there is a visible effect of nonlocal parameter on mode shapes which amplitude increases in the first mode for an increase of the nonlocal parameter. In the second mode, the nonlocal effect is less pronounced. Besides, in the case of cantilever boundary conditions we have a weak influence of nonlocal parameter on normalized mode shape in the first mode and more pronounced effect in the second mode.

TABLE 2. Eigenvalues of the nanobeam in first four modes for neglected mass of a nanoparticle.

$\bar{\mu}$	0.0		0.2		0.4		0.6	
	Ref. [28]	Present	Ref. [28]	Present	Ref. [28]	Present	Ref. [28]	Present
Simply supported nanobeam								
k_1	3.1416	3.14159	2.8908	2.89083	2.4790	2.47903	2.1507	2.15067
k_2	6.2832	6.28319	4.9581	4.95805	3.8204	3.82035	3.1815	3.18150
k_3	9.4248	9.42478	6.4520	6.45202	4.7722	4.77225	3.9329	3.93293
k_4	12.5664	12.56637	7.6407	7.64070	5.5509	5.55086	4.5565	4.55655
Clamped nanobeam								
k_1	4.7300	4.73004	4.2766	4.27661	3.5923	3.59232	3.0837	3.08370
k_2	7.8532	7.85321	6.0352	6.03522	4.5978	4.59781	3.8165	3.81649
k_3	10.9956	10.99561	7.3840	7.38403	5.4738	5.47380	4.5231	4.52315
k_4	14.1372	14.13717	8.4624	8.46242	6.1504	6.15043	5.0505	5.05048
Cantilever nanobeam								
k_1	1.8751	1.87510	1.8919	1.89193	1.9543	1.95428	2.1989	2.19894
k_2	4.6941	4.69409	4.1924	4.19238	3.3456	3.34555	2.4809	2.48090
k_3	7.8548	7.85476	6.0674	6.06738	4.8370	4.83705	. . .	. . .
k_4	10.9955	10.99375	7.3617	7.36170	5.2399	5.23986	. . .	. . .

In Table 3, we have analyzed the eigenvalues of viscoelastic nanobeam for arbitrary attached nanoparticle and fixed value of nonlocal parameter $\bar{\mu} = 0.5$ and four different positions $\bar{x}_m$ of the nanoparticle. In addition, it can be noticed that for simply supported and clamped-clamped boundary conditions, eigenvalues are decreasing for an increase of the mass of nanoparticle and for an increase of the distance $\bar{x}_m$ from the boundary. However, it is obvious that eigenvalues are increasing when a distance $\bar{x}_m$ pass the midpoint of nanobeam. As expected, these eigenvalues are equal to those with the same distance from the midpoint but in opposite direction due to the symmetric boundary conditions. For the cantilever nanobeam, we can observe a decrease of eigenvalues for an increase of the mass and distance $\bar{x}_m$. Nevertheless, this effect is pronounced only in the first mode. For the next three modes eigenvalues are given in Table 3. From the data presented, there is no clear pattern of change of eigenvalues that can be recognized.
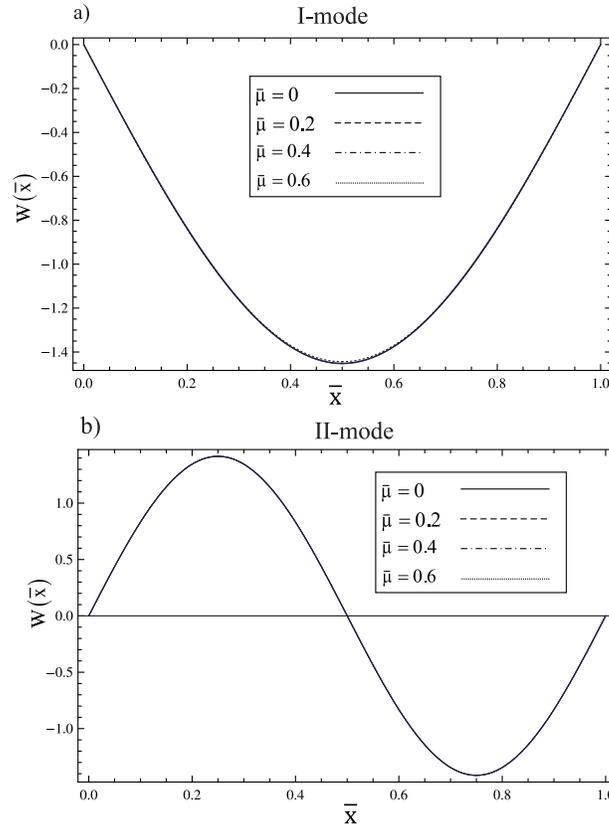
FIGURE 3. Mode shapes of nanobeam with attached nanoparticle at the middle and for simply supported boundary conditions a) first mode; b) second mode.

5.4. Time dependent behavior of nanobeam with attached nanoparticle. In the follow, we plot three characteristic cases of the time function, which defines the dynamic behavior of the nanobeam with internal fractional order damping properties in the first vibration mode i.e. for $n = 1$. In our simulation we take into account the following initial conditions $T(0) = 0$ and $\dot{T}(0) = 1$. Considering Eq. (4.5) it follows that we can expect different time behaviors depending of the sign of the value of $\omega^2 + \beta$ at $\alpha = 0$ and the sign of $\omega^2 - \beta^2/4$ at $\alpha = 1$. On Figures 6 to 8 we illustrate the continuous transition for the time function behavior from the first special case solution to the second one over the domain $0 < \alpha < 1$. For the analysis and illustration of the time dependent behavior we use three typical cases of the ratio of parameters ω and β as given in Table 4, which defines the sign of the value of $\omega^2 - \beta^2/4$ and the shape of the solution. First, in Figure 6 we plot a time function for the first case in Table 4. As it can be noticed, in this case we have a transition from harmonic oscillation at $\alpha = 0$ to damped oscillation for an
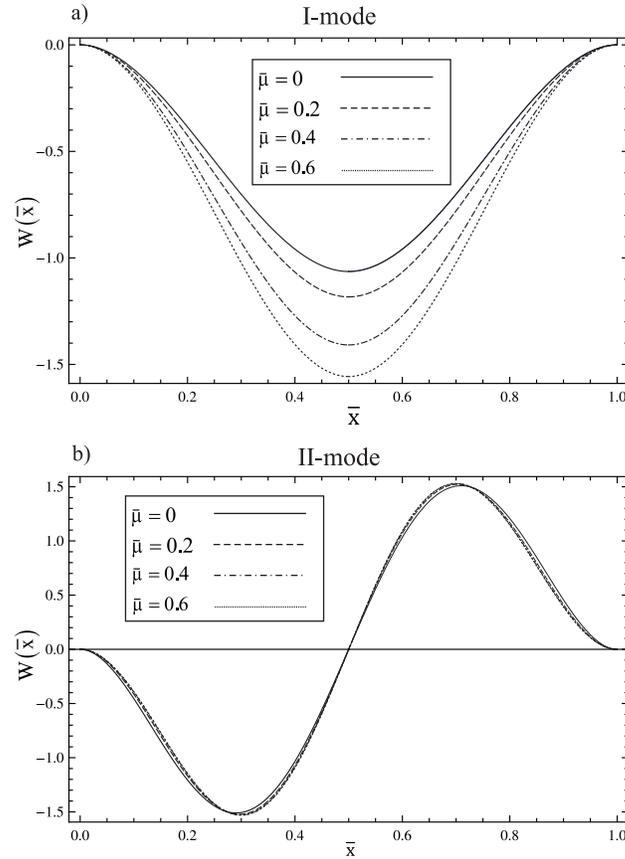
a)   I-mode

b)   II-mode

FIGURE 4. Mode shapes of nanobeam with attached nanoparti-
cle at the middle and for clamped-clamped boundary conditions
a) first mode; b) second mode.

increase of the fractional order parameter α. "Strong" damping properties are the
most pronounced at $\alpha = 1$, where the solution is equal to the solution in the case
of classical viscoelastic model with integer order derivative. The solution for the
second case from Table 4 is shown in Figure 7. One can observe a smooth transition
from harmonic oscillation to aperiodic solution over domain $\alpha \in [0, 1]$. In addition,
in Figure 6b) it can be observed that aperiodic solution starts from $\alpha = 0.6$. The
similar solution can be noticed in the third case from Table 4, which is plotted in
Figure 8. Here, we also have a transition from harmonic oscillation to aperiodic
solution over the domain $\alpha \in [0, 1]$, where the aperiodic solution starts somewhere
around the value $\alpha = 0.5$.

From the previous results, it is clear that fractional order models of viscoelas-
ticity such as fractional Kelvin–Voigt, can be used to describe variety of time de-
pendent behavior on nanoscale level depending on the values of internal damping

parameter and system eigenvalues. Thus, values of system parameters and fractional order parameter defines a continuous transition from harmonic to damped oscillation that can describe much wider range of dynamics behaviors compare to the classical viscoelastic models. Depending of the values of parameters it can also define an aperiodic behavior. Therefore, fractional derivative models are natural environment to describe damping properties, which is important from the practical point of view since such generalized models needs less number of parameters compare to the integer order ones. Thus, nonlocal theory and fractional order viscoelasticity are powerful tools for theoretical analysis of nanostructures and should be both applied for their accurate modeling.

As stated in [17], solution (4.4) of the fractional differential equation in terms of series converges very slowly and even some authors are questioning its physical interpretation, this solution gives good results for small time periods. However, in
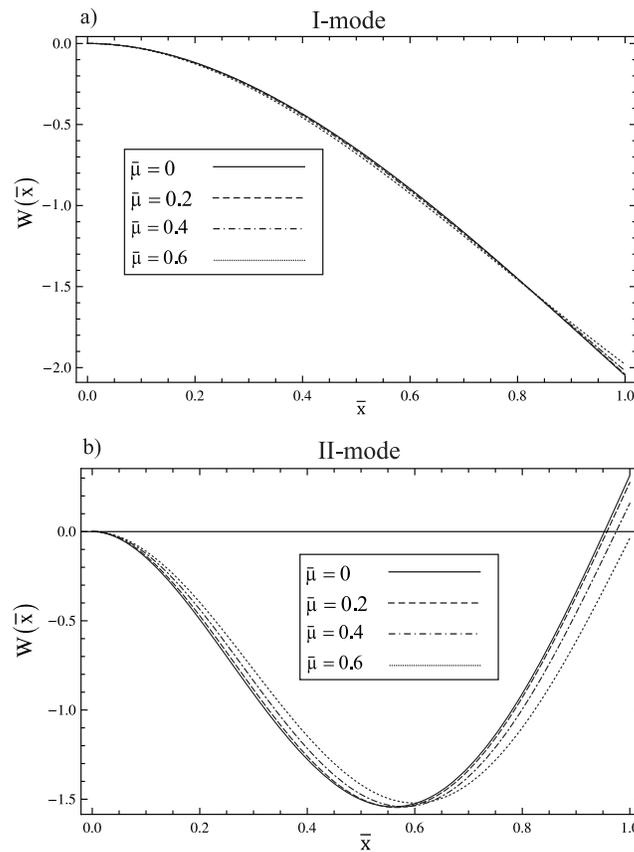

FIGURE 5. Mode shapes of nanobeam with attached nanoparticle at the middle and for cantilever boundary conditions a) first mode; b) second mode.

TABLE 3. Eigenvalues of the nanobeam with attached nanoparticle.

m			1				5	
$\bar{x}_m$	0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8
Simply supported nanobeam								
k_1	2.1963	2.0705	2.0705	2.1963	1.9096	1.6627	1.6627	1.9096
k_2	3.3501	3.4272	3.4272	3.3501	3.1904	3.3969	3.3969	3.1904
k_3	4.2397	4.2745	4.2745	4.2397	4.1833	4.2560	4.2560	4.1833
k_4	4.9687	4.9493	4.9493	4.9687	4.9553	4.9207	4.9207	4.9553
Clamped nanobeam								
k_1	3.2452	2.9797	2.9797	3.2452	2.9511	2.3838	2.3838	2.9511
k_2	4.0554	4.1068	4.1068	4.0554	3.8340	4.0734	4.0734	3.8340
k_3	4.8001	4.9288	4.9288	4.8001	4.6342	4.9255	4.9255	4.6342
k_4	5.4523	5.5024	5.5024	5.4524	5.3924	5.4840	5.4840	5.3924
Cantilever nanobeam								
k_1	2.0075	1.8701	1.6443	1.4374	1.9506	1.5628	1.2369	1.0217
k_2	2.9384	3.0550	3.2662	3.1169	2.9149	4.7597	3.4576	3.1565
k_3	4.1982	4.0221	5.2418	4.3427	3.5931	6.2969	5.0881	4.3133
k_4	5.4996	4.6954	5.9528	5.0789	6.1564	7.2681	5.9983	5.1423

TABLE 4. Three typical cases of the parameters ratio.

	β/ω	$N = \omega^2 - \beta^2/4$
1.	1/2	$N > 0$
2.	2	$N = 0$
3.	3	$N < 0$

the future investigations other available methods for finding the solution of fractional order differential equation need to be considered. In addition, it should be noted that in [30], Eq. (4.1) is considered with Caputo definition of fractional derivative and the solution is found by using the Laplace transform method and compared with the solution for $\alpha = 1$ i.e., integer order derivative case. Also, in [30] the author has discovered nine distinct cases of the solution depending of the model parameters compared to the three distinct cases observed in our work.

6. Concluding remarks

In this communication, we performed the free vibration analysis of a nanobeam with arbitrary attached nanoparticle using the fractional derivative viscoelastic model. Nonlocal theory of Eringen is applied in order to include small-scale effects appearing on the nano-scale level. Equations of motion for the system are derived and the solution is proposed using the method of separation of variables. Equations for eigenvalues and mode shapes are obtained for three typical boundary

conditions. Validation study with calculated complex roots of characteristic equation, which imaginary parts represents damped frequencies of the system, showed a good agreement with the results from molecular dynamics simulations found in the literature. In the numerical analysis, it is shown that eigenvalues are decreasing for an increase of the nonlocal parameter and mass of nanoparticle. In addition, it is revealed that change of the position of nanoparticle significantly changes the eigenvalues. These results are varying depending on the applied boundary conditions. Time dependent behavior of fractional viscoelastic nanobeam is significantly influenced by the fractional order parameter, which defines how "strong" damping properties will be pronounced. Increase of the parameter in the domain $[0, 1]$ significantly increases damping properties of the system. In addition, relations between the values of parameters of the system also defines whether nanobeam will exhibit
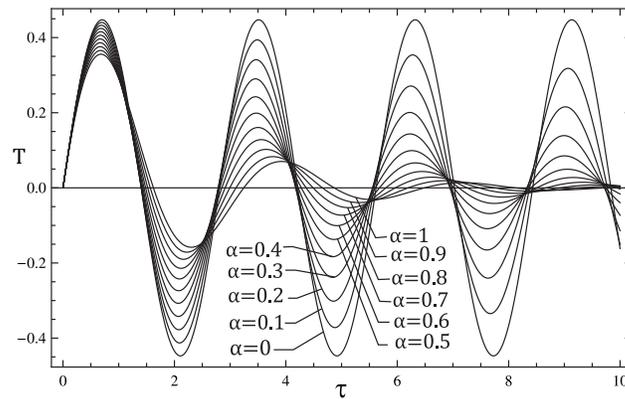

FIGURE 6. Time behavior of fractional viscoelastic nanobeam for different α and ratio $\beta/\omega = 1/2$.
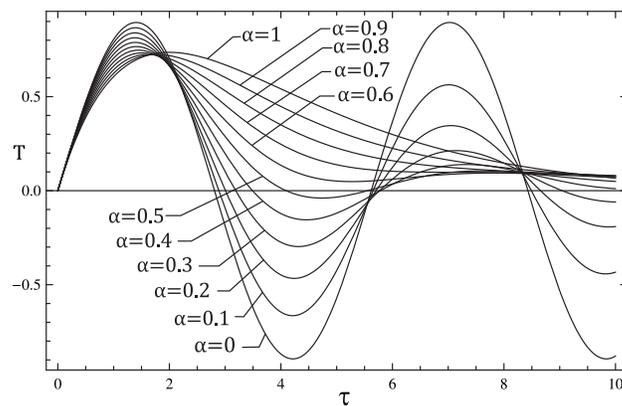

FIGURE 7. Time behavior of fractional viscoelastic nanobeam for different α and ratio $\beta/\omega = 2$.
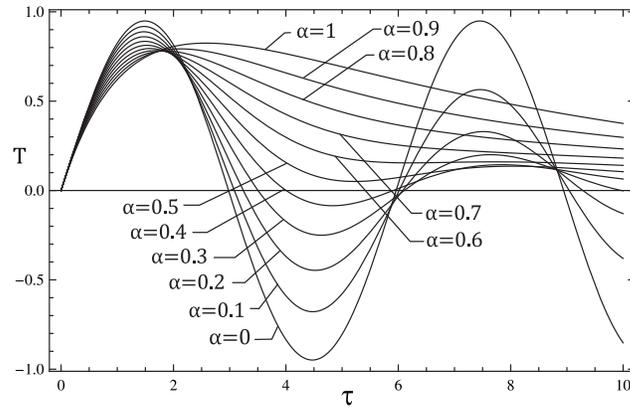
FIGURE 8. Time behavior of fractional viscoelastic nanobeam for different α and ratio $\beta/\omega = 3$.

damped or aperiodic oscillation. This study may be useful for the future analysis of other single or multiple nanostructure based systems with damping or in design procedures of real nanodevices.

References

1. O. Menicke, D. Kaempfer, H. Weickmann, C. Friedrich, M. Vathauer, H. Warth, *Mechanical Properties and electrical condutivity of carbon-nanotube filled polyamide-6 and its blends with acrylonitrile/butadiene/styrene*, Polymer **45** (2004), 739–748.
2. Y. T. Yang, C. Callegari, X. L. Feng, K. L. Ekinci, M. L. Roukes, *Zeptogram-scale nanomechanical mass sensing*, Nano Lett. **6** (2006), 538–586.
3. R. S. Ruoff, D. Qian, W. K. Liu, *Mechanical properties of carbon nanotubes: theoretical predictions and experimental measurements*, C. R. Phys. **4** (2003), 993–1008.
4. G. Cao, X. Chen, J. W. Kysar, *Thermal vibration and apparent thermal contraction of single-walled carbon nanotubes*, J. Mech. Phys. Solids **54** (2006), 1206–1236.
5. A. C. Eringen, D. G. Edelen, *On nonlocal elasticity*, Int. J. Eng. Sci. **10** (1972), 233–248.
6. A. C. Eringen, *On differential-equations of nonlocal elasticity and solutions of Screw Dislocation and Surface-Waves*, J. Appl. Phys. **54** (1983),4703–4710.
7. J. Peddieson, G. R. Buchanan, R. P. McNitt, *Application of nonlocal continuum models to nanotechnology*, Int. J. Eng. Sci. **41** (2003),305–312.
8. J. N. Reddy, *Nonlocal theories for buckling bending and vibration of beams*, Int. J. Eng. Sci. **45** (2007), 288–307.
9. D. Aydogdu, *A general nonlocal beam theory: Its application to nanobeam bending, buckling and vibration*, Physica E **41** (2009), 1651–1655.
10. D. Karličić, M. Cajić, T. Murmu, P. Kozić, S. Adhikari, *Nonlocal effects on the longitudinal vibration of a complex multi-nanorod system subjected to the transverse magnetic field*, Meccanica **50** (2015), 1605–1621.
11. T. Pritz, *Analysis of four-parameter fractional derivative model of real solid materials*, J. Sound Vib. **195** (1996), 103–115.

12. T. Atanacković, S. Pilipović, B. Stanković, D. Zorica, *Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes*, John Wiley & Sons, New York, 2014.

13. Y. A. Rossikhin, M. V. Shitikova, *Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids*, Appl. Mech. Rev. **50** (1997), 15–67.

14. R. C. Koeller, *Application of Fractional Calculus to the Theory of Viscoelasticity*, ASME J. Appl. Mech. **51** (1984), 299–307.

15. K. S. Hedrih, A. Filipovski, *Longitudinal creep vibrations of a fractional derivative order rheological rod with variable cross section*, Facta Univ. Ser. Mech. Automat. Control Robot. **3** (2002), 327–349.

16. K. S. Hedrih, *The transversal creeping vibrations of a fractional derivative order constitutive relation of nonhomogeneous beam*, Math. Probl. Eng. **2006** (2006), 1–18.

17. Y. A. Rossikhin, M. V. Shitikova, *Application of fractional calculus for dynamic problems of solid mechanics: Novel trends and recent results*, Appl. Mech. Rev. **63** (2010), 1–51.

18. J. Freundlich, *Vibrations of a simply supported beam with a fractional viscoelastic material model–supports movement excitation*, Shock Vib. **20** (2013), 1103-1112.

19. Y. A. Rossikhin, M. V. Shitikova, *Analysis of damped vibrations of linear viscoelastic plates with damping modeled with fractional derivatives*, Signal Process. **86** (2006), 2703–2711.

20. S. Hosseini–Hashemi, R. Nazemnezhad, *Surface effects on nonlinear free vibration of functionally graded nanobeams using nonlocal elasticity*, Appl. Math. Model. **38** (2014), 3538–3553.

21. D. Karličić, M. Cajić, P. Kozić, I. Pavlović, *Temperature effects on the vibration and stability behavior of multi-layered graphene sheets embedded in an elastic medium*, Compos. Struct. **131** (2015), 672–681.

22. M. A. Darabi, S. Kazemirad, M. H. Ghayesh, *Free vibrations of beam-mass-spring systems: analytical analysis with numerical confirmation*, Acta Mech. Sinica **28** (2012), 468–481.

23. S. Adhikari, R. Chowdhury, *The calibration of carbon nanotube based bionanosensors*, J. Appl. Phys. **107** (2010), 124322.

24. H. L. Lee, J. C. Hsu, W. J. Chang, *Frequency shift of carbon-nanotube-based mass sensor using nonlocal elasticity theory*, Nanoscale Res. Lett. **5** (2010), 1774–1778.

25. T. Murmu, S. Adhikari, *Nonlocal vibration of carbon nanotubes with attached buckyballs at tip*, Mech. Res. Commun. **38** (2011), 62–67.

26. K. Kiani, H. Ghaffari, B. Mehri, *Application of elastically supported single-walled carbon nanotubes for sensing arbitrarily attached nano-objects*, Curr. Appl. Phys. **13** (2013), 107–120.

27. Y. Lei, S. Adhikari, M. Friswell, *Vibration of nonlocal Kelvin–Voigt viscoelastic damped Timoshenko beams*, Int. J. Eng. Sci. **66** (2013), 1–13.

28. P. Lu, H. P. Lee, C. Lu, P. Q. Zhang, *Dynamic properties of flexural beams using a nonlocal elasticity mode*, J. Appl. Phys. **99** (2006), 073510.

29. T. M. Atanackovic, S. Pilipovic, B. Stankovic, D. Zorica, *Fractional Calculus with Applications in Mechanics: Wave Propagation, Impact and Variational Principles*, John Wiley & Sons, New York, 2014.

30. M. Naber, *Linear fractionally damped oscillator*, Int. J. Differ. Equat. **2010** (2009), 1–12.

31. T. D. Shermergor, *On the use of fractional differentiation operators for the description of elastic-after effect properties of materials*, J. Appl. Mech. Tech. Phy. **7**(6) (1966), 85–87.

32. Y. A. Rossikhin, *Reflections on two parallel ways in the progress of fractional calculus in mechanics of solids*, Appl. Mech. Rev. **63**(1) (2010), 010701.

33. J. A. dos Santos, C. M. Mota Soares, *Nonlocal material properties of single-walled carbon nanotubes*, Int. J. Smart. Nano Mater. **3** (2012), 141–151.

34. Y. Lei, T. Murmu, S. Adhikari, M. Friswell, *Dynamic characteristics of damped viscoelastic nonlocal Euler–Bernoulli beams*, Eur. J. Mech. A-Solid **42** (2013), 125–136.

35. R. Ansari, R. Gholami, H. Rouhi, *Vibration analysis of single-walled carbon nanotubes using different gradient elasticity theories*, Compos. Part B-Eng. **43** (2012), 2985–2989.

36. R. Ansari, M. Oskouie, F. Sadeghi, M. Bazdid-Vahdati, *Free vibration of fractional viscoelastic Timoshenko nanobeams using the nonlocal elasticity theory*, Physica E **74** (2015), 318–327.

ОСЦИЛАЦИЈЕ НЕЛОКАЛНЕ ВИСКОЕЛАСТИЧНЕ НАНОГРЕДЕ ФРАКЦИНОГ РЕДА СА НАНОЧЕСТИЦОМ

Резиме. У раду је предложен другачији приступ у формирању математичког модела за испитивање слободних пригушених трансверзалних осцилација наногреде применом Ерингенове нелокалне теорије и модела вискоеластичног тела фракционог реда. Једначина кретања наногреде, са наночестицом прикаченом на произвољној позицији, је изведена помоћу нелокално вискоеластичне конститутивне једначине са изводима фракционог реда и применом Ојлер–Бернулијеве теорије греда. Предложено је решење методом раздвајања променљивих. Сопствене вредности и облици осциловања су одређени за три типа граничних услова. Диференцијална једначина фракционог реда за временску функцију је решена применом Лапласове трансформације. Понашање временске функције у времену је испитано за различите вредности фракционог параметра и различите односе осталих параметара у моделу. Валидација је изведена поређењем добијених резултата за специјални случај нашег модела са одговарајућим резултатима из литратуре добијених симулацијама молекуларном динамиком.

Department of Mechanics
Mathematical Institute
Serbian Academy of Science and Arts
Belgrade
Serbia
mcajic@mi.sanu.ac.rs

Department of Mechanics
Faculty of Mechanical Enginerring
University of Niš
Niš
Serbia
danilo.karlicic@masfak.ac.ni.rs

Department of Mechanics
Faculty of Mechanical Enginerring
University of Belgrade
Belgrade
Serbia
mihailo.lazarevic@gmail.com